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ON THE HYBRID DISCRETIZATION OF CONTACT AND FRICTION IN ELASTODYNAMICS

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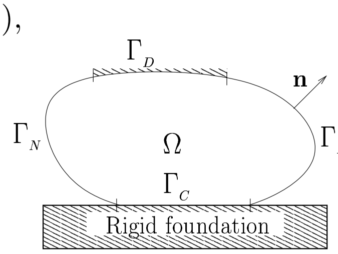
Summary. *We compare two approaches for the discretization of elastodynamic contact problems with friction. The first one we propose is an energy conserving scheme and the second one consists in an equivalent mass matrix leading to a well-posed semi-discretized problem.*

1 Hybrid formulation of the contact with friction problem

The dynamic evolution of a linearly elastic structure in contact with Coulomb friction ($\mathcal{F} \geq 0$ is the friction coefficient) on a rigid obstacle on a part Γ_C of its boundary can be expressed:

$$\left\{\begin{array}{l}
\text{Find } u : [0,T] \longrightarrow V, \lambda_N : [0,T] \longrightarrow X'_N \text{ and } \lambda_T : [0,T] \longrightarrow X'_T \text{ satisfying} \\
< \rho\ddot{u}, v >_{V',V} + a(u,v) = l(v) + < \lambda_N, v_N >_{X'_N,X_N} + < \lambda_T, v_T >_{X'_T,X_T} \quad \forall\, v \in V, \\
\lambda_N \in \Lambda_N, \quad < \mu_N - \lambda_N, u_N >_{X'_N,X_N} \geq 0 \ \ \forall \mu_N \in \Lambda_N, \\
\lambda_T \in \Lambda_T(\mathcal{F}\lambda_N), \quad < \mu_T - \lambda_T, \dot{u}_T >_{X'_T,X_T} \geq 0 \ \ \forall \mu_T \in \Lambda_T(\mathcal{F}\lambda_N), \\
u(0) = u_0, \ \ \dot{u}(0) = u_1,
\end{array}\right. \tag{1}$$



where
$$V = \{v \in H^1(\Omega; I\!R^d) : v = 0 \text{ on } \Gamma_D\},$$

$$a(u,v) = \int_\Omega \mathcal{A}\varepsilon(u) : \varepsilon(v) dx, \ \ l(v) = \int_\Omega f.v dx + \int_{\Gamma_N} g.v d\Gamma,$$

$$X_N = \{v_N|_{\Gamma_C} : v \in V\}, \ \ X_T = \{v_T|_{\Gamma_C} : v \in V\},$$

$$\Lambda_N = \{\mu_N \in X'_N : < \mu_N, v_N >_{X'_N,X_N} \geq 0, \ \forall\, v_N \in X_N, v_N \leq 0\},$$

$$\Lambda_T(G) = \{\mu_T \in X'_T : - < \mu_T, v_T >_{X'_T,X_T} + < G, |v_T| >_{X'_N,X_N} \leq 0, \ \forall\, v_T \in X_T\}.$$

2 Hybrid finite element discretization

The finite element discretization of (1) leads to the following system (see [3, 4]):

$$\left\{\begin{array}{l}
M\ddot{U} + KU = L + B_N^T L_N + B_T^T L_T, \\
L_N \in \Lambda_N^h, \ \ (\tilde{L}_N - L_N)^T B_N U \geq 0 \ \ \forall \tilde{L}_N \in \Lambda_N^h, \\
L_T \in \Lambda_T^h(\mathcal{F} L_N), \ \ (\tilde{L}_T - L_T)^T B_T U \geq 0 \ \ \forall \tilde{L}_T \in \Lambda_T^h(\mathcal{F} L_N), \\
U(0) = U_0, \dot{U}(0) = U_1.
\end{array}\right. \tag{2}$$

where U is the vector of displacement d.o.f., M is the mass matrix, K the stiffness matrix, Λ_N^h and $\Lambda_T^h(\mathcal{F} L_N)$ are some discretizations of Λ_N and $\Lambda_T(\mathcal{F} L_N)$ respectively, $\mathcal{F}$ is the friction coefficient and B_N, B_T are the matrices representing the discrete trace operators. Unfortunately, this problem is not well-posed. It admits a infinite number of solutions corresponding to the choice of a restitution coefficient for each contact node (see [6]).

3 Time discretization with a midpoint scheme

The midpoint scheme applied to Problem (2) with a constant loading can be written:

$$\left\{ \begin{array}{l} U^{n+1} = U^n + \Delta t V^{n+\frac{1}{2}},\ U^{n+\frac{1}{2}} = \dfrac{U^n + U^{n+1}}{2},\ V^{n+1} = V^n + \Delta t A^{n+\frac{1}{2}},\ V^{n+\frac{1}{2}} = \dfrac{V^n + V^{n+1}}{2}, \\ MA^{n+\frac{1}{2}} + KU^{n+\frac{1}{2}} = L + B_N^T L_N^{n+\frac{1}{2}} + B_T^T L_T^{n+\frac{1}{2}}, \\ L_N^{n+\frac{1}{2}} \in \Lambda_N^h\ ,\ \ (\tilde{L}_N - L_N^{n+\frac{1}{2}})^T B_N U^{n+\frac{1}{2}} \geq 0\ \ \forall \tilde{L}_N \in \Lambda_N^h\ , \\ L_T^{n+\frac{1}{2}} \in \Lambda_T^h(\mathcal{F} L_N^{n+\frac{1}{2}})\ ,\ \ (\tilde{L}_T - L_T^{n+\frac{1}{2}})^T B_T U^{n+\frac{1}{2}} \geq 0\ \ \forall \tilde{L}_T \in \Lambda_T^h(\mathcal{F} L_N^{n+\frac{1}{2}})\ . \end{array} \right.$$

From this scheme one obtains: $\Delta t < MA^{n+1/2} + KU^{n+1/2} - L - B_N^T L_N^{n+\frac{1}{2}} - B_T^T L_T^{n+\frac{1}{2}}, V^{n+\frac{1}{2}} >= 0$, which directly implies that the total energy $J(U,V) = \dfrac{1}{2} V^T M V + \dfrac{1}{2} U^T K U - L^T U$ satisfies:

$$J(U^{n+1}, V^{n+1}) = J(U^n, V^n) + \Delta t < B_N^T L_N^{n+\frac{1}{2}} + B_T^T L_T^{n+\frac{1}{2}}, V^{n+\frac{1}{2}} > .$$

The term $< B_T^T L_T^{n+\frac{1}{2}}, V^{n+\frac{1}{2}} >$ is non-positive and represents the frictional dissipation of energy. But the term $< B_N^T L_N^{n+\frac{1}{2}}, V^{n+\frac{1}{2}} >$ is of arbitrary sign. The numerical tests show that this scheme is not convergent. The more Δt is small the more the energy is growing. We propose here two strategies to obtain a stable scheme.

4 An energy conserving scheme with a contact condition in terms of velocity

With an appropriate choice of Λ_N^h and a Lagrange finite element method, the contact condition can be written $\tilde{\lambda}_N^i \leq 0,\ \ U.N_i \leq 0,\ \ (\tilde{\lambda}_N^i)(U.N_i) = 0$, where on each finite element node in potential contact $\tilde{\lambda}_N^i$ and $U.N_i$ are the equivalent contact force and the normal displacement respectively. The idea is to replace this expression of the contact condition with the following equivalent expression in terms of normal velocity:

$$\left\{ \begin{array}{l} U.N_i < 0 \implies \tilde{\lambda}_N^i = 0, \\ U.N_i \geq 0 \implies \dot{U}.N_i \leq 0,\ \ \tilde{\lambda}_N^i \leq 0,\ \ (\dot{U}.N_i)(\tilde{\lambda}_N^i) = 0. \end{array} \right.$$

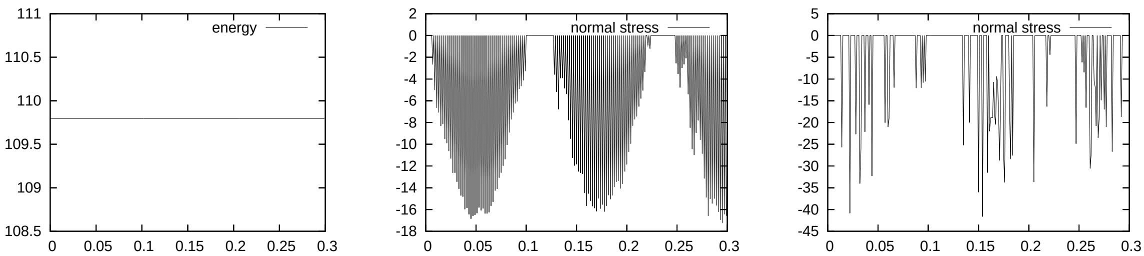
The proposed scheme is based on a midpoind scheme for the elastodynamic part and a central difference scheme for the contact condition. It is strictly energy conserving. Of course a nodal friction condition can be added, and this is stable when a central difference scheme is also used for the friction condition. The expression for the frictionless problem is $(n \geq 1)$:

$$\begin{cases} U^0 \text{ and } V^0 \text{ given }, U^1 = U^0 + \Delta t V^0 + \Delta t z(\Delta t) \quad \text{with } \lim\limits_{\Delta t \to 0} z(\Delta t) = 0, \\ M\left(\dfrac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2}\right) + K\left(\dfrac{U^{n+1} + 2U^n + U^{n-1}}{4}\right) = L + \sum\limits_i \tilde{\lambda}_N^{i,n} N_i, \\ V^n = (U^{n+1} - U^{n-1})/2\Delta t, \\ U^n.N_i < 0 \implies \tilde{\lambda}_N^{i,n} = 0, \\ U^n.N_i \geq 0 \implies V^n.N_i \leq 0, \ \tilde{\lambda}_N^{i,n} \leq 0, \ (V^n.N_i)(\tilde{\lambda}_N^{i,n}) = 0. \end{cases}$$

5 Equivalent mass matrix

The major difficulty with the elastodynamic contact problems comes from the fact that nodes on the contact boundary have their own inertia. This leads to instabilities especially for energy conserving schemes. We propose here to introduce a new distribution of the mass matrix, conserving the total mass, the moments of inertia and the center of gravity, and so that there is no inertia for the contact nodes. If M_0 is the modified mass matrix, it is requiered that $N_i^T M_0 N_j = 0, \ \forall i, j$. Then it is possible to prove (see [1, 2]) that Problem (2) with M_0 instead of M is well-posed, has a Lipschitz solution and is energy conserving when there is no friction. The consequence is that any classical time integration scheme converges. For instance, a Newmark scheme with $\beta = \gamma = 1/2$ seems to be a good choice.

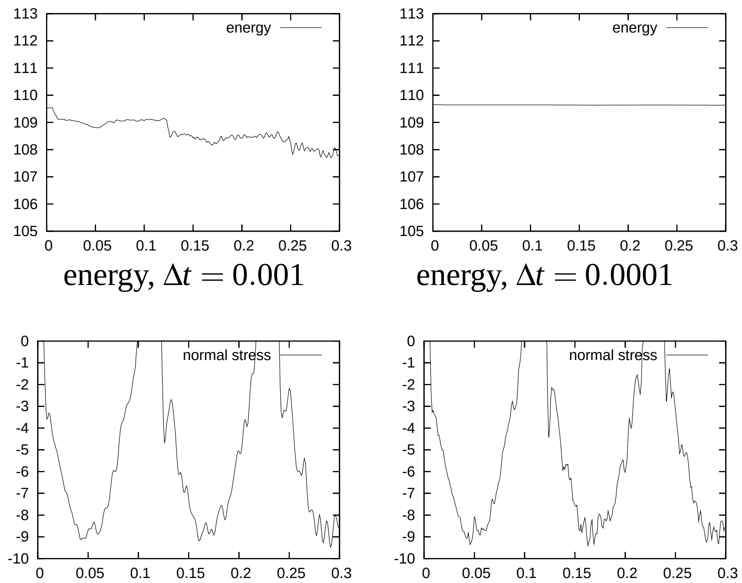
6 Numerical tests



energy, $\Delta t = 0.001$ and $\Delta t = 0.0001$ | contact stress, $\Delta t = 0.001$ | contact stress, $\Delta t = 0.0001$

Figure 1: energy and contact pressure evolution with the energy conserving scheme

The presented numerical test is a simulation in a two dimensional case of an elastic disc under its own weight bouncing on a rigid foundation. Figures 1 and 2 give the total energy and the contact stress at a particular contact node for the two proposed schemes. The first scheme, although stable due to the energy conserving, is very oscillating in contact stress. The conservation of energy makes the contact nodes oscillating. Moreover there is a small interpenetration. This method is probably more adapted to the dynamics of rigid bodies. The second approach with an equivalent mass matrix is more satisfactory. The energy is not strictly conserved, but there is only small variations, and the contact stress is well approximated and there is no interpenetration.



energy, $\Delta t = 0.001$ energy, $\Delta t = 0.0001$

contact stress, $\Delta t = 0.001$ contact stress, $\Delta t = 0.0001$

Figure 2: energy and contact pressure evolution with the equivalent mass matrix and a Newmark scheme with $\beta = \gamma = 1/2$

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