Quasi-symmetrical formulations for contact and friction between deformable bodies: application to 3D forging

L. Fourment and S. Popa*

CEMEF, Ecole des mines de Paris, B.P. 207, 06 904 Sophia Antipolis Cedex, France

e-mail: lionel.fourment@ensmp.fr

Web page: http://www-cemef.cma.fr/welcome-us.html

INTRODUCTION

To properly handle the contact conditions without introducing spurious numerical constraints, the master / slave approach is inescapable but it results into a non-symmetric formulation for non-coinciding meshes [1]. From a theoretical standpoint, this unsatisfactory treatment of the contact area results into a decrease of the convergence rate of the finite element method [2]. From a more practical standpoint, severe problems arise when the discretization of the master surface is much finer than the slave surface. In metal forming, the workpiece is always the slave while the tools are the masters. So aiming at accurate tool stress calculations require masters meshes that are locally much finer than the corresponding mesh on the slave-workpiece. With a standard formulation, parts of the tool contact surface may result to be numerically unloaded, so providing very inaccurate finite element solution where high accuracy is required. A symmetric formulation has been proposed in [3], but it introduces spurious constraints. In [1], an accurate calculation of the contact conditions between the contacting bodies in proposed, while in [2], a L^2 enhanced projection of the displacement field on the contact surface is developed. Both algorithms are written in 2D for an integrated formulation. Their extension in 3D seems quite uneasy. We then proposed a quasi-symmetric formulation [4]. It can be compared to [3] but the contact Lagrange multipliers are not duplicated on both contact surfaces. It so allows avoiding introducing spurious constraints, while keeping a simple and almost symmetric formulation. The implementation is not too complex. It is carried out into the FORGE3® finite element software, where a nodal (node to facet) contact formulation is utilized and the contact conditions are handled by a penalty method. A series of patch tests that have been proposed in [1] and [2], allows evaluating the convergence rate of this formulation and its robustness.

QUASI-SYMMETRIC FORMULATION

Contact formulation

When body *B* is in contact with body *A*, the contact conditions are written on the bodies interface $\partial\Omega_{AB}$ as:

$$\text{on } \partial\Omega_{AB}, \quad h_A(u_B) = (u_B - u_A) \cdot n_B - \delta^{BA} \leq 0 \tag{1}$$

where u_A and u_B respectively are the displacements of *A* and *B*, δ^{BA} is the signed distance between *B* and *A*, and $h_A(v_B)$ is the gap function. Using an implicit Euler scheme for time discretization, displacements $u^{t+\Delta t}$ at time $t+\Delta t$ can be written as a function of the velocity field $v^{t+\Delta t}$ at time $t+\Delta t$:

$$u^{t+\Delta t} = x^{t+\Delta t} - x^t = v^{t+\Delta t}\Delta t \tag{2}$$

So, the unilateral contact equation (1) becomes:

$$h_A(v_B) = (v_B^{t+\Delta t} - v_A^{t+\Delta t}) \cdot n_B{}^t - \frac{\delta^{BA}}{\Delta t} \leq 0 \tag{3}$$

The following Lagragian is then introduced to handle the contact constraints:

$$\Lambda_B = \int_{\partial\Omega_B^{contact}} h_A(v_B)\lambda^B ds_B \tag{4}$$

where λ^B are the contact Lagrange multipliers on body *B*. $\partial\Omega_B^{contact}$ is potential contact surface. Λ_A can be defined in a similar way. At the continuous level, these two Lagrangians are equivalent: $\Lambda = \Lambda_A = \Lambda_B$

Quasi-symmetric formulation

At a discrete level, when the finite element meshes of $\partial\Omega_A^{contact}$ and $\partial\Omega_B^{contact}$ do not coincide, this equality is not satisfied. Using both Lagrangians yields an over-constraint problem [1]. Using only Λ_B provides the standard master / slave formulation (5), which is not symmetric and which shortcomings have been emphasised in the introduction.

$$\Lambda^{M/S} = \Lambda_B \tag{5}$$

Using a double pass algorithm, i.e. the mean of both Lagrangians (6), similarly provides an over-constraint problem [1]

$$\Lambda^{SYM} = \frac{1}{2}\left(\Lambda_A + \Lambda_B\right) \tag{6}$$

In order to avoid introducing unnecessary contact constraints, the Lagrange multipliers should belong to a variational space which is equivalent to the λ^B one, as in the Mortar approach. On the other hand, in order to obtain a more symmetric formulation, both Lagrangian should be considered. Therefore, in the Quasi-Symmetric formulation, the Lagrange multipliers are defined only on $\partial\Omega_B^{contact}$ as in (5), and a symmetric Lagrangian is written, as in (6), but here λ^A is replaced by $\overline{\lambda^B}$, the orthogonal projection of λ^B onto $\partial\Omega_A$:

$$\Lambda^{QS} = \frac{1}{2}\left(\Lambda_B + \overline{\Lambda_A}\right) \quad \text{where:} \quad \overline{\Lambda_A} = \int_{\partial\Omega_A^{contact}} h_B(v_A)\overline{\lambda^B} ds \tag{7}$$

Node to facet and Penalty framework

This work is carried out in the FORGE3® software which uses a nodal (node to facet) contact formulation and where the contact inequations are handled by a penalty formulation. The Quasi-Symmetric formulation is then derived in this more specific frame. After finite element discretization, these inequations are written as:

$$\text{For any } k \in \partial\Omega_A, l \in \partial\Omega_B, \quad \begin{cases} h_A(V_k) = \sum_{i=1}^{3}\left[\left(V_{ik} - \sum_{l \in f_A^k} N_l^A(\zeta_A^k) \cdot V_{il}\right) \cdot n_{ik}^B\right] - \dfrac{\delta_k^{BA}}{\Delta t} \\ h_B(V_l) = \sum_{i=1}^{3}\left[\left(V_{il} - \sum_{k \in f_B^l} N_k^B(\zeta_B^l) \cdot V_{ik}\right) \cdot n_{il}^A\right] - \dfrac{\delta_l^{AB}}{\Delta t} \end{cases} \tag{8}$$

where f_A^k denotes the facet of $\partial\Omega_A$ containing the orthogonal projection $\pi^A(k)$ of node k, ζ_A^k are the coordinates of this projection, N_l^A the linear interpolation functions, V_{il} is the ith component of the velocity field at node l. The Quasi-Symmetric Lagrangian (7) is then written:

$$\Lambda^{QS} = \frac{1}{2}\sum_{k \in \partial\Omega_B^{contact}} h_A(V_k)\lambda_k^B S_k^B + \frac{1}{2}\sum_{l \in \partial\Omega_A^{contact}} h_B(V_l)\overline{\lambda^B}_l S_l^A \tag{9}$$

In the nodal contact formulation, $S_k^B = \int_{\partial\Omega_B} N_k^B ds$ is the surface of $\partial\Omega_B$ that is associated to node k. With such discrete formulation, the main issue is to define $\overline{\lambda^B}$, the orthogonal projection of λ^B onto $\partial\Omega_A^{contact}$. Actually, λ^B is only define at the nodes of $\partial\Omega_B^{contact}$. In order to evaluate it at any point of $\partial\Omega_B^{contact}$, the discrete values λ_k^B are extrapolated using the velocity finite element functions. Then, $\overline{\lambda^B}_l$ is approximated by $\lambda^B_{\pi^B(l)}$, the value of λ^B at $\pi^B(l)$.

$$\overline{\lambda^B}_l = \lambda^B_{\pi^B(l)} \qquad \text{and} \qquad \lambda^B_{\pi^B(l)} = \sum_{k' \in f_B^l} N_{k'}^B(\zeta_B^l)\lambda^B_{k'} \tag{10}$$

Equation (9) is then written again as:

$$\Lambda^{QS} = \frac{1}{2} \sum_{k \in \partial\Omega_B^{contact}} \lambda^B_k S_k^B \left(h_A(V_k) + \sum_{\substack{l \in \partial\Omega_A \\ s.t. k \in f_B^l}} \frac{S_l^A}{S_k^B} N_k^B(\zeta_B^l) h_B(V_l) \right) \tag{11}$$

Equation (11) clearly shows that this formulation is actually a master/slave one, as Lagrange multipliers are only defined on $\partial\Omega_B$. On the other hand, the contact conditions are significantly different from the standard formulation. The part of equation (11) into brackets contains an averaged contribution of nodes of $\partial\Omega_B^{contact}$ and of $\partial\Omega_A^{contact}$, providing its quasi-symmetric character. From the Quasi-Symmetric Lagrangian (11), it is easy to derive the corresponding penalty functional that is used in the penalty formulation:

$$\Phi^{QS} = \rho \sum_{k \in \partial\Omega_B^{contact}} S_k^B \left[h^A(V_k) + \sum_{\substack{l \in \partial\Omega_A \\ s.t. f_B^1 \ni k}} \frac{S_l^A}{S_k^B} N_k^B(\zeta_B^l) \cdot h^B(V_l) \right]^{+2} \tag{12}$$

where ρ is the penalty coefficient and $\left[x\right]^+ = \dfrac{x + |x|}{2}$ is the positive part of x.

CONVERGENCE RATE OF THE QS FORMULATION

The patch test utilized in [1] is considered to evaluate the stability of the formulation. Two cubes with specific meshes (see Figure 1) are upsetted by an imposed displacement of *1mm* or an imposed pressure of σ_{zz}=*65Mpa* (see Figure 1). The material is either elastic (E_{master}=*3*10⁴ MPa* and E_{slave}=*1.3*10⁴ Mpa* ; v_{master}=v_{slave}=*0*) or newtonian (viscoplastic with a linear coeffcient: K_{master}=K_{slave}=*200 MPa*). Figure 1 shows that, in the vertical direction of imposed stresses or imposed displacements, there is only one element in the mesh of both bodies. Consequently, the constant stress value is always properly imposed in the master side of the contact interface.
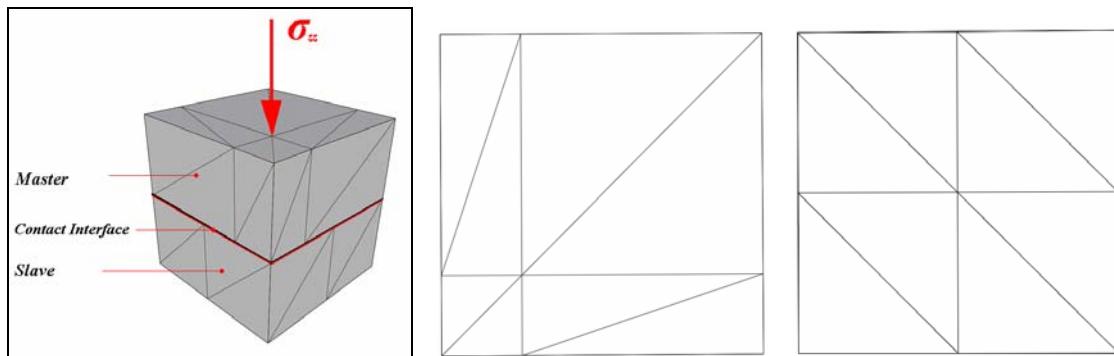

Figure 1: *Patch test with imposed pressure, master and slave interface meshes*

The exact solution of this problem is a constant stress field in the *z* direction on both interfaces and particularly to the slave one. With coincident meshes the error (distance to the constant stress value) due to a finite element formulation is less then 10^{-8}, so it can be neglected. The contact between the cubes is bilateral sticking. For the elastic cubes the reference solution is the analytical one. For the newtonian cases, a numerical reference solution is obtained with a single cube made of the reunion of the two cubes.

Case number	1. Elastic cubes Imposed pressure		2. Elastic cubes Imposed displ.		3. Newt. cubes Imposed pressure		4. Newt. cubes Imposed displ.	
Contact	QS	Std MS	QS	Std MS	QS	Std MS	QS	Std MS
Error	3.6%	8.0%	4,8%	10.0%	5.3%	10.3%	7.2%	15.0%
Max. error	8.3%	22.0%	10.9%	24.8%	12.6%	30.9%	17.9%	34.5%

Tableau 1: *Results of the patch test for the different material bahaviors, boundary conditions, and contact algorithms.*

A standard convergence test is conducted on a pseudo 2D (plane deformations) upsetting of two identical elastic cubes, using coincident and non-coincident meshes as shown in Figure 2.
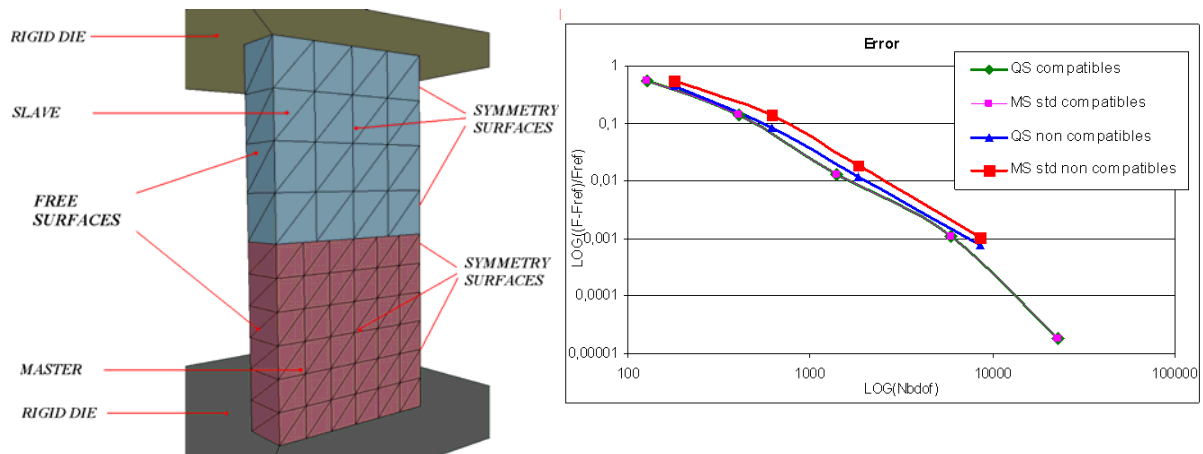

Figure 2: *Convergence test: convergence rates of the different contact algorithms with coinciding and non coinciding meshes.*

Several other similar tests have been conducted. The quasi-symmetrical formulation (QSF) always provides better results than the standard master/slave one (MSF), with a slightly better convergence rate. Shortcomings of the MSF are circumvented with the QSF, so allowing handling contact and friction conditions when the slave surface is very coarse compared to the master surface. It is shown the QSF is almost symmetric, even when the discretizations of both contacting surfaces are quite different.

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