VIII International Conference on Computational Plasticity
COMPLAS VIII
E. Oñate and D.R.J. Owen (Eds)

FINITE ELEMENT FORMULATIONS FOR GEOMETRICAL AND MATERIAL NONLINEAR SHELLS BASED ON A HU–WASHIZU FUNCTIONAL

Friedrich Gruttmann* **and Werner Wagner**†

*Institut für Werkstoffe und Mechanik im Bauwesen
Technische Universität Darmstadt
Petersenstr. 12, 64287 Darmstadt, Germany
e-mail: gruttmann@iwmb.tu-darmstadt.de, web page: http://www.iwmb.tu-darmstadt.de

†Institut für Baustatik
Universität Karlsruhe (TH)
Kaiserstraße 12, 76131 Karlsruhe, Germany
e-mail: ww@bs.uka.de, web page: http://www.bs.uni-karlsruhe.de

Key words: Reissner–Mindlin shell theory, Hu–Washizu variational principle, elasto-plasticity, large load steps.

Summary. *The paper deals with the nonlinear finite element analysis of thin shells. Numerical tests show the advantages of the developed mixed hybrid quadrilateral element. The essential feature of the element formulation is the robustness in nonlinear computations with large rigid body motions. It allows very large load steps in comparison to standard displacement models or enhanced strain models.*

1 INTRODUCTION

Low order shell elements based on a standard displacement interpolation are usually characterized by locking phenomena and thus lead to unacceptable stiff results when reasonable finite element meshes are employed. To overcome these problems mixed element formulations have been successfully applied. In this context enhanced strain methods[1] based on a three–field variational functional are mentioned, where in a second step the stress field is eliminated using some orthogonality conditions. The application to nonlinear shells[2] can be done in a straight forward way enhancing the shell strains derived from the Green–Lagrangean strain tensor.

In this paper we start with a Hu-Washizu functional with independent displacements, stresses and strains. The associated Euler–Lagrange equations are the static and geometric field equations, the constitutive equations and the static boundary conditions. The inextensible director kinematic accounts for transverse shear deformations and finite rotations. In contrast to the mentioned enhanced strain elements the stress field is not eliminated from the set of variational equations. Based on a previous publication[3], where appropriate interpolation functions for the independent stress resultants and shell strains have been formulated, we present an improved strain approximation.

2 BASIC EQUATIONS

With ξ^i a convected coordinate system of the considered shell body is introduced. The arbitrary reference surface Ω is defined by the thickness coordinate $\xi^3 = 0$. The shell is loaded statically by surface loads $\bar{\mathbf{p}}$ on Ω and by boundary loads $\bar{\mathbf{t}}$ on the boundary Γ_σ. Hence, the Hu–Washizu functional can be written as

$$\Pi(\mathbf{v}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = \int_{(\Omega)} [W(\boldsymbol{\varepsilon}) + \boldsymbol{\sigma}^T(\boldsymbol{\varepsilon}_G(\mathbf{v}) - \boldsymbol{\varepsilon})]\, dA - \int_{(\Omega)} \mathbf{u}^T \bar{\mathbf{p}}\, dA - \int_{(\Gamma_\sigma)} \mathbf{u}^T\, \bar{\mathbf{t}}\, ds \rightarrow \text{stat.} \qquad (1)$$

where, $\mathbf{v} = [\mathbf{u}, \boldsymbol{\omega}]^T$, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, denote the independent displacements, stress resultants and shell strains. The displacement vector of the reference surface follows with $\mathbf{u} = \mathbf{x} - \mathbf{X}$, where $\mathbf{X}(\xi^1, \xi^2)$ and $\mathbf{x}(\xi^1, \xi^2)$ denote the position vectors of the initial and current shell reference surface. Furthermore $\boldsymbol{\omega}$ is the vector of rotational parameters. The shell strains are organized in a vector $\boldsymbol{\varepsilon}_G(\mathbf{v}) = [\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}, \kappa_{11}, \kappa_{22}, 2\kappa_{12}, \gamma_1, \gamma_2]^T$, where the membrane strains $\varepsilon_{\alpha\beta}$, curvatures $\kappa_{\alpha\beta}$ and shear strains γ_α can be derived from the Green-Lagrangean strain tensor. The work conjugate stress resultants $\boldsymbol{\sigma} = [n^{11}, n^{22}, n^{12}, m^{11}, m^{22}, m^{12}, q^1, q^2]^T$ with membrane forces $n^{\alpha\beta} = n^{\beta\alpha}$, bending moments $m^{\alpha\beta} = m^{\beta\alpha}$ and shear forces q^α are integrals of the Second Piola–Kirchhoff stress tensor. Elastic and inelastic finite strain constitutive models can be taken into account, since the strain energy W can be formulated as an arbitrary nonlinear function of the independent strains. The stationary condition is derived in a standard way by variation with respect to the independent tensor fields.

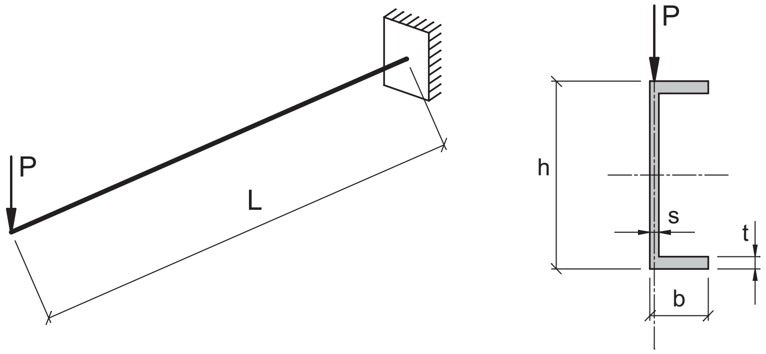
The associated finite element formulation for quadrilaterals is verified within the isoparametric concept. A map of the coordinates $\{\xi, \eta\} \in [-1, 1]$ from the unit square to the mid–surface in the initial and current configuration is applied. Furthermore an orthogonal basis system is generated at the nodes within the mesh input, whereas the actual frame is obtained using an orthogonal transformation. The position vectors and director vectors of the initial and current configuration are approximated using bi–linear functions. In order to obtain a stable element formulation it is crucial to choose the shape functions for the independent stress resultants and shell strains in a proper way. The interpolation of the membrane forces and bending moments corresponds to the approach for plane stress elements introduced by Pian and Sumihara[4]. In this paper we apply an improved interpolation of the shell strains with two parts

$$\boldsymbol{\varepsilon}^h = [\mathbf{N}_\varepsilon^1, \mathbf{N}_\varepsilon^2] \begin{bmatrix} \hat{\boldsymbol{\varepsilon}}_1 \\ \hat{\boldsymbol{\varepsilon}}_2 \end{bmatrix}. \qquad (2)$$

The vector $\hat{\boldsymbol{\varepsilon}}_1$ contains with 14 parameters the same number as for the stress resultant interpolation. The second vector $\hat{\boldsymbol{\varepsilon}}_2$ contains a variable number of parameters β. Numerical investigations show that two parameters for the membrane part and two parameters for the bending part with associated shape functions are sufficient, thus $\beta = 4$. Further functions do not essentially improve the element behaviour. All matrices are specified in detail in Wagner and Gruttmann[3,5].

3 EXAMPLE: CHANNEL–SECTION BEAM

As numerical example we consider a clamped channel–section beam with a tip load at the free end. We assume linear elastic ideal plastic material behaviour with parameters E, ν and initial yield stress y_0 according to Fig. 1. The shell discretization consists of 36 elements along the length direction, 6 elements along the web and 2 elements for each flange, in total 360 elements. An arc–length scheme with displacement control is applied to calculate the load carrying behaviour up to a tip displacement of $w = 250$ cm with subsequent unloading. The associated load is computed with the present shell element and compared with the results of the EAS–element[2] with four enhanced parameters for the membrane interpolation.



$$
\begin{aligned}
L &= 900 \ cm \\
h &= 30 \ cm \\
b &= 10 \ cm \\
s &= 1.0 \ cm \\
t &= 1.6 \ cm \\
E &= 21000 \ kN/cm^2 \\
\nu &= 0.3 \\
y_0 &= 36 \ kN/cm^2
\end{aligned}
$$

Figure 1: Channel–section beam with geometrical and material data

displacement step w: 50 ⟶ 53 cm		
Iterat.	EAS–element	present element
1	1.0000000E+00	1.0000000E+00
2	6.3761516E+03	5.5798017E+03
3	1.4983476E+02	2.8338326E+01
4	8.3009989E+01	5.0787877E-03
5	1.0505898E+01	3.4263339E-08
6	9.8385135E+00	
7	5.7130968E+00	
8	1.2059626E+00	
9	2.5993786E-01	
10	3.0121923E-03	
11	1.9727645E-05	
12	2.2762804E-08	

displacement step w: 50 ⟶ 120 cm		
Iterat.	EAS–element	present element
1	no convergence	1.0000000E+00
2		4.7604831E+06
3		1.5814845E+06
4		1.6610973E+05
5		1.2818501E+04
6		3.5064220E+01
7		1.6863763E-01
8		4.2263405E-06
9		1.9314382E-08
10		
11		
12		

Table 1: Norm of the residual vector for a small and large displacement step

The EAS–element is much more sensitive and allows only displacement steps of $\Delta w \approx 1 - 3$ cm, whereas with the present element displacement steps of $\Delta w \approx 50 - 70$ cm are possible, see table 1. The results of both models agree very good in the total range of the computed load deflection curves, see figure 2. Without the second part of the strain interpolation ($\beta = 0$) the computed load deflection behaviour is slightly to stiff. Figure 2 contains a plot of the von Mises stresses at the ultimate state.

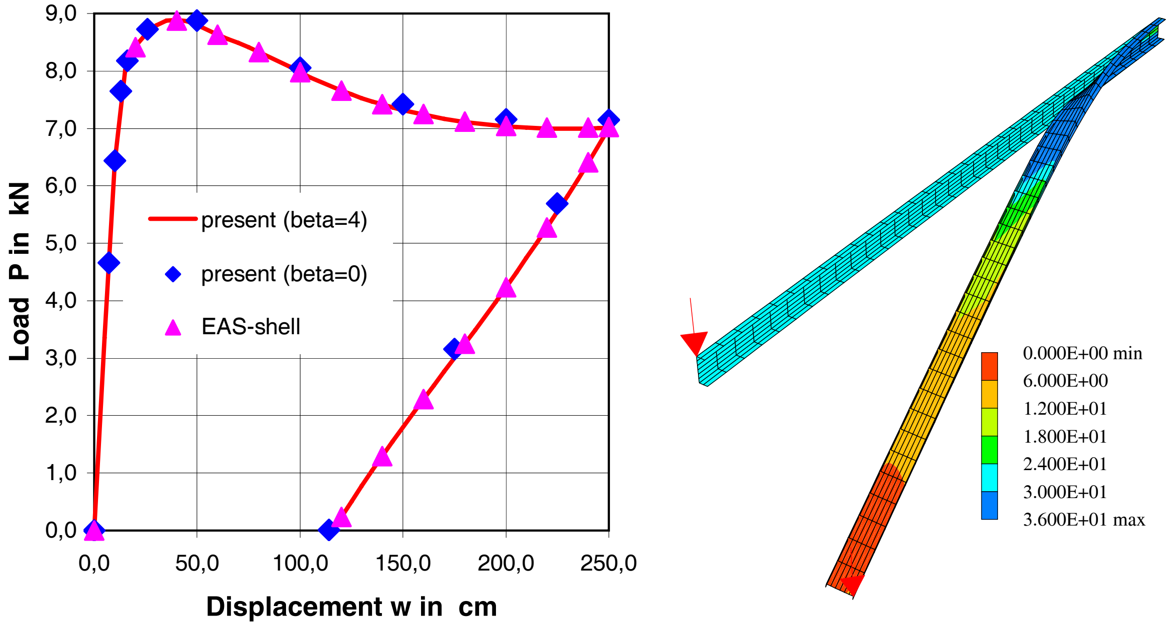


Figure 2: Load deflection curves and v.Mises stresses at the ultimate state

REFERENCES

[1] J.C. Simo and M.S. Rifai. A class of mixed assumed strain methods and the method of incompatible modes, *Int. J. Num. Meth. Engng.*, **29**, 1595–1638, 1990.

[2] R. Sauer. *Eine einheitliche Finite–Element–Formulierung für Stab– und Schalentragwerke mit endlichen Rotationen.* Bericht 4, Institut für Baustatik, Universität Karlsruhe (TH), 1998.

[3] W. Wagner and F. Gruttmann. A robust nonlinear mixed hybrid quadrilateral shell element, *Int. J. Num. Meth. Engng.*, in press, 2005.

[4] T.H.H. Pian and K. Sumihara. Rational approach for assumed stress finite elements, *Int. J. Num. Meth. Engng.*, **20** 1685-1695, 1984.

[5] F. Gruttmann and W. Wagner. Structural analysis of composite laminates using a mixed hybrid shell element, *Computational Mechanics*, in press, 2005.