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A DISCONTINUOS GALERKIN METHOD FOR THE DYNAMIC ANALYSIS OF FULLY-SATURATED POROUS MEDIA

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Summary. *We present a time-discontinuous Galerkin method (TDG) for the dynamic analysis of fully-saturated porous media. The method consists of a finite element discretization in space and time simultaneously. The discrete basis functions are continuous in space and discontinuous in time. The continuity across the time interval is weakly enforced by the flux function. The resulting Bubnov-Galerkin method is stable and requirs no extra stabilization term.*

1 INTRODUCTION AND GOVERNING EQUATIONS

In our days the Discontinuous Galerkin method has established itself as a viable method for solving partial differential equations and has found a wide variety of applications, c. f. [1] and the literature therein for an overview of the state of the art. In this work we consider a time-discontinuous Galerkin method for a two-phase porous medium with incompressible constituents. The model equations are based on the well studied Theory of Porous Media (TPM), c. f. [2] with restriction to the geometrical linear case. The four fields formulation of the concerned problem was proposed by *Diebels* and *Ehlers* [3] in 1996 as follows

$$\mathbf{u}_S' = \mathbf{v}_S \quad \text{in } \Omega \times \mathcal{I}, \quad (1)$$

$$(n^S \rho^{SR} + n^F \rho^{FR}) \mathbf{v}_S' + n^F \rho^{FR} \mathbf{w}_F' = \operatorname{div}\left(\mathbf{T}_E^S - p\,\mathbf{I}\right) - \left(n^S \rho^{SR} + n^F \rho^{FR}\right) \mathbf{b} \quad \text{in } \Omega \times \mathcal{I}, \quad (2)$$

$$\rho^{FR} \mathbf{v}_S' + \rho^{FR} \mathbf{w}_F' + \frac{n^{FR} \gamma^{FR}}{k^F} \mathbf{w}_F + \operatorname{grad} p = \rho^{FR} \mathbf{b} \quad \text{in } \Omega \times \mathcal{I}, \quad (3)$$

$$\operatorname{div}(\mathbf{u}_S' + n^F \mathbf{w}_F) = 0 \quad \text{in } \Omega \times \mathcal{I}, \quad (4)$$

where $\mathcal{I}$ denotes the time interval $[t_0, T]$ and Ω is the spatial domain of the body, $(\cdot)_\alpha'$ represents the material time derivative with respect to the constituent φ^α. The variables are the displacement $\mathbf{u}_S$, and the velocity $\mathbf{v}_S$ of the solid phase, the seepage velocity $\mathbf{w}_F$=$\mathbf{v}_F - \mathbf{v}_S$ concerning the relative movement of the fluid phase with respect to the solid phase, as well as the fluid pore pressure p. The permeability coefficient k^F measures the resistance of the medium to fluid flow, $n^\alpha = \mathrm{d}v^\alpha / \mathrm{d}v$ represents the volume fraction

of the constituent φ^α and $\rho^{\alpha R}$ represents the effective material density with $\alpha \in \{S, F\}$ (S for solid, F for fluid). Material incompressibility is assumed for both constituents, i. e. $\rho^{\alpha R}$ =const. The effective (true) weight of the fluid is denoted by γ^{FR}, $\mathbf{T}_E^S$ represents the Cauchy extra stress of the solid phase given by a constitute equation, and $\mathbf{b}$ is the body force. An elaborate interpretation of the equations can be found in references [3, 4]. The boundary of the domain is decomposed into Dirichlet and Neumann parts. Due to the superimposed constituents of the solid and fluid phases in the mixture the boundaries of both constituents are described separately as follows

$$\begin{aligned} \mathbf{u}_S(\mathbf{x},t) &= \tilde{\mathbf{u}}_S \quad \text{for} \quad \mathbf{x} \in \Gamma_D^S \times \mathcal{I}, \\ p(\mathbf{x},t) &= \tilde{p} \quad \text{for} \quad \mathbf{x} \in \Gamma_D^F \times \mathcal{I}, \\ (\mathbf{T}_E^S - p\,\mathbf{I})(\mathbf{x},t)\cdot\mathbf{n} &= \bar{\bar{\mathbf{t}}} \quad \text{for} \quad \mathbf{x} \in \Gamma_N^S \times \mathcal{I}, \\ \mathbf{w}_F(\mathbf{x},t)\cdot\mathbf{n} &= \bar{\bar{w}} \quad \text{for} \quad \mathbf{x} \in \Gamma_N^F \times \mathcal{I}, \end{aligned} \tag{5}$$

where $\partial\Omega = \Gamma_D^S \bigcup \Gamma_N^S = \Gamma_D^F \bigcup \Gamma_N^F$ with $\Gamma_D^S \bigcap \Gamma_N^S = \emptyset$ and $\Gamma_D^F \bigcap \Gamma_N^F = \emptyset$. The initial conditions are prescribed for both constituents as $\mathbf{u}_{S,0}$, $\mathbf{v}_{S,0}$, $\mathbf{w}_{F,0}$ and p_0 in a consistent way.

2 SPACE-TIME FORMULATION

In this section we introduce the time-discontinuous Galerkin formulation in detail. The space-time domain ($Q = \Omega \times \mathcal{I}$) is intuitively constructed by adding an time axis which is orthogonal to the spatial domain, such that a spatially one-dimensional problem results in two-dimensional elements, and a two-dimensional problem results in three-dimensional elements, etc. Let $t_0 <, \cdots, < t_n < t_{n+1} <, \cdots, < T$ be a sequence of discrete time level t_n, consequently we obtain the time interval $\mathcal{I}^n = (t_n, t_{n+1}]$. Hereby we introduce the definition of a space-time slab $Q^n = \mathcal{I}^n \times \Omega$. The space-time elements are constructed on every such slab. The space-time elements for one/two dimensional spatial domains are shown exemplarily in form of linear elements in Fig. 1. The approximate solutions are solved sequentially on every slab, which leads to a method analog to the general time difference method where the solution on every time step t_n is solved consecutively (method of lines). The weak forms of the considered problem are obtained by multiplying certain

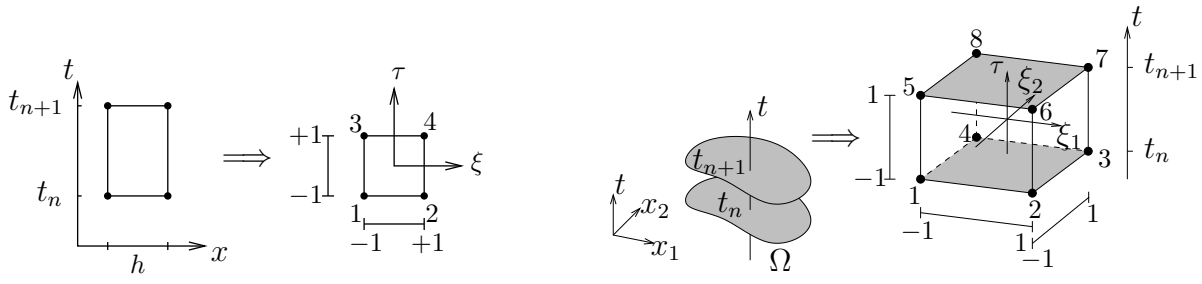

Figure 1: Space-time elements for one/two dimensional space

test functions to the governing equations and then integrating over the space-time slab

Q^n. Owning to the virtue of partial integration in time, we obtain excluded values on the border of time interval, as briefly shown below

$$\int_{Q^n} \mathbf{F}' \cdot \delta\mathbf{F} \, \mathrm{d}v\mathrm{d}t = -\int_{Q^n} \mathbf{F} \cdot \delta\mathbf{F}' \, \mathrm{d}v\mathrm{d}t + \int_{\Omega} \underbrace{\bar{\mathbf{F}} \cdot \delta\mathbf{F}|_{t_n}^{t_{n+1}}}_{= \bar{\mathbf{F}}_{n+1} \cdot \delta\mathbf{F} - \bar{\mathbf{F}}_n \cdot \delta\mathbf{F}} \mathrm{d}v \tag{6}$$

in which $\mathbf{F}$ could be any concerned quantity in the weak form (e. g. $\mathbf{u}_S$, $\mathbf{v}_S$, $\mathbf{w}_F$ or p) and $\delta\mathbf{F}$ represents its corresponding test function. The subscript n or $n+1$ represents the time level t_n or t_{n+1}, respectively. It is obvious that in general (continuous case) $\bar{\mathbf{F}} = \mathbf{F}_n$. However, under the discontinuous assumption inconsistent values before and after t_n are allowed. In order to deal with such inconsistent values, we introduce here the numerical flux concerning the *upwind* treatment on the interface as

$$\bar{\mathbf{F}}(\mathbf{x}, t_n) = \begin{cases} \mathbf{F}_0(\mathbf{x}), & \text{if} \quad t_n = t_0, \\ \mathbf{F}_n^-(\mathbf{x}), & \text{otherwise} \end{cases}, \quad \text{where} \quad \mathbf{F}_n^-(\mathbf{x}) = \lim_{\epsilon \to 0^+} \mathbf{F}(\mathbf{x}, t_n - \epsilon). \tag{7}$$

where $\mathbf{F}_0$ is the respective initial value (e. g. $\mathbf{u}_{S,0}$, $\mathbf{v}_{S,0}$ or $\mathbf{w}_{F,0}$). Such concept of numerical flux stems from the finite volume methods and has become an important member in the discontinuous Galerkin family in the recent 10 years. One simple reason for accepting such assumption in a time-dependent problem is that the value at the certain time level t_n must equal the value of its immediate past $t_n - \epsilon$ (ϵ is a infinite small positive value). Therefore, it is natural to start the procedure with $\bar{\mathbf{F}} = \mathbf{F}_0$, which is the initial value at t_0. An survey of the flux like treatment can be found in [1] and the literature therein. The governing weak form on the time slab Q^n is given here for the balance of momentum of the mixture eq. (2) exemplarily

$$\begin{aligned} &\int_{Q^n} \Big\{ -(n^S\rho^{SR} + n^F\rho^{FR})\mathbf{v}_S \cdot \delta\mathbf{v}_S' - \rho^F \mathbf{w}_F \cdot \delta\mathbf{v}_S' + (\mathbf{T}_E^S - p\,\mathbf{I}) : \operatorname{grad}\delta\mathbf{v}_S \Big\} \, \mathrm{d}v \, \mathrm{d}t \\ &+ \int_{\Omega} \Big\{ (n^S\rho^{SR} + n^F\rho^{FR})(\mathbf{v}_{S,n+1}^- - \bar{\mathbf{v}}_{S,n}) \cdot \delta\mathbf{v}_S + n^{FR}\rho^F(\mathbf{w}_{F,n+1}^- - \bar{\mathbf{w}}_{F,n}) \cdot \delta\mathbf{v}_S \Big\} \, \mathrm{d}v \\ &= \int_{\Gamma_N^S \times \mathcal{I}^n} \Big\{ \Big((\mathbf{T}_E^S - p\,\mathbf{I}) \cdot \mathbf{n} \Big) \cdot \delta\mathbf{v}_S \Big\} \, \mathrm{d}a \, \mathrm{d}t + \int_{Q^n} \Big\{ (n^S\rho^{SR} + n^F\rho^{FR})\mathbf{b} \cdot \delta\mathbf{v}_S \Big\} \, \mathrm{d}v \, \mathrm{d}t. \end{aligned} \tag{8}$$

The weak forms of the remaining set of equations (1,3-4) can be calculated similarly.

3 NUMERICAL EXAMPLES

We consider here a one-dimensional consolidation problem with a constant external load on the drained surface at $x = L$. The geometry and the material parameters are shown in Fig. 2. For the spatial discretization we choose biquadratic ansatz functions for $\mathbf{u}_S$ and $\mathbf{v}_S$, and bilinear ansatz functions for $\mathbf{w}_F$ and p. The chosen polynomials

in the time domain are standard linear *Lagrange* polynomials. We depicted the curves of the seepage velocity $w_F(x_P)$ at the point P obtained by the TDG and the implicit *Euler* method in Fig. 2. It is observed, that the numerical dissipation spoiled the results dramatically in the implicit Euler method if we apply a large time step ($\Delta t = 1$ s). The traveling wave was completely damped out, and can be obtained within the implicit *Euler* method only for much smaller time steps. The TDG shows a much favorable property, even with a coarse discretization ($\Delta t = 1$ s), the traveling wave is not damped out. Under the coarse discretization, the TDG shows a visible jump property at the end of each time step, which is well cured with refined time step length ($\Delta t = 0.1$ s). Although such jumps are unphysical, they render a more accurate numerical solution in practical finite element calculations. Furthermore, adaptive space-time strategies could be developed, e. g. a simple gradient-based error indicator could be based on the jumps in analogy to the Z^2 error indicator.

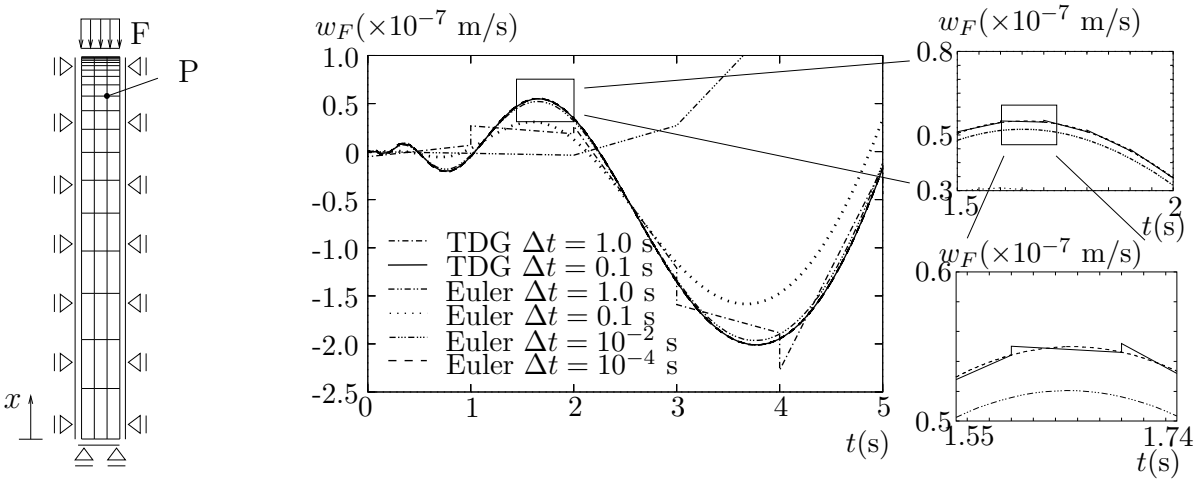

Figure 2: Seepage velocity (w_F) at point P

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