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AN EFFICIENT FINITE ELEMENT FORMULATION BASED ON THE STRONG DISCONTINUITY APPROACH FOR MODELING MATERIAL FAILURE AT FINITE STRAINS

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Summary. *A novel, fully three-dimensional finite element formulation suitable for the modeling of locally embedded strong discontinuities at finite strains is presented. Following the Strong Discontinuity Approach (SDA), the proposed numerical model is based on an additive decomposition of the displacement gradient into a conforming and an enhanced part. While the conforming part is associated with a smooth deformation mapping, the enhanced part captures the final failure kinematics of solids. These localized deformations are approximated by means of a discontinuous displacement field. In contrast to previous works, the presented finite element formulation does not require the employment of the static condensation technique. The proposed algorithm is based and a predictor/corrector procedure which is formally identical to the return-mapping algorithm of classical (local and continuous) computational plasticity models. Hence, subroutines designed for standard models can be applied with only minor modifications necessary.*

1 KINEMATICS ASSOCIATED WITH STRONG DISCONTINUITIES

In what follows, a body $\Omega \subset \mathbb{R}^3$ is separated by a hyperplane $\partial_s\Omega \subset \Omega$ of class C^1 (piecewise). Based on this partition, an approximation of the displacement field of the type

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + [\![\boldsymbol{u}]\!]\ (H_s - \varphi), \quad \text{with} \quad \bar{\boldsymbol{u}} \in \mathcal{C}^\infty(\Omega, \mathbb{R}^3),\ \varphi \in \mathcal{C}^\infty(\Omega, \mathbb{R}) \tag{1}$$

is adopted. In Equation (1), H_s denotes the indicating function of the subset Ω^+, $[\![\boldsymbol{u}]\!]$ the displacement discontinuity and φ a smooth ramp function (see [1]). By applying the generalized derivative $D(\bullet)$, the deformation gradient is assumed to be of the type

$$\boldsymbol{F} := \mathbf{1} + \mathrm{GRAD}\bar{\boldsymbol{u}} - ([\![\boldsymbol{u}]\!] \otimes \mathrm{GRAD}\varphi) + ([\![\boldsymbol{u}]\!] \otimes \boldsymbol{N})\ \delta_s, \tag{2}$$

with δ_s representing the DIRAC-delta distribution and $\boldsymbol{N}$ the normal vector of $\partial_s\Omega$. Since the enhanced strains are modeled in an incompatible fashion, it is admissible to neglect the gradient of the displacement jump ($\mathrm{GRAD}\,[\![\boldsymbol{u}]\!] = \mathbf{0}$), cf. [1, 2]. In the proposed EAS concept, only the field $\bar{\boldsymbol{u}}$ is approximated globally conforming, i.e. by using standard shape functions.

The additive decomposition (2) of the deformation gradient is not well-suited for the development of constitutive equations. Following [3], Equation (2) is rewritten into a multiplicative decomposition as

$$\boldsymbol{F} = \bar{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}}, \qquad \text{with} \qquad \begin{array}{rcl} \bar{\boldsymbol{F}} & = & \mathbf{1} + \mathrm{GRAD}\bar{\boldsymbol{u}} - [\![\boldsymbol{u}]\!] \otimes \mathrm{GRAD}\varphi \\ \tilde{\boldsymbol{F}} & = & \mathbf{1} + \boldsymbol{J} \otimes \boldsymbol{N}\, \delta_{\mathrm{s}}, \qquad \boldsymbol{J} := \bar{\boldsymbol{F}}^{-1} \cdot [\![\boldsymbol{u}]\!] \,. \end{array} \tag{3}$$

2 CONSTITUTIVE EQUATIONS

Since the strains in $\Omega^{\pm}$ are regular distributions, standard (local stress-strain) constitutive equations can be applied. As a prototype, a hyperelastic material response governed by the energy functional

$$\Psi(\boldsymbol{C}) = \lambda\, \frac{J^2 - 1}{4} - \left(\frac{\lambda}{2} + \mu\right) \ln J + \frac{1}{2}\, \mu\, \left(\mathrm{tr}\boldsymbol{C} - 3\right), \quad J := \det \boldsymbol{F} \tag{4}$$

is adopted, with $\boldsymbol{C}$, λ and μ denoting the right CAUCHY-GREEN tensor and the LAMÉ constants, respectively.

Inelastic deformations are taken into account by means of a cohesive law of the type

$$\boldsymbol{T}|_{\partial_{\mathrm{s}}\Omega} =: \boldsymbol{T}_{\mathrm{s}} = \boldsymbol{T}_{\mathrm{s}}([\![\boldsymbol{u}]\!]) \tag{5}$$

in terms of the traction vector $\boldsymbol{T} = \boldsymbol{P} \cdot \boldsymbol{N}$ depending on the first PIOLA stresses $\boldsymbol{P}$ and the normal vector $\boldsymbol{N}$. That is, a cohesive model connecting the traction vector $\boldsymbol{T}$ and the displacement jump $[\![\boldsymbol{u}]\!]$ is used.

The coupling between the constitutive equation (4) and the traction separation law (5) is provided by the condition of traction continuity. With the positive definiteness of a norm $|| \bullet ||$ and applying a pull back operation to the traction vector, i.e. $\bar{\boldsymbol{T}} = \bar{\boldsymbol{F}}^T \cdot \boldsymbol{T}$, this continuity condition reads

$$\phi(\bar{\boldsymbol{T}}^+, \bar{\boldsymbol{T}}_{\mathrm{s}}([\![\boldsymbol{u}]\!])) := ||\bar{\boldsymbol{T}}^+ - \bar{\boldsymbol{T}}_{\mathrm{s}}([\![\boldsymbol{u}]\!])|| = 0. \tag{6}$$

Here, $\bar{\boldsymbol{T}}^+$ denotes the right hand side limit of the traction vector $\bar{\boldsymbol{T}}$ at $\boldsymbol{X}_0 \in \partial_{\mathrm{s}}\Omega$. Since the identity $\bar{\boldsymbol{T}} = \boldsymbol{C} \cdot \boldsymbol{S} \cdot \boldsymbol{N}$ holds ($\boldsymbol{S}$ is the second PIOLA-KIRCHHOFF stress tensor and $\boldsymbol{C} \cdot \boldsymbol{S}$ are the MANDEL stresses), the condition of traction continuity is formally identical to the necessary condition of yielding known from standard finite strain plasticity theory. As a consequence, it is possible to develop cohesive laws following the same ideas as applied in classical plasticity models. More precisely, we introduce the space of admissible stresses $\mathbb{E}_{\bar{\boldsymbol{T}}} := \{(\bar{\boldsymbol{T}}^+, \boldsymbol{q}) \in \mathbb{R}^{3+n} | \phi(\bar{\boldsymbol{T}}^+, \boldsymbol{q}) \leq 0\}$ and consider evolution equations of the type

$$\dot{\boldsymbol{J}} = \lambda\, \partial_{\bar{\boldsymbol{T}}^+} g, \qquad \dot{\boldsymbol{\alpha}} = \lambda\, \partial_{\boldsymbol{q}} h. \tag{7}$$

In Equations (7), g and h are potentials defining the evolution equations of the displacement jump $\boldsymbol{J}$ and those of the displacement like internal variables $\boldsymbol{\alpha}$ conjugated with $\boldsymbol{q}$, and λ is a plastic multiplier. Setting $\phi(\bar{\boldsymbol{T}}^+, \bar{\boldsymbol{T}}_{\mathrm{s}}([\![\boldsymbol{u}]\!])) = ||\bar{\boldsymbol{T}}^+ - \bar{\boldsymbol{T}}_{\mathrm{s}}([\![\boldsymbol{u}]\!])||$, $g = h = \phi$ and choosing the internal stress-like variables to $\boldsymbol{q} = \bar{\boldsymbol{T}}_{\mathrm{s}}$, the standard continuity condition is obtained. Hence, the condition of traction continuity is included in the more general concept defined by the space of admissible stresses. For further details, cf. [4].

3 NUMERICAL IMPLEMENTATION

The numerical implementation of all SDAs is based on the weak form of equilibrium together with the weak form of traction continuity, i.e.

$$\begin{aligned} \int_{\Omega^{\mathrm{e}}} \mathrm{GRAD}\boldsymbol{\eta}_0 : \boldsymbol{P}\, \mathrm{d}V &= \int_{\Omega^{\mathrm{e}}} \boldsymbol{B}\cdot\boldsymbol{\eta}_0\, \mathrm{d}V + \int_{\Gamma_{\boldsymbol{P}}} \boldsymbol{T}^{*}\cdot\boldsymbol{\eta}_0\, \mathrm{d}A \\ \frac{1}{V^{\mathrm{e}}}\int_{\Omega^{\mathrm{e}}} \boldsymbol{P}\cdot\boldsymbol{N}\, \mathrm{d}V &= \frac{1}{A_{\mathrm{s}}}\int_{\partial_{\mathrm{s}}\Omega} \boldsymbol{T}_{\mathrm{s}}\, \mathrm{d}A \end{aligned} \tag{8}$$

Here, $\boldsymbol{\eta}_0$, $\boldsymbol{B}$, $\boldsymbol{T}^*$, V^{e}, and A_{s} denote a continuous test function, body forces, tractions prescribed on the NEUMANN boundary $\Gamma_{\boldsymbol{P}}$, the volume of the respective finite element e and the area of the singular surface $\partial_{\mathrm{s}}\Omega$, respectively. Since the displacement jump is assumed as spatially constant, Equation (8) can be rewritten as

$$\phi(\mathrm{ave}(\boldsymbol{T}), \boldsymbol{T}_{\mathrm{s}}) = 0, \qquad \text{with} \qquad \mathrm{ave}(\bullet) := \frac{1}{V^{\mathrm{e}}}\int_{\Omega^{\mathrm{e}}} \boldsymbol{P}\cdot\boldsymbol{N}\, \mathrm{d}V. \tag{9}$$

As a consequence, the condition of traction continuity is again formally identical to the necessary condition of yielding known from standard plasticity theory, and the concept as proposed in the previous section can be applied. That is, we introduce the space of admissible stresses $\mathbb{E}_{\bar{\boldsymbol{T}}} := \{\mathrm{ave}(\bar{\boldsymbol{T}}), \boldsymbol{q}) \in \mathbb{R}^{3+n} | \phi(\mathrm{ave}(\bar{\boldsymbol{T}}), \boldsymbol{q}) \leq 0\}$ and define the evolution equations according to

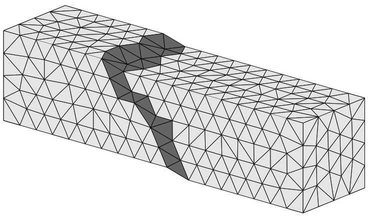
$$\dot{\boldsymbol{J}} = \lambda\, \partial_{\mathrm{ave}(\bar{\boldsymbol{T}})} g \qquad \dot{\boldsymbol{\alpha}} = \lambda\, \partial_{\boldsymbol{q}} h. \tag{10}$$

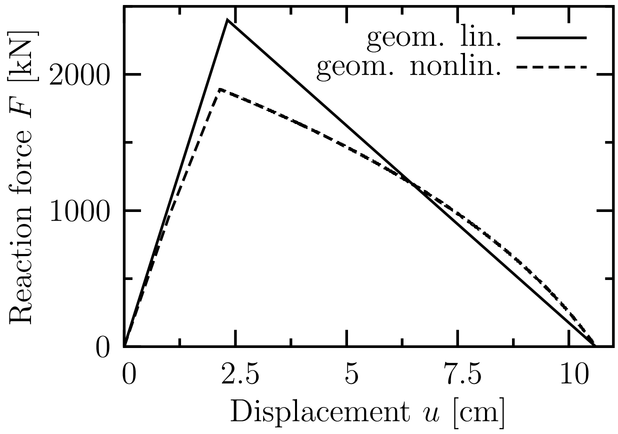
For further details, refer [4].

The constitutive equations defining the proposed cohesive model are formally identical to those of standard plasticity model. Hence, the return-mapping algorithm can be applied. For that purpose, a trial step characterized by purely elastic deformations is defined by $\boldsymbol{J}_{n+1} = \boldsymbol{J}_n$ and $\boldsymbol{q}_{n+1} = \boldsymbol{q}_n$. If unloading is signaled, the trial step represents already the solution. Otherwise, a backward-EULER integration is applied to integrate the evolution equations (10). The solution of the resulting set of nonlinear algebraic equations is computed using NEWTON's method. Although some steps involved in the derivation of the algorithm are by no means trivial, details are omitted due to the shortage of space. However, in the case of linearized kinematics, details may be found in [4].

4 NUMERICAL EXAMPLE

The applicability and the performance of the presented 3D finite element formulation are demonstrated by means of a numerical analysis of shear band propagation on a steel made bar (length = 8 cm, cross section 2 cm × 2 cm). It is assumed that a slip band forms if $\phi = ||\bar{\boldsymbol{T}}^{+} - (\bar{\boldsymbol{T}}^{+}\cdot\boldsymbol{N})\ \boldsymbol{N}||_2 - q(\alpha) > 0$ (for the trial state) and $\boldsymbol{N}$ is computed





by means of a bifurcation analysis, cf. [5]. According to the snapshot, a typical mode-II failure is predicted. The angle between the normal vector $\boldsymbol{N}$ of the corresponding slip band and the direction of the major principle stress normal is about 45°. By comparing the results obtained from the geometrically linear model with those computed from the finite strain formulation, the importance of accounting for large deformations is apparent.

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REFERENCES

[1] J. Simo and J. Oliver. A new approach to the analysis and simulation of strain softening in solids. In Z.P. Bažant, Z. Bittnar, M. Jirásek, and J. Mazars, editors, *Fracture and Damage in Quasibrittle Structures*, pages 25–39. E. &F.N. Spon, London, 1994.

[2] J. Mosler. On the modeling of highly localized deformations induced by material failure: The strong discontinuity approach. *Archives of Computational Methods in Engineering*, 11(4):389–446, 2004.

[3] F. Armero and K. Garikipati. An analysis of strong discontinuities in multiplicative finite strain plasticity and their relation with the numerical simulation of strain localization in solids. *International Journal for Solids and Structures*, 33:2863–2885, 1996.

[4] J. Mosler. A novel algorithmic framework for the numerical implementation of locally embedded strong discontinuities. *Computer Methods in Applied Mechanics and Engineering*, 2004. in press.

[5] J. Mosler. Numerical analyses of discontinuous material bifurcation: Strong and weak discontinuities. *Computer Methods in Applied Mechanics and Engineering*, 194(9-11):979–1000, 2005.