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AN UNSTRUCTURED MESH FRAMEWORK FOR SIMULATION OF ALL-SCALE ATMOSPHERIC FLOWS

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Abstract. *Higher resolutions, necessary to capture detailed flow features in multi-scale geophysical flows are not available in routinely used Cartesian-grid models, because they would require a many-fold increase in the number of computational points. Flexible unstructured and adaptive meshes may offer an effective alternative. The paper describes a general unstructured/hybrid mesh framework suitable for the development of all-scale atmospheric flow models. The framework is based on a finite volume, edge-based space discretisation and employs a class of Non-Oscillatory Forward in Time (NFT) integrators for the governing PDEs, built on the Multidimensional Positive Definite Advection Transport Algorithm (MPDATA). Numerical examples include non-hydrostatic orographic flows in weakly and strongly stratified regimes, and their hydrostatic analog on the sphere.*

1 INTRODUCTION

A rigid connectivity of structured grids routinely used in the simulation of atmospheric/oceanic circulations (viz. rotating stratified flows) imposes severe limitations on mesh adaptivity to flow features and/or the complex geometry of physical domains. In contrast, for many problems, a wide range of scales in atmospheric flows, heterogeneous distribution of regions of interest, and/or complex geometry can be accommodated efficiently with fully-unstructured mesh technology. The realization of limitations of structured grids, and of a need for flexible mesh adaptivity, has stimulated recent interest within the atmospheric/oceanic science community in the development of unstructured-mesh solvers.

The paper describes a general unstructured/hybrid mesh framework suitable for the development of all-scale atmospheric flow models. The key elements of the proposed framework include the finite volume edge-based non-oscillatory advection schemes MPDATA [1], preconditioned nonsymmetric Krylov-subspace elliptic solver [2, 3], and a class of non-oscillatory forward-in-time (NFT) algorithms for integrating governing PDEs. The NFT algorithms, reviewed recently in [4], have been successfully utilised in the developments of unstructured mesh based solvers for compressible [5] and incompressible fluid equations [6]. Herein, this work is extended to stratified rotating flows. For global simulations the framework is generalised to spherical geometry. Unconventionally for unstructured-mesh global models, the latter employs unholonomic geospherical frame [7].

The developed framework allows for investigating optimal mesh-point distributions, including a flexible use of various mesh adaptivity techniques, for which we use an explicit analytic form of the error estimator naturally arising from MPDATA [8]. For illustration, a selection of unstructured mesh based local-area and global models is presented.

2 A GENERAL NFT UNSTRUCTURED MESH FRAMEWORK

The unstructured mesh based framework employs a flexible template for fluid flows, offered by a high resolution NFT scheme. NFT methods have been put forward in the early nineties in the context of geophysical flows and structured grid solvers [9, 10]. NFT labels a class of second-order-accurate, either semi-Lagrangian (trajectory wise) or Eulerian (finite-volume wise), algorithms that rely on strengths of two-time-level nonlinear advection techniques that suppress/reduce/control numerical oscillations characteristic of higher-order linear schemes. The underlying theory derives from the truncation-error analysis of uncentered two-time level approximations for an archetype inhomogeneous PDE for fluids

$$\frac{\partial \phi}{\partial t} + \nabla \bullet (\mathbf{V}\phi) = R \ , \tag{1}$$

in abstraction from any particular assumptions on the nature of physical forcings R, and their relation to dependent-variables densities ϕ, but driven solely by the requirement of the second-order accuracy for arbitrary flows $\mathbf{V}(\mathbf{x}, t)$. A resulting template algorithm for

(1) takes a simple compact form

$$\phi_i^{n+1} = \mathcal{A}_i(\phi^n + 0.5\delta t R,\ \mathbf{V}^{n+1/2}) + 0.5\delta t R^{n+1} \ , \tag{2}$$

where n and i refer to the temporal and spatial position on the mesh, $\mathcal{A}$ symbolizes any two-time level nonoscillatory transport operator (viz. advection scheme) assumed second-order accurate for a homogeneous case of (1) with time-independent $\mathbf{V}$. Furthermore, $\mathbf{V}^{n+1/2}$ is a $\mathcal{O}(\delta t^2)$ estimate of $\mathbf{V}$ at $t + 0.5\delta t$, and $R^{n+1} = R(t+\delta t) + \mathcal{O}(\delta t^3)$ can either be an explicit estimate or depend implicitly on all relevant problem variables ϕ.

In NFT schemes there are two elements defining the solver: i) the choice of the transport operator $\mathcal{A}$; and ii) the approach for evaluating R^{n+1} at the rhs. For the former, our method of choice is MPDATA [1], for its genuine multidimensionality free of splitting errors, easy accuracy-sustaining generalization to unstructured meshes, and suitability for DNS, LES, and ILES. In general, the evaluation of R^{n+1} depends on the problem at hand, the form of governing PDEs and a selection of dependent variables. In particular, implicit representations of R^{n+1} [5] lead to an associated elliptic problem.

3 A LOCAL AREA NON-HYDROSTATIC MODEL

First, we illustrate how the approach can be used for developments of local area non-hydrostatic models. Here, the unstructured mesh NFT framework is applied to a stratified flow past a two-dimensional cosine hill with the fixed half-width L equal to the wavelength of the vertically propagating hydrostatic mountain wave $\lambda_z = 2\pi U_0/N$; where U_0 and N denote, respectively, the ambient wind and buoyancy frequency. To depict suitability of the approach for weakly forced (linear) and strongly forced (nonlinear) flow response, we show the solutions at Froude number $Fr \gtrsim 2$ and $\lesssim 1$, corresponding to the dimensionless mountain heights of $h_0/L \lesssim 0.125$ and $h_0/L \gtrsim 0.25$, respectively; here, $Fr = U_0/Nh_0$. The solutions are obtained by the incompressible Boussinesq model, for which the governing conservation laws of mass, momentum and thermodynamic properties take the form

$$\begin{aligned}
&\nabla \bullet (\mathbf{V}\rho_o) = 0 \ , \\
&\frac{\partial \rho_o V^I}{\partial t} + \nabla \bullet (\mathbf{V}\rho_o V^I) = -\rho_o \frac{\partial \widetilde{p}}{\partial x^I} + g\rho_o \frac{\theta'}{\theta_o}\delta_{I2} \ , \\
&\frac{\partial \rho_o \theta}{\partial t} + \nabla \bullet (\mathbf{V}\rho_o \theta) = 0 \ .
\end{aligned} \tag{3}$$

Here, ρ and θ denote the density and potential temperature[1], V^I $(I = 1, 2)$ refers to the velocity components in the horizontal and the vertical, $\widetilde{p} = p'/\rho_o$; whereas subscripts $_o$ denote constant values of the static reference state, while primes denote perturbations with respect to the static ambient state characterised by constant stratification $S_o = N^2/g$. The template algorithm (2) is applied first to the entropy equation, thus providing readily

[1]The potential temperature in (3) amounts to the specific entropy, because $s = c_p \ln \theta$.

an $\mathcal{O}(\delta t^3)$ estimate of the buoyancy force on the rhs of the vertical momentum equation. In consequence, in the corresponding realisations of (2) for momenta, the components of the pressure are the only unknowns on the rhs. Applying the discrete counterpart of the mass continuity constraint, to these realisations of (2), formulates the discrete elliptic problem for pressure, subsequently solved with a preconditioned nonsymmetric Krylov-subspace solver [2, 3]. Combining the updated buoyancy and pressure-gradient components on the rhs of respective (2) for momenta completes the solution.

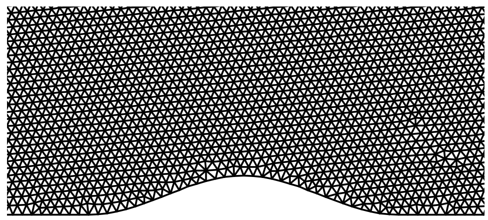

Figure 1: Details of triangular mesh

Figure (1) shows the unstructured mesh details in the vicinity of the cosine-shaped hill. Note that although the underlying mesh consists of triangles, the model uses a corresponding dual mesh with finite volumes of polygonal shapes. Figure (2) shows the isentropes in a developing flow for the linear and nonlinear regimes. In the former case, the gravity wave propagates in the vertical, in accord with linear-theory predictions. For $Fr \lesssim 1$ the wave breaking is evident in the lee.

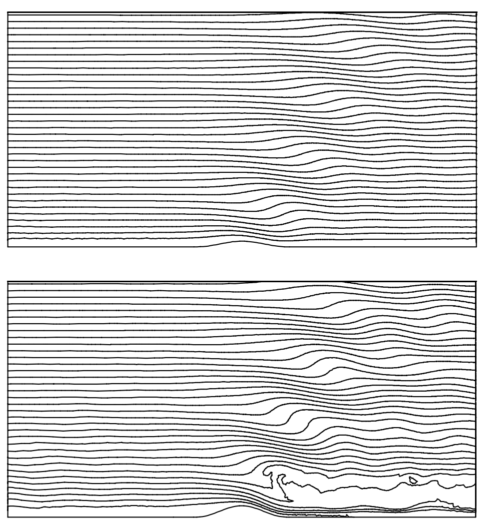

Figure 2: Isentropes simulated using the two-dimensional non-hydrostatic model; $Fr \gtrsim 2$ and $Fr \lesssim 1$, in the upper and lower plate, respectively.

4 A GLOBAL HYDROSTATIC MODEL

Our second application of the unstructured mesh NFT framework involves extension to global flows. For the flows on the sphere the framework accommodates the geospherical unholonomic formulation of [11], where the integrations of PDEs governing atmospheric dynamics were generalised for time-dependent curvilinear coordinates, to enable dynamic grid adaptivity by means of continuous mappings. Reference [7] details derivation of the unstructured-mesh NFT framework applied to the transformed conservation laws cast in curvilinear coordinates on the sphere. One immediate benefit of our approach is that it makes the task of mesh generation particularly straightforward compared to procedures required by methods using meshes defined directly on a sphere; cf. [7] for a discussion.

Consistently with the theoretical formulation in [11], the archetype PDE (1) is written compactly on a spherical surface, while extended to a vector of dependent variables $\mathbf{\Phi}$

$$\frac{\partial G\mathbf{\Phi}}{\partial t} + \nabla \cdot (\mathbf{V}\mathbf{\Phi}) = G\mathbf{R} \ . \tag{4}$$

In (4) the Jacobian $G \equiv |g_{pq}|^{1/2}$ is defined in terms of the metric tensor g_{pq} of the spherical coordinate system $\mathbf{x} = (x^1, x^2) \equiv (x, y) \equiv (\lambda, \varphi)$ with the metric form $ds^2 = g_{pq}dx^p dx^q = g_{11}dx^1dx^1 + g_{22}dx^2dx^2 \equiv (h_x d\lambda)^2 + (h_y d\varphi)^2$, where $h_x = \sqrt{g_{11}} = r\cos\varphi$, $h_y = \sqrt{g_{22}} = r$ with λ, φ and r denoting, respectively, the longitude and latitude angles and the sphere's radius. Correspondingly, the NFT template algorithm (2) takes a modified form

$$\forall_{i,n} \quad \mathbf{\Phi}_i^{n+1} = \mathcal{A}_i(\mathbf{\Phi}^n + 0.5\delta t\mathbf{R}^n,\ \mathbf{V}^{n+1/2}, G) + 0.5\delta t\mathbf{R}_i^{n+1} \ , \tag{5}$$

where "advective velocity" $\mathbf{V}$ is a product of the Jacobian G and the contravariant velocity of the transformed frame.

The particular set of the PDEs adopted describes rotating stratified fluids under shallow hydrostatic atmosphere approximation, cast in a hybrid Eulerian-Lagrangian form with a material coordinate ζ monotonically increasing with height. In technical terms, it extends the shallow-water models to continuously stratified 3D fluids [7]:

$$\begin{aligned}
\frac{\partial G\mathcal{D}}{\partial t} + \nabla \cdot (G\mathbf{v}^*\mathcal{D}) \quad &= 0 \ , \\
\frac{\partial GQ_x}{\partial t} + \nabla \cdot (G\mathbf{v}^*Q_x) \quad &= G\left(-\frac{1}{h_x}\mathcal{D}\frac{\partial M}{\partial x} + fQ_y - \frac{1}{G\mathcal{D}}\frac{\partial h_x}{\partial y}Q_xQ_y\right) \ , \\
\frac{\partial GQ_y}{\partial t} + \nabla \cdot (G\mathbf{v}^*Q_y) \quad &= G\left(-\frac{1}{h_y}\mathcal{D}\frac{\partial M}{\partial y} - fQ_x + \frac{1}{G\mathcal{D}}\frac{\partial h_x}{\partial y}Q_x^2\right) \ , \\
\frac{\partial M}{\partial \zeta} = \Pi \ .
\end{aligned} \tag{6}$$

Here, the position vector $\mathbf{x} = (x, y, \zeta)$, the generalised density is $\mathcal{D} \equiv \partial p/\partial\zeta$, and $\mathbf{Q} = (Q_x, Q_y)$ is a momentum vector, a product of $\mathcal{D}$ and a physical velocity measurable in a local Cartesian frame tangent to the sphere's surface. The $\mathbf{v}^* = \dot{\mathbf{x}}$ on the lhs is the contravariant velocity. On the rhs, the pressure gradient force depends on the Montgomery potential (or Bernoulli function) $M = gH + \zeta\Pi$, where H is the height of a material surface, and Π is a transform of the pressure p. An isopycnic model is used in the calculation presented here, hence $\zeta = \rho^{-1}$ is the specific volume, so $\Pi \equiv p$ [12]. The remaining terms on the rhs include the Coriolis and the metric forces familiar from meteorological applications.

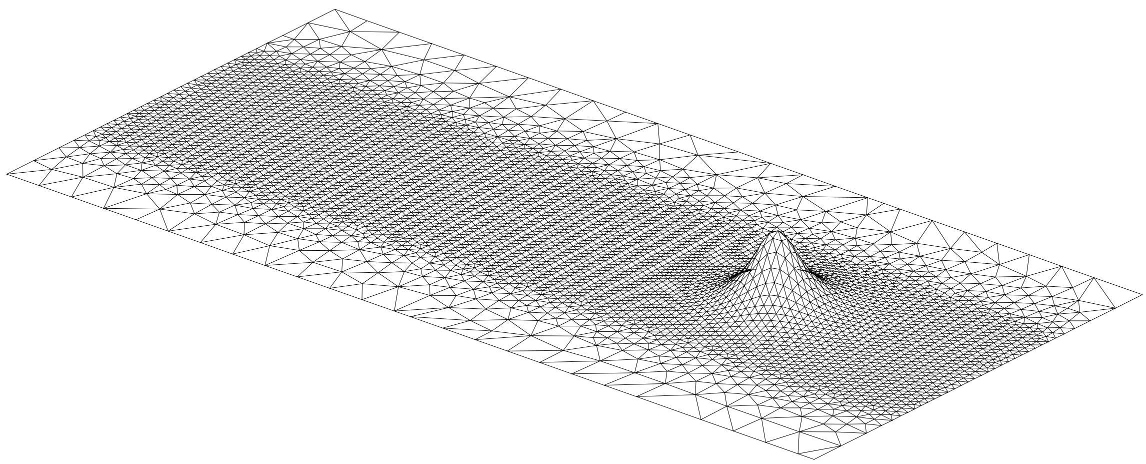

Figure 3: Perspective display of the mesh (4532 points) used in 3D orographic global flow experiments; the vertical scale is exaggerated for clarity of visualisation.

For continuity with the example discussed in the preceding section, we illustrate a strongly stratified 3D mesoscale flow past an isolated hill. To simulate mesoscale motions without incurring the computational expense associated with discretising the Earth's surface with mesoscale resolutions, we reduce the planet's radius hundredfold [13]. The analytic axisymmetric hill is placed at the equator, and its height decays as $h(x, y) = h_0[1. + (l/\mathcal{L})^2]^{-3/2}$, where $l(x, y)$ denotes the distance from the centre of the hill on the spherical surface, and $\mathcal{L} = 12.4 \cdot 10^3$ m is the hill's profile half-width. The calculations were conducted for $h_0 = 500$ m, corresponding to $Fr = 0.5$. Figure (3) shows the computational mesh, whereas Figure (4) illustrates flow features, after four hours. The results evince flow blocking on the lower upwind side of the hill and intense lee eddies, characteristic of strongly stratified 3D flows [15, 14].

5 REMARKS

The paper demonstrates applicability of the general NFT framework to simulation of atmospheric flows using fully unstructured meshes. Novel numerical illustrations confirm that the edge-based discretisation sustains accuracy of structured grids.

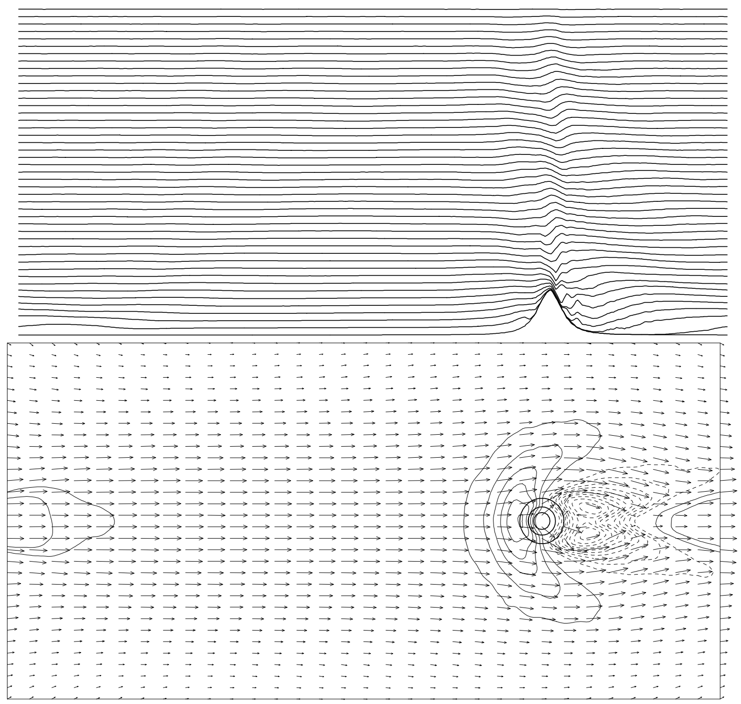

Figure 4: $Fr = 0.5$ flow on the sphere; isopycnal elevations after 4h of simulated time in the vertical equatorial plane, and in the horizontal plane for the isopycnal surface with undisturbed height ≈ 0.125wavelength of the mountain wave.

The key of the present development is a derivation of approach which provides a simple equivalence between the structured grid methodologies used in EULAG [4] and the edge-based framework. Consequently, the proven elements of the EULAG non-hydrostatic model translate directly to unstructured meshes. This not only provides a particularly efficient numerical development path, but also facilitates a meaningful comparison between the performance of structured and unstructured meshes.

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REFERENCES

[1] P.K. Smolarkiewicz and J. Szmelter, MPDATA: An Edge-Based Unstructured-Grid Formulation, *J. Comput. Phys.*, **206**, 624-649 (2005).

[2] P.K. Smolarkiewicz, V. Grubišić, L.G. Margolin, On forward-in-time differencing for

fluids: stopping criteria for iterative solutions of anelastic pressure equations, *Monthly Weather Rev.*, **125** 647-654 (1997).

[3] P.K. Smolarkiewicz, C. Temperton, S.J. Thomas, and A.A. Wyszogrodzki. Spectral Preconditioners for nonhydrostatic atmospheric models: extreme applications. *Proceedings of the ECMWF Seminar Series on Recent developments in numerical methods for atmospheric and ocean modelling, Reading, UK*, 203-220 (2004).

[4] J.M. Prusa, P.K. Smolarkiewicz and A.A Wyszogrodzki, EULAG, a Computational Model for Multiscale Flows, *Comput. Fluids*, **37**, 1193–1207 (2008).

[5] P.K. Smolarkiewicz and J. Szmelter, Iterated Upwind Schemes for Gas Dynamics, *J. Comput. Phys.*, **228**, 33–54 (2009).

[6] J. Szmelter P.K. and Smolarkiewicz , An iterative upwind method for incompressible and compressible fluids, *IACM/ECCOMAS Congress, Venice, Italy*, (2008).

[7] J. Szmelter and P.K. Smolarkiewicz, An Edge Based Unstructured Mesh Discretisation in Geospherical Framework, *J. Comput. Phys.*, in press.

[8] J. Szmelter and P.K. Smolarkiewicz, MPDATA Error Estimator for Mesh Adaptivity, *Int. J. Numer. Methods Fluids*, **50**, 1269–1293 (2006).

[9] P.K. Smolarkiewicz, On forward-in-time differencing for fluids, *Monthly Weather Rev.*, **119**, 2505–2510 (1991).

[10] P.K. Smolarkiewicz and L.G. Margolin, On forward-in-time differencing for fluids: extension to a curvilinear framework, *Monthly Weather Rev.*, **121**, 1847-1859 (1993).

[11] J.M. Prusa, P.K. Smolarkiewicz, An all-scale anelastic model for geophysical flows: dynamic grid deformation, *J. Comput. Phys.*, **190**, 601-622 (2003).

[12] C. Schär, P.K. Smolarkiewicz, A synchronous and iterative flux-correction formalism for coupled transport equations, *J. Comput. Phys.*, **128**, 101–120 (1996).

[13] N.P. Wedi and P.K. Smolarkiewicz, A Framework for testing global nonhydrostatic models, *Q.J. Roy. Meteorol. Soc.*, **135**, 469–484 (2009).

[14] P.K. Smolarkiewicz, R. Rotunno, Low Froude number flow past three-dimensional obstacles. Part I: Baroclinically generated lee vortices, *J. Atmos. Sci.* **46** 1154-1164 (1989).

[15] C.R. Hunt, W.H. Snyder, Experiments on stably and neutrally stratified flow over a model three-dimensional hill, *J. Fluid Mech.* **96** 671-704 (1980).