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# VARIATIONAL FORMULATION OF LOCALIZATION PHENOMENON

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## 1 INTRODUCTION

The approximation of strong discontinuities simulates the fracture process, experimented by a continuum, during deformation process, as a jump in the displacement field. To solve the strong discontinuities approach, finite elements with embedded discontinuities have been studied and developed in recent years. They capture the jump of the displacement field in the bulk of the element, nevertheless, some of them show problems such as mesh dependence and stress locking.

A comparative study of formulations of finite elements with embedded discontinuities carried out by [3], classified three families of element formulations, and concluded that the statically and kinematically optimal nonsymmetric formulation guarantees at the elemental level both the traction continuity across the discontinuity interface and the free rigid body relative motions of the two portions of the element split up into by the discontinuity, but the traction continuity is introduced in strong form, which makes the resulting formulation non-symmetrical. The advantages of this formulation deals with a very natural traction continuity condition and its capability of properly representing complete separation at late stages of the fracturing process, without any locking effects. However, the loss of symmetry of the tangential stiffness matrix presents numerical instability problems which may cause the solution to diverge; consequently, leading to erroneous results. To overcome this problem, [2] developed a symmetrical element but the problem of stress locking is still present.

Hence in this work, a variational formulation, based on the functional energy of [1], generalized to a continuum exhibiting the strain-localization phenomenon is developed, as is also the approximated solution using finite elements with embedded discontinuities, whose stiffness matrix are symmetrical,thus minimizing the computing time and reducing the problem of numerical instability.

## 2 VARIATIONAL FORMULATION OF STRONG DISCONTINUITIES

Let $\Omega$ the domain of a continuum, whose boundary is $\Gamma$ (fig. 1a), divided in two subdomains, $\Omega = \Omega^- + \Omega^+$ and two boundaries $\Gamma = \Gamma^- + \Gamma^+$. The boundary conditions are the prescribed surface tractions $\bar{\mathbf{t}}$ on $\Gamma_\sigma = \Gamma_\sigma^- + \Gamma_\sigma^+$ and the prescribed displacement $\bar{\mathbf{u}}$ on $\Gamma_u = \Gamma_u^- + \Gamma_u^+$,respectively, such that $\Gamma_\sigma \cup \Gamma_u = \Gamma$ and $\Gamma_\sigma \cap \Gamma_u = \emptyset$.

The problem described above can be solved by two approaches: 1) Discrete. The behavior of the crack borders is described by a traction-separation relation , independent from the constitutive behavior at the bulk of the material (fig. 1*b*), and 2) Continuous. Whose material behavior is described by a stress-strain non linear constitutive equation equipped with softening (fig. 1*c*).

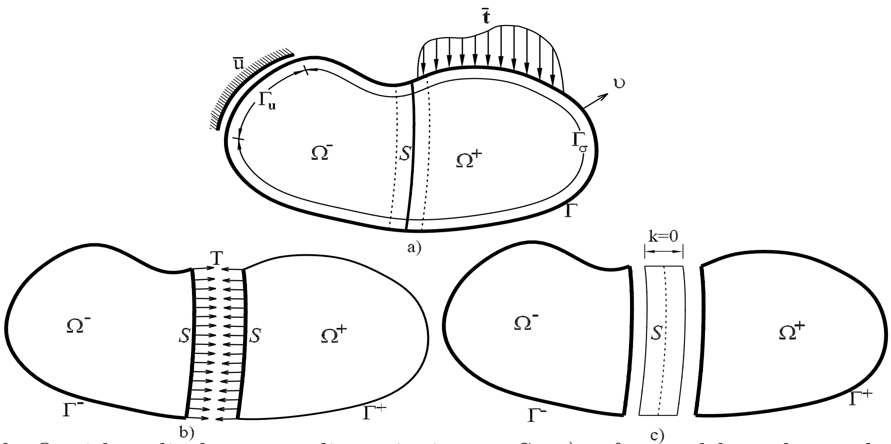


Figure 1: Body $\Omega$ with a displacement discontinuity on $S$: a) referenced boundary value problem, b) Discrete approach and c) Continuous approach.

The energy functional of the continuum described above for the Discrete approach is defined by:

$$\Pi^{\Omega}_{\mathrm{VM}\mathbf{T}}\left(\mathbf{u}, \sigma_{\Omega\backslash S}, \hat{\varepsilon}, [|\mathbf{u}|]\right) = \int_{\Omega\backslash S} [\sigma : (\hat{\varepsilon}^u - \hat{\varepsilon})\, d\Omega + W(\hat{\varepsilon}) d\Omega - \mathbf{b}\cdot\mathbf{u}]\, d\Omega - \int_{\Gamma_\sigma} \bar{\mathbf{t}}\cdot\mathbf{u} d\Gamma - \int_{\Gamma_u} \mathbf{t}\cdot(\mathbf{u}-\bar{\mathbf{u}}) d\Gamma + \int_S T\cdot [|\mathbf{u}|]\, dS \quad (1)$$

and for the Continuous approach by

$$\Pi^{\Omega}_{\mathrm{V}}\left(\mathbf{u}, \sigma, \hat{\varepsilon}, \mathbf{t}, [|\mathbf{u}|], \sigma_{\mathbf{S}}, \breve{\varepsilon}\right) = \int_{\Omega\backslash S} [\sigma : (\hat{\varepsilon}^u - \hat{\varepsilon})\, d\Omega + W(\hat{\varepsilon}) d\Omega - \mathbf{b}\cdot\mathbf{u}]\, d\Omega - \int_{\Gamma_\sigma} \bar{\mathbf{t}}\cdot\mathbf{u} d\Gamma - \int_{\Gamma_u} \mathbf{t}\cdot(\mathbf{u}-\bar{\mathbf{u}}) d\Gamma + \int_S \left[\sigma_S \colon \left(\breve{\varepsilon}^{[|\mathbf{u}|]} - \breve{\varepsilon}\right) + W(\breve{\varepsilon})\right] dS \quad (2)$$

Using the fundamental lemma of variational calculus, it can be shown that the first variation of the energy functional of eq. (1) of the Discrete approach, $\delta\,\Pi^{\Omega}_{\mathrm{VM}\mathbf{T}} = 0$, yields the weak form of the following equations.

$$
\begin{array}{llll}
a) & \hat{\varepsilon}^{\mathbf{u}}(\mathbf{x},t) - \hat{\varepsilon}(\mathbf{x},t) = 0 & \text{in } \Omega \setminus S & \text{Kinematical compatibility} \\
b) & \sigma^{\hat{\varepsilon}}(\mathbf{x},t) - \sigma(\mathbf{x},t) = \mathbf{0} & \text{in } \Omega \setminus S & \text{Constitutive equation} \\
c) & \nabla\cdot\sigma(\mathbf{x},t) + \mathbf{b}(\mathbf{x},\mathbf{t}) = \mathbf{0} & \text{in } \Omega \setminus S & \text{External equilibrium} \\
d) & \sigma(\mathbf{x},t)\cdot\nu = \bar{\mathbf{t}}(\mathbf{x},t) & \text{on } \Gamma_\sigma & \text{External equilibrium} \\
 & \sigma(\mathbf{x},t)\cdot\nu = \mathbf{t}(\mathbf{x},t) & \text{on } \Gamma_u & \\
e) & \mathbf{u}(\mathbf{x},t) = \bar{\mathbf{u}}(\mathbf{x},t) & \text{on } \Gamma_u & \text{Essential boundary condition} \\
f) & \underbrace{\sigma_{\Omega^-}\cdot\mathbf{n}}_{=[|\sigma|]_{S^-}\cdot\mathbf{n}} - T = \mathbf{0} & \text{on } S & \text{Inner traction continuity} \\
g) & \underbrace{\sigma_{\Omega^+}\cdot\mathbf{n}}_{=[|\sigma|]_{S^+}\cdot\mathbf{n}} - T = \mathbf{0} & \text{on } S & \text{Inner traction continuity}
\end{array}
\tag{3}
$$

The first variation of the energy functional of eq. (2) of Continuous approach, $\delta\,\Pi^{\Omega}_{\mathrm{V}} = 0$, yields the weak form of the eqs.$(3c - e)$ and

$$
\begin{array}{lll}
\check{\varepsilon}^{\mathbf{u}}(\mathbf{x},t) - \check{\varepsilon}(\mathbf{x},\mathbf{t}) = \mathbf{0} & \text{on } S & \text{Kinematical compatibility} \\
\sigma^{\hat{\varepsilon}}(\mathbf{x},t) - \sigma(\mathbf{x},\mathbf{t}) = \mathbf{0} & \text{on } S & \text{Constitutive equation} \\
f)\;\; \sigma_{\Omega^-}\cdot\mathbf{n} - \sigma_S\cdot\mathbf{n} = [|\sigma|]_S\cdot\mathbf{n} = \mathbf{0} & \text{on } S & \text{Inner traction continuity} \\
\quad\;\; \sigma_{\Omega^+}\cdot\mathbf{n} - \sigma_S\cdot\mathbf{n} = [|\sigma|]_S\cdot\mathbf{n} = \mathbf{0} & &
\end{array}
\tag{4}
$$

In both cases the inner and outer traction continuity on $S$ is fulfilled.

## 3 APPROXIMATION BY FEM

The energy functionals of eqs. (1) and (2) are approximated by the Finite Element Method, considering each field as independent. Then, the stiffness matrix of a finite element with an embedded discontinuity for the Discrete approach is given by

$$
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\mathbf{d}\sigma_{\Omega\setminus S}} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{[|\mathbf{u}|]\sigma_{\Omega\setminus S}} \\
\mathbf{0} & \mathbf{0} & \mathbf{K}_{\hat{\mathbf{e}}\hat{\mathbf{e}}} & \mathbf{K}_{\hat{\mathbf{e}}\sigma_{\Omega\setminus S}} \\
K_{\sigma_{\Omega\setminus S}\mathbf{d}} & \mathbf{K}_{\sigma_{\Omega\setminus S}[|\mathbf{u}|]} & \mathbf{K}_{\sigma_{\Omega\setminus S}\hat{\mathbf{e}}} & \mathbf{0}
\end{bmatrix}
\cdot
\begin{Bmatrix}
\dot{\mathbf{d}} \\ [|\dot{\mathbf{u}}|] \\ \dot{\hat{\mathbf{e}}} \\ \dot{\sigma}_{\Omega\setminus S}
\end{Bmatrix}
=
\begin{Bmatrix}
\dot{\mathbf{F}}_{\mathbf{ext}} \\ -\dot{\mathbf{F}}_S \\ \mathbf{0} \\ \mathbf{0}
\end{Bmatrix}
\tag{5}
$$

where

$$
\begin{array}{ll}
\mathbf{K}_{\hat{\mathbf{e}}\hat{\mathbf{e}}} = \int_{\Omega\setminus S} \mathbf{N}_{\hat{\mathbf{e}}}^T \cdot \mathbf{C} \cdot \mathbf{N}_{\hat{\mathbf{e}}}\, d\Omega & \mathbf{K}_{\hat{\mathbf{e}}\sigma_{\Omega\setminus S}} = -\int_{\Omega\setminus S} \mathbf{N}_{\hat{\mathbf{e}}}^T \cdot \mathbf{N}_{\sigma_{\Omega\setminus S}}\, d\Omega = \mathbf{K}^T_{\sigma_{\Omega\setminus S}\hat{\mathbf{e}}} \\
\mathbf{K}_{\mathbf{d}\sigma_{\Omega\setminus S}} = \int_{\Omega\setminus S} \mathbf{B}^T \cdot \mathbf{N}_{\sigma_{\Omega\setminus S}}\, d\Omega = \mathbf{K}^T_{\sigma_{\Omega\setminus S}\mathbf{d}} & \dot{\mathbf{F}}_{\mathbf{ext}} = \int_{\Omega\setminus S} \mathbf{N}^T \cdot \dot{\mathbf{b}}\, d\Omega + \int_{\Gamma_\sigma} \mathbf{N}^T \cdot \dot{\bar{\mathbf{t}}}\, d\Gamma \\
\mathbf{K}_{[|\mathbf{u}|]\sigma_{\Omega\setminus S}} = -\int_{\Omega\setminus S} \nabla\varphi^T \cdot \mathbf{N}_{\sigma_{\Omega\setminus S}}\, d\Omega = \mathbf{K}^T_{\sigma_{\Omega\setminus S}[|\mathbf{u}|]} & \dot{\mathbf{F}}_S = \int_S \dot{T}\, dS
\end{array}
\tag{6}
$$

and, for the Continuous approach by

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\mathbf{d}\sigma_{\Omega\setminus S}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{[|\mathbf{u}|]\sigma_{\Omega\setminus S}} & \mathbf{K}_{[|\mathbf{u}|]\sigma_S} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{\hat{\mathbf{e}}\hat{\mathbf{e}}} & \mathbf{0} & \mathbf{K}_{\hat{\mathbf{e}}\sigma_{\Omega\setminus S}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\check{\mathbf{e}}\check{\mathbf{e}}} & \mathbf{0} & \mathbf{K}_{\check{\mathbf{e}}\sigma_S} \\ K_{\sigma_{\Omega\setminus S}\mathbf{d}} & \mathbf{K}_{\sigma_{\Omega\setminus S}[|\mathbf{u}|]} & \mathbf{K}_{\sigma_{\Omega\setminus S}\hat{\mathbf{e}}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{\sigma_S[|\mathbf{u}|]} & \mathbf{0} & \mathbf{K}_{\sigma_S\check{\mathbf{e}}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{Bmatrix} \dot{\mathbf{d}} \\ [|\dot{\mathbf{u}}|] \\ \dot{\hat{\mathbf{e}}} \\ \dot{\check{\mathbf{e}}} \\ \dot{\sigma}_{\Omega\setminus S} \\ \dot{\sigma}_S \end{Bmatrix} = \begin{Bmatrix} \dot{\mathbf{F}}_{\mathbf{ext}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \quad (7)$$

where

$$\begin{array}{ll} \mathbf{K}_{\check{\mathbf{e}}\check{\mathbf{e}}} = \int_S \mathbf{N}_{\check{\mathbf{e}}}^T \cdot \mathbf{C}^d \cdot \mathbf{N}_{\check{\mathbf{e}}} dS & \mathbf{K}_{[|\mathbf{u}|]\sigma_S} = \int_S \mathbf{N}_{\sigma_S} \cdot \mathbf{n} dS = \mathbf{K}_{\sigma_S[|\mathbf{u}|]}^T \\ \mathbf{K}_{\check{\mathbf{e}}\sigma_S} = -\int_S \mathbf{N}_{\check{\mathbf{e}}}^T \cdot \mathbf{N}_{\sigma_S} dS = \mathbf{K}_{\tilde{\mathbf{e}}\sigma_S}^T & \end{array} \quad (8)$$

Some representative numerical simulations of the onset and propagation of fracture phenomenon illustrate the performance of the presented formulation [4].

## 4 CONCLUSIONS

- The developed finite elements with embedded discontinuities fulfil the conditions of traction continuity and rigid body relative motions of the portions of the elements split up into by the discontinuity; also the stiffness matrices are symmetric.
- The advantages of this formulation are: 1) the possibility of having an approximation of four independent fields fields for Discrete approach and six for the Continuous approach, 2) the stiffness matrices developed above are symmetrical, characteristic which reduces the problem of numerical instability and 3) it can be shown that there is a bridge between Continuous and Discrete approaches.

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