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HIGH ORDER EXTENDED FINITE ELEMENT METHOD FOR CRACKED DOMAINS

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Summary. *This paper deals with the numerical modelling of fractures in bidimensional elasticity. Using the XFEM method, we suggest some improvements we have developped in order to obtain an optimal rate of convergence even if high order polynomial functions are used in the FEM discretization.*

1 INTRODUCTION

Computer simulation of fracture processes remains a challenge for many industrial modelling problems. In a classical fi nite element method, the non-smooth displacement near the crack tip is captured by refi ning the mesh locally. The number of degrees of freedom may drastically increase, especially in three dimensional applications. Moreover, the incremental computation of a crack growth needs frequent remeshings. Reprojecting the solution on the updated mesh is not only a costly operation but also it may have a troublesome impact on the quality of results.

In the last few years, Moës, Dolbow and Belytschko[1] introduced a numerical methodology which has been developed by the name of XFEM - eXtended Finite Element Method. Not only fi nite elements are enriched with the asymptotic crack tip displacement solutions, but also with

a step function which takes into account the jump of the displacement across the crack. Then, the finite element mesh can be defined independently of the crack geometry. The partition of unity chosen to localize the enrichment functions is linked to the mesh and is generally defined using linear shape functions.

An advantage of the XFEM method is to obtain more accurate numerical results than classical finite element one. However, the rate of convergence remains of order $\sqrt{h}$ with respect to the mesh parameter h and is independant of the degree of the finite element method (see Fig. 1 and Stazi et al[2]). Then, the aim of this paper is to introduce some improvements to this method in order to reach the optimal rate of convergence.

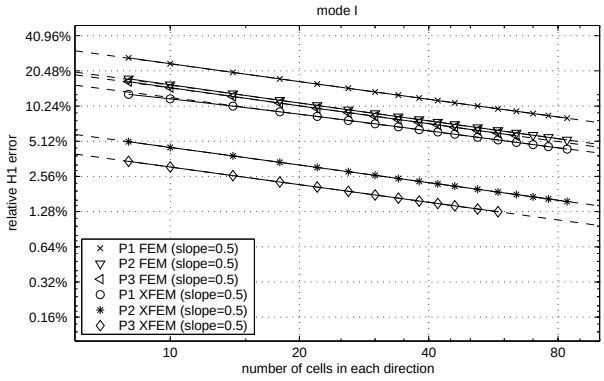

Figure 1: Convergence comparison of classical finite elements and XFEM

2 A NEW ENRICHMENT METHOD NEAR THE CRACK TIP

2.1 The classical XFEM approach

Let us consider a bidimensional linear elasticity problem on a cracked domain. On a regular triangulation of the (non-cracked) domain Ω, we define the set of vector valued functions φ_i which form a basis of the discrete space. The associated finite element function is supposed to be the classical Lagrange polynomial of order k.

In the classical XFEM[1], the enriched space is built with the following functions :

$$v_h = \sum_{i=1}^{N} a_i \varphi_i + \sum_{i \in I_H} b_i H \psi_i + \sum_{j=1}^{4} \sum_{i \in I_F} c_{ij} F_j \psi_i \tag{1}$$

where ψ_i are the basis functions of the vector P_1 FEM, I_H is the set of nodes whose basis functions support is entirely splitted by the crack, so which are enriched by the global discontinuous step function H. This function is equal to $+1$ on one side of the crack and -1 on the other. Finally, I_F is the set of nodes which contain the crack tip in the (interior of) support of their basis function. Finally, the expressions of F_j are, in polar coordinates relatively to the crack tip :

$$F_1 = \sqrt{r}\sin\frac{\theta}{2}\ ,\ F_2 = \sqrt{r}\cos\frac{\theta}{2}\ ,\ F_3 = \sqrt{r}\sin\frac{\theta}{2}\cos\theta\ ,\ F_4 = \sqrt{r}\cos\frac{\theta}{2}\cos\theta \tag{2}$$

2.2 First improvements

First, an original quadrature rule was defined, in order to obtain a better accuracy in the computation of the elementary stiffness matrix for triangles which contains the crack tip. In fact, these integrals can be written as : $\int_T \nabla(F_i\,\varphi_j)\ .\ \nabla(F_k\,\varphi_l)\ dx$. It is easy to see that ∇F_i has a singularity of order $r^{-1/2}$ and that expressing the previous integral in polar coordinates will cancel this singularity. So, in practice, for numerical integration, the triangle, which contains the crack tip, is divided in a few number of subtriangles such that the crack tip is a vertex of them. On each subtriangle, a polar integration is used. We show[3] that this integration method gives excellent results with a very lower number of integration points than a classical Gauss quadrature.

Second, when P_k polynomial functions are used for the classical FEM (first term of (1)), the same shape functions will be used for the approximation of the jump displacement : $\sum_{i\in I_H} b_i H\psi_i$ will be replaced by $\sum_{i\in I_H} b_i H\varphi_i$.

Finally, in the classical XFEM method, only the nodes the nearest to the crack tip are enriched; consequently the support of the additional basis functions vanishes when h goes to zero. So we propose to enrich a whole fixed area around the crack tip independently of h. In practice, the size of this area is about $1/10th$ of the domain size.

With all this modifications, the expected optimal rate of convergence is nearly reached (see Fig. 2). But the condition number of the linear system and the number of unknowns increase[3] .

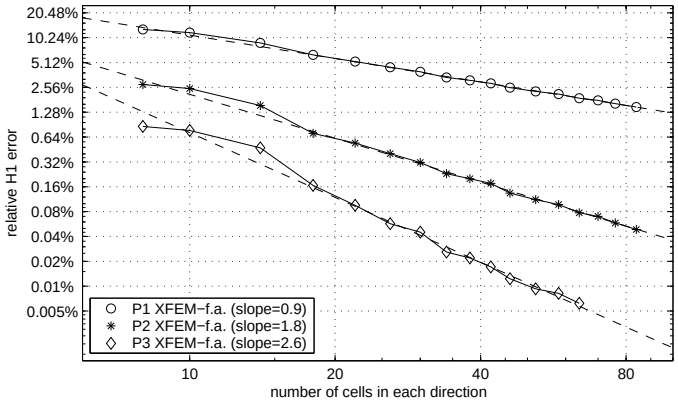

Figure 2: Convergence of XFEM with a fixed enrichment area

2.3 Bonding condition

So, a bonding condition is introduced on the enrichment area around the crack tip : for each singular shape function, the equality of the corresponding degrees of freedom is prescribed. Doing so, the total number of unknowns is roughly the same as the classical FEM, even if the size of the stiffness matrix is increased. Moreover, the condition number of this matrix is drastically improved. Unfortunately, the numerical tests show an important lake of optimality. A simplifi ed analysis shows that this is due to the transition layer between the enriched area and the rest of domain[3].

To overcome it, we propose a nodal matching condition at the interface and numerical tests show that the optimal convergence rate is then obtained (see Fig. 3) even for high order polynomial functions.

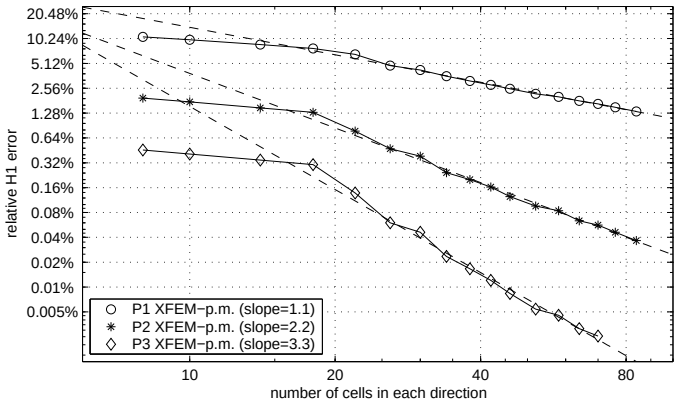

Figure 3: Convergence of XFEM with the pointwise matching condition

All the numerical tests concern a bidimensional plate, using degree one, two and three polynomials for the classical FEM approximation, on triangular meshes, in the frame of the Getfem library[4].

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