VIII International Conference on Computational Plasticity
COMPLAS VIII
E. Oñate and D.R.J. Owen (Eds)

A NOTE ON THE IMPLEMENTATION OF RETURN MAPPING ALGORITHMS AND CONSISTENT TANGENT OPERATORS IN ISOTROPIC ELASTOPLASTICITY

Nunziante Valoroso*, **Luciano Rosati**†, **Roberto Serpieri**†

*Istituto per le Tecnologie della Costruzione
Consiglio Nazionale delle Ricerche
Viale Palmiro Togliatti 1473, 00155 Roma, Italy
e-mail: nunziante.valoroso@itc.cnr.it

†Dipartimento di Scienza delle Costruzioni
Università di Napoli Federico II
via Claudio 21, 80125 Napoli, Italy
e-mail: rosati@unina.it

Key words: Computational Plasticity, Return Mapping, Consistent Tangent.

Summary. *A novel intrinsic implementation of return mapping algorithms and consistent tangent operators for general three-invariant isotropic plasticity models is presented.*

1 INTRODUCTION

It is now well understood that accurate and stable algorithms for integrating the rate constitutive equations in elastoplasticity are of major importance for carrying out efficient stress computation schemes; furthermore, the paramount role of the consistent tangent[1] has been put forward by several authors. Nevertheless, in many cases the exact consistent linearization may be demanding or computationally expensive to obtain.

A first source of difficulty in obtaining consistent tangent operators lies in the evaluation of the gradient of the plastic flow i.e., for standard models, in computing the second derivatives of the yield function. This is however only a preliminary task to accomplish since the complete linearization requires the inversion of the jacobian associated with the local stress computation scheme. This topic has been previously discussed by the authors and an intrinsic representation of the consistent tangent and its explicit expression with no use of matrix operations has been arrived at[2]; these ideas have been further elaborated by extending the treatment to the principal axis formulation of isotropic plasticity[3].

Objective of this work is to present an implementation of the return mapping algorithm and of the consistent tangent that aims to take proper advantage of the isotropic properties of the model. In particular, we provide an entirely intrinsic representation of all the tensor variables that enter the stress computation algorithm and, by properly exploiting the linearized form of the residual equations, we derive a novel intrinsic expression of the consistent tangent which, besides being more compact and effective with respect to other existing ones, is also amenable to a direct specialization to the plane stress case[4].

2 CONSTITUTIVE MODEL AND TIME-DISCRETE FORMULATION

Let $\boldsymbol{\varepsilon}$ be the strain measure at a point $\boldsymbol{X}$ of a structural model, and assume the usual additive decomposition into elastic ($\mathbf{e}$) and plastic ($\mathbf{p}$) shares. For linear isotropic elasticity one has the constitutive law for the Cauchy stress:

$$\boldsymbol{\sigma} = \mathbb{E}(\boldsymbol{\varepsilon} - \mathbf{p}) = [2G\,(\mathbf{1} \boxtimes \mathbf{1}) + \lambda(\mathbf{1} \otimes \mathbf{1})]\,(\boldsymbol{\varepsilon} - \mathbf{p}) \tag{1}$$

where G and λ are the Lamé's moduli; the definitions along with the matrix form of the dyadic and square tensor products can be found elsewhere[2]. Kinematic and isotropic hardening of the model are governed by:

$$\boldsymbol{\beta} = \mathbb{H}_{kin}\boldsymbol{\eta} = h_{kin}(\mathbf{1} \boxtimes \mathbf{1})\boldsymbol{\eta}; \qquad \vartheta = h_{iso}(\zeta) \tag{2}$$

where h_{kin} is the kinematic hardening modulus and h_{iso} a nonlinear isotropic hardening function. Denoting by $\boldsymbol{\tau} = \boldsymbol{\sigma} - \boldsymbol{\beta}$ the relative stress and I_1, J_2, J_3 the principal invariants, we consider a general isotropic yield function in the form:

$$\tilde{\boldsymbol{\phi}}(\boldsymbol{\sigma}, \boldsymbol{\beta}, \vartheta) = \boldsymbol{\phi}(\boldsymbol{\tau}, \vartheta) = \boldsymbol{\varphi}(I_1, J_2, J_3) - \vartheta - Y_o \tag{3}$$

where Y_o depends upon the initial yields limit of the material.

Setting $\mathbb{E}_H = \mathbb{E} + \mathbb{H}_{kin}$, $\mathbf{n}_H = \mathrm{d}_{\boldsymbol{\tau}}\boldsymbol{\phi}$ and $H_{iso} = h'_{iso}$, for plastic loading ($\boldsymbol{\phi}^{tr} > 0$) one has the residual equations[5]:

$$\begin{cases} \mathbf{r}_{\mathbf{e}}^{(k)} = \mathbb{E}_H^{-1}\left(\boldsymbol{\tau}^{(k)} - \boldsymbol{\tau}^{tr}\right) + \gamma^{(k)}\mathbf{n}_H^{(k)} = \mathbf{0} \\ \mathbf{r}_{\zeta}^{(k)} = H_{iso}^{-1}\left(\vartheta^{(k)} - \vartheta^{tr}\right) - \gamma^{(k)} = 0 \\ \mathbf{r}_{\boldsymbol{\phi}}^{(k)} = \boldsymbol{\phi}(\boldsymbol{\tau}^{(k)}, \vartheta^{(k)}) = 0 \end{cases} \tag{4}$$

which are solved for the increment of the plastic parameter to get, for the $k-$th iterate:

$$\delta\gamma_{(k)}^{(k+1)} = \frac{\mathbf{r}_{\boldsymbol{\phi}}^{(k)} - \mathbb{G}_H^{(k)\,-1}\mathbf{r}_{\mathbf{e}}^{(k)} \cdot \mathbf{n}_H^{(k)} + H_{iso}^{(k)}\mathbf{r}_{\zeta}^{(k)}}{\mathbb{G}_H^{(k)\,-1}\mathbf{n}_H^{(k)} \cdot \mathbf{n}_H^{(k)} + H_{iso}^{(k)}} \tag{5}$$

where $\mathbb{G}_H^{(k)} = \mathbb{E}_H^{-1} + \gamma^{(k)}\,\mathrm{d}^2_{\boldsymbol{\tau}\boldsymbol{\tau}}\boldsymbol{\phi}^{(k)}$ is the rank-four elastoplastic compliance tensor.

It is worth emphasizing that, as put forward by (5), the solution of the local return mapping algorithm is accomplished by computing the inverse of the rank-four tensor $\mathbb{G}_H^{(k)}$ although this operation is not strictly required for computing the increments of the state variables. The reason underlying this choice is mainly the fact that the tensor $\mathbb{G}_H^{-1}$ has to be computed at the end of the constitutive iterations in order to build up the expression of the consistent tangent[1].

To achieve an effective implementation of the return mapping, one has to express all the tensors appearing in (5) in intrinsic, matrix-free format; in particular, this is required only for the elastic strain residual $\mathbf{r}_{\mathbf{e}}^{(k)}$, since the following representation formulas hold[2]:

$$\mathbf{n}_H^{(k)} = n_{H1}^{(k)}\mathbf{1} + n_{H2}^{(k)}\mathbf{S}^{(k)} + n_{H3}^{(k)}[\mathbf{S}^{(k)}]^2$$

and

$$\begin{aligned}[\mathbb{G}_H^{-1}]^{(k)} = \;& i_1^{(k)}(\mathbf{1} \boxtimes \mathbf{1}) + i_2^{(k)}(\mathbf{S}^{(k)} \boxtimes \mathbf{1} + \mathbf{1} \boxtimes \mathbf{S}^{(k)}) + i_3^{(k)}(\mathbf{S}^{(k)} \boxtimes \mathbf{S}^{(k)}) \\ & + i_4^{(k)}(\mathbf{1} \otimes \mathbf{1}) + i_5^{(k)}(\mathbf{S}^{(k)} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{S}^{(k)}) + i_6^{(k)}([\mathbf{S}^{(k)}]^2 \otimes \mathbf{1} + \mathbf{1} \otimes [\mathbf{S}^{(k)}]^2) \\ & + i_7^{(k)}(\mathbf{S}^{(k)} \otimes \mathbf{S}^{(k)}) + i_8^{(k)}([\mathbf{S}^{(k)}]^2 \otimes \mathbf{S}^{(k)} + \mathbf{S}^{(k)} \otimes [\mathbf{S}^{(k)}]^2) + i_9^{(k)}([\mathbf{S}^{(k)}]^2 \otimes [\mathbf{S}^{(k)}]^2)\end{aligned}$$

$\mathbf{S}^{(k)}$ being the deviator of $\boldsymbol{\tau}^{(k)}$.

3 INTRINSIC RESIDUAL REPRESENTATION

Recalling definition $(4)_1$ and that $\boldsymbol{\tau}^{(k)} = \mathbb{E}\boldsymbol{\varepsilon} - \mathbb{E}_H \mathbf{p}^{(k)}$, the trial stress $\boldsymbol{\tau}^{tr}$ can be expressed as:

$$\boldsymbol{\tau}^{tr} = \boldsymbol{\tau}^{(k)} + \gamma^{(k)}\, \mathbb{E}_H \mathbf{n}_H^{(k)} \tag{6}$$

whence, owing to the isotropy of $\boldsymbol{\phi}$ and of the elastic constitution, it can be considered as a nonlinear isotropic tensor function of $\boldsymbol{\tau}^{(k)}$. As such, the trial stress (6) is amenable to the following representation:

$$\boldsymbol{\tau}^{tr} = \delta_1^{(k)}\mathbf{1} + \delta_2^{(k)}\mathbf{S}^{(k)} + \delta_3^{(k)}[\mathbf{S}^{(k)}]^2 \tag{7}$$

where the coefficients are computed by projecting (7) onto the basis $\mathbf{1}$, $\mathbf{S}^{(k)}$, $[\mathbf{S}^{(k)}]^2$, i.e.

$$[\mathbf{A}^{(k)}][\boldsymbol{\delta}^{(k)}] = [\mathbf{b}^{(k)}] \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 0 & 2J_2^{(k)} \\ 0 & 2J_2^{(k)} & 3J_3^{(k)} \\ 2J_2^{(k)} & 3J_3^{(k)} & 2[J_2^{(k)}]^2 \end{bmatrix} \begin{bmatrix} \delta_1^{(k)} \\ \delta_2^{(k)} \\ \delta_3^{(k)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}^{tr} \cdot \mathbf{1} \\ \boldsymbol{\tau}^{tr} \cdot \mathbf{S}^{(k)} \\ \boldsymbol{\tau}^{tr} \cdot [\mathbf{S}^{(k)}]^2 \end{bmatrix} \tag{8}$$

It is not difficult to show that $\det(\mathbf{A}^{(k)})$ vanishes whenever the stress deviator has non-trivial coalescent eigenvalues; in this case one has then two possible solutions depending on whether the Lode angle equals 0 or $\pi/3$.

4 A NOVEL EXPRESSION OF THE CONSISTENT TANGENT TENSOR

The consistent tangent can be derived by comparing the two equations:

$$\mathrm{d}_{\boldsymbol{\varepsilon}}\boldsymbol{\sigma} = \mathbb{E} - \mathbb{E}\,\mathrm{d}_{\boldsymbol{\varepsilon}}\mathbf{p}; \qquad \mathrm{d}_{\boldsymbol{\varepsilon}}\boldsymbol{\tau} = \mathbb{E} - \mathbb{E}_H\,\mathrm{d}_{\boldsymbol{\varepsilon}}\mathbf{p}$$

which yield:

$$\mathrm{d}_{\boldsymbol{\varepsilon}}\boldsymbol{\sigma} = \mathbb{E}_{tan} = \mathbb{E} - \mathbb{E}\mathbb{E}_H^{-1}\mathbb{E} + \mathbb{E}\mathbb{E}_H^{-1}\,\mathrm{d}_{\boldsymbol{\varepsilon}}\boldsymbol{\tau} \tag{9}$$

The derivative $\mathrm{d}_{\boldsymbol{\varepsilon}}\boldsymbol{\tau}$ is easily obtained from the linearization of the residual equations at the local converged state, i.e. from the system:

$$\begin{bmatrix} -\mathbb{E}_H^{-1}\mathbb{E}\delta\boldsymbol{\varepsilon}^* \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbb{G}_H & \mathbf{0} & \mathbf{n}_H \\ \mathbf{0} & H_{iso}^{-1} & -1 \\ (\mathbf{n}_H)^T & -1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{d}_{\boldsymbol{\varepsilon}}\boldsymbol{\tau}\; \delta\boldsymbol{\varepsilon}^* \\ \mathrm{d}_{\boldsymbol{\varepsilon}}\vartheta \cdot \delta\boldsymbol{\varepsilon}^* \\ \mathrm{d}_{\boldsymbol{\varepsilon}}\gamma \cdot \delta\boldsymbol{\varepsilon}^* \end{bmatrix} = \mathbf{0} \qquad \forall \delta\boldsymbol{\varepsilon}^*$$

which, solved for $\mathrm{d}_{\boldsymbol{\varepsilon}}\boldsymbol{\tau}$ gives:

$$\mathrm{d}_{\boldsymbol{\varepsilon}}\boldsymbol{\tau} = \mathbb{D}_{tan}\,\mathbb{E}_H^{-1}\mathbb{E} = \left[\mathbb{G}_H^{-1} - \frac{\mathbb{G}_H^{-1}\mathbf{n}_H \otimes \mathbb{G}_H^{-1}\mathbf{n}_H}{\mathbb{G}_H^{-1}\mathbf{n}_H \cdot \mathbf{n}_H + H_{iso}}\right]\mathbb{E}_H^{-1}\mathbb{E}$$

Substitution of the previous expression in (9) supplies the novel expression of the consistent tangent as:

$$\mathbb{E}_{tan} = \mathbb{E} - \mathbb{E}\mathbb{E}_H^{-1}\mathbb{E} + (\mathbb{E}\mathbb{E}_H^{-1} \boxtimes \mathbb{E}\mathbb{E}_H^{-1})\,\mathbb{D}_{tan} \tag{10}$$

whereby a representation formula analogous to that of $\mathbb{G}_H^{-1}$ is finally arrived at.

5 CONCLUDING REMARKS

A novel implementation of the constitutive algorithm and consistent tangent for general three-invariant elastoplastic models has been provided.

The solution update is entirely performed in intrinsic form by suitably exploiting basic theorems for nonlinear isotropic tensor functions; this allows one to obtain the representation formula for the elastic strain residual and the closed-form expression of the local elastoplastic tangent $\mathbb{G}_H^{-1}$, this last one being required for the derivation of the consistent tangent tensor. Accordingly, the only tensor-to-matrix mapping operation needed in the proposed implementation is the one relevant to the construction of the consistent tangent matrix to be assembled later at the global level.

The present approach can be shown to be applicable with no modification to other problems formulated in terms of any reduced set of stress components, and in particular to the plane stress case[4], where no special assumption needs to be considered, as done for earlier algorithmic treatments.

REFERENCES

[1] J. C. Simo and R. L. Taylor. Consistent tangent operators for rate-independent elastoplasticity. *Comp. Meth. in Appl. Mech. Engng.*, 48:101–118, 1985.

[2] V. Palazzo, L. Rosati, and N. Valoroso. Solution procedures for J_3 plasticity and viscoplasticity. *Comp. Meth. in Appl. Mech. Engng.*, 191(8):903–939, 2001.

[3] L. Rosati and N. Valoroso. A return map algorithm for general isotropic elasto/viscoplastic materials in principal space. *Int. J. Num. Meth. Engng.*, 60(2):461–498, 2004.

[4] N. Valoroso and L. Rosati. Consistent derivation of the constitutive algorithm for plane-stress isotropic plasticity. Part I: Theoretical formulation. Part II: Computational issues. 2005. *Submitted.*

[5] J. C. Simo and T. J. R. Hughes. *Computational Inelasticity.* Springer, Berlin, 1998.

[6] N. Valoroso and L. Rosati. Matrix-free implementation of isotropic tensor functions in classical plasticity. 2005. *In preparation.*