VIII International Conference on Computational Plasticity
COMPLAS VIII
E. Oñate and D. R. J. Owen (Eds)

IMPLEMENTATION OF THE DISCRETE AND CONTINUUM APPROXIMATIONS OF EMBEDDED DISCONTINUITIES IN THE FINITE ELEMENT METHOD

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Key words: Finite Element Method, Embedded discontinuities, Crack, Strain localization.

1 INTRODUCTION

The embedded discontinuity approximation in the finite element method has proved to be a useful methodology for the numerical analysis of the cracking process in solids. In this methodology the elements crossed by discontinuities use an enhanced displacement approximation to simulate the presence of discontinuities. Three types of approximations for embedded discontinuities can be identified: Discrete Approximation (D. A.), Continuum Approximation with Strong Discontinuities and Continuum Approximation with Weak Discontinuities (C. A. W. D.). This paper focuses on the first and third approximations.

The numerical implementation of these approximations is done by completing the following three steps: (1) implementation of the stiffness matrix, which is formulated from the equilibrium equations at the body and at the discontinuity, (2) implementation of the constitutive model, and (3) implementation of the discontinuity tracking routine.

2 STIFFNESS MATRIX

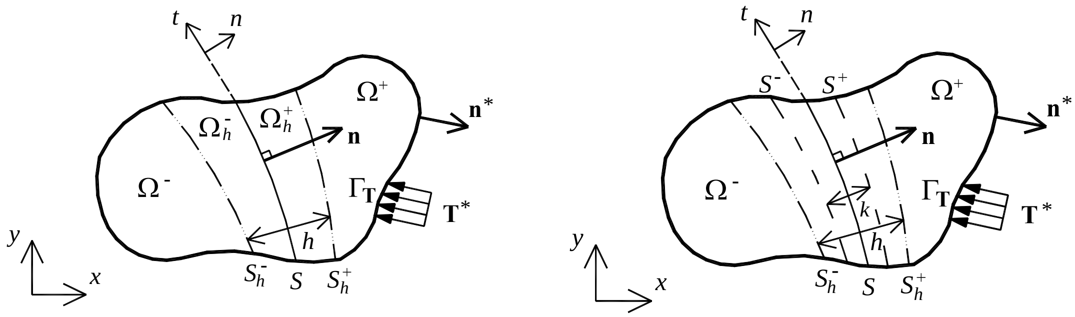
Consider a solid and homogenous body whose material points are labeled by the global coordinate system, $\mathbf{x}$. The body has a domain, Ω, and a boundary, Γ. A discontinuity (*e.g.* crack) is introduced in the body for the D. A. and a strain localization zone (*e.g.* cracking band) is introduced in the body for the C. A. W. D., as illustrated in figure 1.

The displacement field $\mathbf{u}(\mathbf{x})$ is a function of the displacement jump $[[\mathbf{u}]](\mathbf{x})$ and is defined as:

D.A. $$\mathbf{u}(\mathbf{x}) = \hat{\mathbf{u}}(\mathbf{x}) + M_S(\mathbf{x})\,[[\mathbf{u}]](\mathbf{x}) \quad (1)$$

C.A.W.D. $$\mathbf{u}(\mathbf{x}) = \hat{\mathbf{u}}(\mathbf{x}) + M_K(\mathbf{x})\,[[\mathbf{u}]](\mathbf{x}) \quad (2)$$

where $\hat{\mathbf{u}}(\mathbf{x})$ is the regular displacement, $M_S(\mathbf{x})$ is a function defined as: $M_S(\mathbf{x})= H_S(\mathbf{x})-\varphi^h(\mathbf{x})$, $M_K(\mathbf{x})$ is a function defined as: $M_K(\mathbf{x})= H_K(\mathbf{x})-\varphi^h(\mathbf{x})$, $H_S(\mathbf{x})$ is a step function, $H_K(\mathbf{x})$ is a ramp function, $\varphi^h(\mathbf{x})$ is a continuous function that satisfies: $\varphi^h(\mathbf{x})=0\ \forall \mathbf{x}\in\Omega^-$ and $\varphi^h(\mathbf{x})=1\ \forall \mathbf{x}\in\Omega^+$.



(a) Discrete Approximation (D. A.) (b) Continuum approximation with Weak Discontinuities (C. A. W. D.)
Figure 1: Body with a discontinuity

The displacement field $\mathbf{u}(\mathbf{x})$ can be also defined as:

D.A. $$\mathbf{u}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}) + H_S(\mathbf{x})\,[[\mathbf{u}]](\mathbf{x}) \tag{3}$$

C.A.W.D. $$\mathbf{u}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}) + H_K(\mathbf{x})\,[[\mathbf{u}]](\mathbf{x}) \tag{4}$$

where $\bar{\mathbf{u}}(\mathbf{x})$ is the "continuous" displacement, which is defined as:

$$\bar{\mathbf{u}}(\mathbf{x}) = \hat{\mathbf{u}}(\mathbf{x}) - \varphi^h(\mathbf{x})\,[[\mathbf{u}]](\mathbf{x}) \tag{5}$$

In both approximations, the application of the Principal of Virtual Work to a body with a discontinuity leads to an equation associated to the global equilibrium of the body:

$$\int_\Omega \nabla\delta\hat{\mathbf{u}} : \boldsymbol{\sigma}\ d\Omega = \int_\Omega \delta\hat{\mathbf{u}}\cdot\mathbf{b}\ d\Omega + \int_\Gamma \delta\hat{\mathbf{u}}\cdot\mathbf{T}^*\ d\Gamma \tag{6}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{b}$ are the body forces and $\mathbf{T}^*$ are the prescribed tractions on the surface.

In the numerical implementation of Embedded Discontinuities in the Finite Element Method, the approximations of the regular displacements, $\hat{\mathbf{u}}_e$, and the "continuous" displacements, $\bar{\mathbf{u}}_e$, are defined as:

$$\hat{\mathbf{u}}_e = \mathbf{N}\,\hat{\mathbf{u}}_i \tag{7}$$
$$\bar{\mathbf{u}}_e = \mathbf{N}\,\bar{\mathbf{u}}_i = \mathbf{N}\,\hat{\mathbf{u}}_i - \mathbf{N}\,\boldsymbol{\Phi}\,[[\mathbf{u}]]_{x,y} \tag{8}$$

where $\mathbf{N}$ are the standard shape functions; $\hat{\mathbf{u}}_i$ and $\bar{\mathbf{u}}_i$ are the regular and "continuous" displacement vectors; $[[\mathbf{u}]]_{x,y}$ is the displacement jump in the global coordinate system; $\boldsymbol{\Phi}$ is a matrix that depends on the relative location of the nodes with respect to the discontinuity[1].

The finite element equilibrium equations are obtained from (6). The stiffness matrix is obtained from the first term in (6), which is a function of the stress tensor, $\boldsymbol{\sigma}$, that is calculated as: $\boldsymbol{\sigma}=\mathbf{D}\,\boldsymbol{\varepsilon}$ ($\mathbf{D}$ is the constitutive matrix and $\boldsymbol{\varepsilon}$ is the strain vector); the strain is calculated as $\boldsymbol{\varepsilon}=\mathbf{B}\,\bar{\mathbf{u}}_i$ in the continuous part, so an equation that relates $\bar{\mathbf{u}}_i$ and $\hat{\mathbf{u}}_i$ is required as $\hat{\mathbf{u}}_i$ is the independent variable. The required equation can be obtained form the traction equilibrium equation at the discontinuity (D. A.) or at the strain localization border (C. A. W. D.); in particular, an equation that relates the displacement jump and the “continuous” displacement can be obtained from the equilibrium of tractions at the discontinuity[1]:

$$\text{D.A.} \qquad [[\, \dot{\mathbf{u}} \,]]_{n,t} = [\mathbf{D}^{d}]^{-1}\mathbf{n}_{\mathrm{T}}\mathbf{D}^{e}\mathbf{B}\,\dot{\hat{\mathbf{u}}}_{i} \tag{9}$$

$$\text{C.A.W.D.} \qquad [[\, \dot{\mathbf{u}} \,]]_{x,y} = [\,\frac{1}{k}\,\mathbf{n}_{\mathrm{T}}\mathbf{D}^{wd}\mathbf{n}_{e}]^{-1}\mathbf{n}_{\mathrm{T}}(\mathbf{D}^{e}\text{-}\mathbf{D}^{wd})\,\mathbf{B}\,\dot{\hat{\mathbf{u}}}_{i} \tag{10}$$

where $[[\mathbf{u}]]_{n,t}$ is the displacement jump in the local coordinate system; k is the strain localization zone width; $\mathbf{n}_{\mathrm{T}}$ is the stress transformation matrix; $\mathbf{n}_{e}$ is a matrix associated to the strains; $\mathbf{D}^{e}$, $\mathbf{D}^{d}$ y $\mathbf{D}^{wd}$ are the constitutive matrices of the continuous part, the discontinuity and the strain localization zone, respectively; the dot above the variable means time derivative.

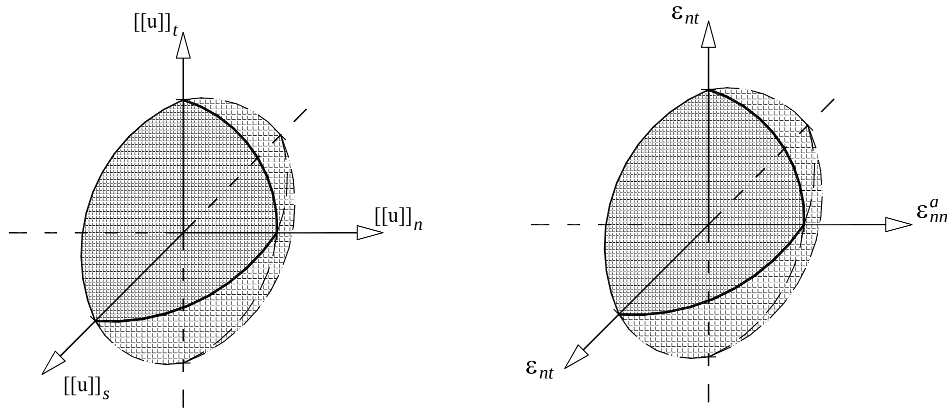
The required equation that relates $\bar{\mathbf{u}}_{i}$ and $\hat{\mathbf{u}}_{i}$ or $\nabla\,\bar{\mathbf{u}}_{i}$ and $\nabla\,\hat{\mathbf{u}}_{i}$ is obtained from the algebraic manipulation of (8), (9) and (10). This equation can be substituted in (6) in order to establish the equilibrium equation and the stiffness matrix in terms of $\hat{\mathbf{u}}_{i}$. Several sets of equations that relate $\bar{\mathbf{u}}_{i}$ and $\hat{\mathbf{u}}_{i}$ or $\nabla\,\bar{\mathbf{u}}_{i}$ and $\nabla\,\hat{\mathbf{u}}_{i}$ can be found; each has numerical advantages and drawbacks[1], however, all come from the same equations.

The stiffness matrix is non-symmetric and its numerical implementation can be carried out by modifying the constitutive matrix or the stiffness matrix. In both cases the standard constitutive and the stiffness matrix should be calculated and then multiplied by a matrix that introduces the effect of the discontinuity.

3 CONSTITUTIVE MODEL

Two constitutive models must be established for the analysis of bodies with discontinuities: one for the continuous part and another for the discontinuity. A lineal elastic model is commonly used for the continuous part of the body and a non-linear model is used for the discontinuity.

The differences between both types of discontinuity approximations are emphasized when the constitutive model is defined. In the D. A. a constitutive matrix of dimension two is defined, which relates the displacement jump, $[[\mathbf{u}]]$, and the traction at the discontinuity, $\mathbf{T}$. In the C. A. W. D. a constitutive matrix of dimension three is defined, which relates the strain, $\boldsymbol{\varepsilon}$, and the stress, $\boldsymbol{\sigma}$, at the strain localization zone. Fernández and Ayala[2] developed damage models for both types of approximations, which can be equivalent given the fulfillment of a two equations. Figure 2 illustrates the failure surfaces for both models, which are defined for the conditions $[[u]]_{n}\geq 0$ and $[[\varepsilon]]_{nn}\geq 0$ due to physical consistency (a crack can only be closed or open); $[[\boldsymbol{\varepsilon}]]$ is the strain jump. The evaluation of the constitutive equation requires the resolution of a non-linear equation to obtain the value of $[[\mathbf{u}]]$ which satisfies the equilibrium at the discontinuity.



(a) Discrete Approximation (D. A.) (b) Continuum Approximation with Weak Discontinuities (C. A. W. D.)
Figure 2: Failure surface of the constitutive model

4 DISCONTINUITY TRACKING

The discontinuity tracking is an important part of the numerical implementation because the $\mathbf{\Phi}$ matrix from (8) is defined by the relative position of the nodes with respect to the discontinuity. The discontinuity in the body is numerically approximated by continuous lineal segments; the failure criterion used defines the orientation of each segment.

The tracking can be carried out based on a local or a global criterion. In the local criterion, the orientation is calculated using the information given by the corresponding element; this criterion is easy to implement, but can introduce a misalignment of the discontinuity due to the spurious strains[1]. The spurious strains are those that an adjacent element produces and which are not consistent with the kinematics of discontinuities. In the global criterion[3], the orientation is calculated using the information given by a group of elements; this criterion can minimize the effects of these spurious strains.

5 CONCLUSIONS

The main differences and similarities of the Discrete Approximation and the Continuum Approximation with Weak Discontinuities are presented. The numerical simulation of the fracture process of a body using these two approximations can yield to the same result, given the use of suitable constitutive models.

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