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A NEW NUMERICO-ANALYTICAL METHOD FOR THE SOLUTION OF ELASTOPLASTIC EQUATIONS BASED ON THE SPLITTING OF CONSTITUTIVE EQUATIONS

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Summary. *The new numerico-analytical method for the integration static and dynamic problems of elastoplastic equations suggested. The method is based on the splitting of constitutive equations for the models independent on the time scale as well as for the strain rate dependent models. The proposed method has significant computational advantages as compared with the standard iteration methods.*

1 INTRODUCTION

The distinctive feature of the constitutive equations of the plasticity theory is that in addition to differential relations, these equations involve also a finite relation that constrains the stress tensor invariants. Owing to this, various mathematical formulations are allowed for the constitutive relations of the model. The most widespread statement involves differentiation of the plasticity conditions to reduce the problem to a system of differential equations. A drawback of this approach is that the artificial differentiation leads to an increase in the order of the system of constitutive equations and substantially complicates solution.

We will show that there is another approach which is using decomposition of the constitutive equations, this allows one to simplify the solution of the problem.

The idea of the decomposition method is simple and well known. An additive operator $A(u)$ can be replaced by a multiplicative operator on a small time interval Δt.

$$\begin{gathered}\frac{\partial u}{\partial t} = A(u) = (A_1 + A_2 + \cdots + A_n)\, u, \\ u^{n+1}(t+\Delta t) = \left[E + (A_1 + A_2 + \cdots + A_n)\Delta t\right] u^n(t),\end{gathered} \tag{1}$$

where A_n are matrix differential operators with respect to the spatial variables, E is the identity operator, and u is a vector. The solution of the difference equation of (1) can be represented in the form of a multiplicative operator

$$u^{n+1}(t+\Delta t) = (E+A_1\Delta t)(E+A_2\Delta t)\cdots(E+A_n\Delta t)u^n(t) = (E+A\Delta t)u^n(t)+O(\Delta t^2). \tag{2}$$

This allows one to reduce the numerical integration of systems of differential equations of the form of (1) to the successive solution of problems for each of the operators A_n, with the initial conditions being obtained as the solution of the previous problem for A_{n-1}.

2 SPLITTING OF ELASTOPLASTIC EQUATIONS

The system of equations governing the behavior of a hypoelastoplastic medium subject to finite strains involves the conservation laws for mass, momentum, and energy. These equations should be supplemented with the constitutive equations of the hypoelastoplastic medium. In the Eulerian variables, these equations have the form

$$\frac{D\sigma_{ij}}{Dt} = \frac{d\sigma_{ij}}{dt} + \Omega_{ik}\sigma_{kj} + \Omega_{ki}\sigma_{jk} = D_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl}), \tag{3}$$

where $D\sigma_{ij}/Dt$ is the Jaumann derivative, D_{ijkl} is the tensor of elastic moduli, and $\Omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ is the rotation rate tensor. The plasticity condition in general form is expressed by a finite relation between the invariants of the stress J_i, strain E_i and strain rate $\dot{E}_i$ tensors and the internal parameters of the medium χ_k

$$F(J_i, E_i, \dot{E}_i, \chi_k) = 0, \qquad (i = 1,2,3, \quad k = 1,2,\cdots,n) \tag{4}$$

The plastic strain rate tensor is defined by the associated flow law

$$\dot{\varepsilon}^p_{ij} = \dot{\Lambda}\frac{\partial F}{\partial \sigma_{ij}}, \tag{5}$$

$$\dot{\chi}_k = f_k(J_i, \chi_k). \tag{6}$$

The dot stands for the material derivative with respect to time. Relation (3) follows from the additivity of strains combined with Hooke's law and provides the equation for stresses.

The decomposition general scheme looks as follows. To be decomposed is only Eq. (3).

The predictor is taken at $\dot{\varepsilon}^p_{ij} = 0$, which corresponds to the elastic material. In this case, for each step Δt, one should solve the elastic problem

$$\rho\frac{\partial v_i}{\partial t} = \sigma_{ij,j}, \qquad \frac{\partial \sigma_{ij}}{\partial t} + \Omega_{ik}\sigma_{kj} + \Omega_{ki}\sigma_{jk} = \frac{1}{2}D_{ijkl}(v_{i,j} + v_{j,i}), \tag{7}$$

subjected to the initial conditions obtained at the previous step for the complete problem.

The corrector is taken at $\dot{\varepsilon}_{ij} = 0$ in Eq. (3). In this case, Eqs. (3) and (5) lead to the stress relaxation equation

$$\frac{d\sigma_{ij}}{dt} = -\frac{d\Lambda}{dt}D_{ijkl}\frac{\partial F}{\partial \sigma_{kl}}. \tag{8}$$

For the elastoplastic medium, the relaxation is completed before the elastic unloading has occurred. For the elastoviscoplastic medium, the relaxation is completed before the steady-state plasticity condition has held.

$$F(I_i, \dot{I}_i, \chi_i)|_{\dot{I}_i=0} = 0, \tag{9}$$

For the classical **elastoplastic medium**, the properties of which are independent of the change in the time scale, one can eliminate the time t from Eqs. (8)

$$\frac{d\sigma_{ij}}{d\Lambda} = -D_{ijkl}\frac{\partial F}{\partial \sigma_{kl}}. \tag{10}$$

This system of differential equations on small time interval Δt can be always linearized and solved in close analytical form with accuracy $O(\Delta t)$, the same as the main decomposition procedure has. Analytical solution significantly simplifies the solution.

Solve the equations of (10) and (6) subjected to the initial conditions $\Lambda = \Lambda_0$, $\sigma_{ij} = \sigma_{ij}^e$, and $\chi_i = \chi_i^e$ (resulting from the solution of the elastic problem) we fined functions

$$\sigma_{ij} = \sigma_{ij}(\Lambda, \sigma_{ij}^e, \chi_i^e), \quad \chi_{ij} = \chi_{ij}(\Lambda, \sigma_{ij}^e, \chi_i^e). \tag{11}$$

Substitute (11) into the plasticity condition of (4) to obtain the equation for Λ

$$F\big[I_i(\Lambda), \chi_i(\Lambda), J_i^e, \chi_i^e\big] = 0, \tag{12}$$

By solving this equation and substituting the resulting Λ into (11) we obtain the final solution of the problem.

For the *isotropic hardening Mises plasticity*

$$J_2 = \left(\frac{1}{2}s_{ij}s_{ij}\right)^{1/2} = k_0 + 2\mu_1\chi^\beta. \tag{13}$$

we obtain the power-law nonlinear equation to determine the correction coefficient x

$$J_2^e x - k_0 + 2\mu\left[\xi^e + \frac{(J_2^e)^\alpha}{2\alpha\mu}(1 - x^\alpha)\right]^\beta = 0, \qquad x = e^{-2\mu(\Lambda - \Lambda_0)}. \tag{14}$$

For the linear hardening law, $\beta = 1$, $\alpha = 1$, the solution is obtained in closed form

$$x = \left[k_0 + \frac{\mu_1}{\mu}(J_2^e + 2\mu_1\chi^e)\right]\left(1 + \frac{\mu_1}{\mu}\right)^{-1}(J_2^e)^{-1}. \tag{15}$$

For the case of perfect plasticity, $\mu_1 = 0$, we have $x = k_0/J_2^e$, $s_{ij}^{n+1} = s_{ij}^e k_0/J_2^e$. This is the familiar Wilkins correction rule that applies only to perfectly plastic media. The formal extension of this rule to the hardening medium by setting $x = (k_0 + 2\mu_1\chi^e)/J_2^e$ leads to an erroneous result.

Another widely used plasticity theory is that due to *Prager and Drucker.*

$$F(J_1, J_2, k) = J_2 + aJ_1 - k = 0, \qquad (0 < a < 1) \tag{16}$$

The Eq. (12) gives the following expression for the correction coefficient x

$$J_2^e x + a\left[J_1^e - \frac{9Ka}{\mu}J_2^e(1 - x)\right] - k_0 = 0, \qquad J_1 = J_1^e - \frac{9Ka}{\mu}J_2^e(1 - x) \tag{17}$$

The correction coefficient involves an additional term due to the influence of the first invariant. One can see that the Wilkins correction rule does not apply in this case.

For the **elastoviscoplastic medium**, the plasticity condition is of differential type.

$$F_1(\sigma_{ij}^e, \chi_i^e, \Lambda, \dot{\Lambda}) = 0. \tag{18}$$

Solve this relation for $\dot{\Lambda}$ to obtain the equation

$$\tau \frac{d\Lambda}{dt} = \varphi\big[F\big(J_i(\Lambda), \chi_i(\Lambda)\big)\big] = \varphi(F), \tag{19}$$

where τ is the relaxation time and $\varphi(0) = 0$. The right-hand side of Eq. (19) involves the function $F = F\big(J_i(\Lambda), \chi_i(\Lambda)\big)$ that corresponds to the equilibrium plastic state for $\tau = 0$ of Eq. (12).

This decomposition scheme is stable, if the predictor scheme for the solution of the elastic problem is stable and there exist solutions of Eqs. (12) and (19).

Let consider the viscoplasticity condition of Mises type (13)

$$\tau \dot{E}_2^p = \varphi\left[F(J_i, \chi_i) - k_0\right], \quad \varphi(0) = 0. \tag{20}$$

Than differential Eq. (19) will be in the following form

$$\frac{dx}{dt} = -\frac{2\mu}{\tau J_2^e}\left\{J_2^e x - k_0 + 2\mu_1\left[\chi^e + \frac{J_2^e}{2\alpha\mu}(1 - x^\alpha)\right]^\beta\right\}^{1/n}. \tag{21}$$

For the perfect elastoviscoplastic medium we obtain

$$\frac{\Delta t}{\tau} = -\frac{1}{2\mu}\int_1^x \left(x - \frac{k_0}{J_2^e}\right)^{-1} dx, \qquad x - \frac{k_0}{J_2^e} = \left(1 - \frac{k_0}{J_2^e}\right)\exp\left(-\frac{\Delta t}{\tau}\cdot\frac{2\mu}{J_2^e}\right),$$

$$x \to x_*^p = \frac{k_0}{J_2^e} \quad \text{when} \quad \tau \to 0.$$

This implies that the solution converges to the equilibrium one for $(\Delta t/\tau)(2\mu/J_2^e) \gg 1$ and, hence, the difference scheme is absolutely stable at the corrector stage.

3 CONCLUSIONS

The proposed method has significant computational advantages as compared with the standard iteration methods. At each integration step, this method involves only one solution of an elasticity problem and the solution of one equation for the correction coefficients at the corrector stage. In contrast to this, the traditional methods require solving a system of n non-linear constitutive equations ($n \geq 6$) at each point of the body in the case of explicit scheme. For the implicit schemes the advantage is much greater.

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