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A SHALLOW WATER MODEL WITH VISCOSITY AND DEPENDENCE ON DEPTH

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Abstract. *In this work, we study the anisotropic Navier-Stokes equations in a shallow domain. We use asymptotic analysis (as in our previous works[1,2]) to obtain a new shallow water model with viscosity that allows us to give not only the depth-averaged horizontal velocity, but the three components of velocity for all z. There are two major novelties in the new shallow water model that we have obtained: the new diffusion terms and the dependence on depth of the velocities.*

1 INTRODUCTION

In this work we wish to obtain a shallow water model from Navier-Stokes equations but without averaging in depth. With this aim we introduce a small adimensional parameter ε related to the depth of the domain where we shall work, tipically a river, a lake or an ocean's region. Let us consider the domain

$$\Omega^\varepsilon = \left\{(x^\varepsilon, y^\varepsilon, z^\varepsilon) \in \mathbb{R}^3 /\ (x^\varepsilon, y^\varepsilon) \in D \subset \mathbb{R}^2,\ z^\varepsilon \in (H^\varepsilon(x^\varepsilon, y^\varepsilon), s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon))\right\} \tag{1}$$

where $(x^\varepsilon, y^\varepsilon)$ are the horizontal coordinates, z^ε is the vertical coordinate, D is the

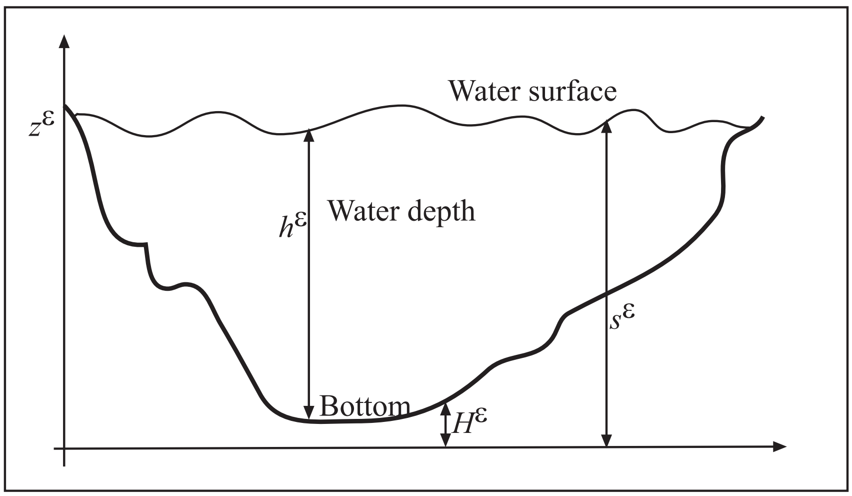

Figure 1: Domain Ω^ε

projection on the XY plane of Ω^ε, $z^\varepsilon = H^\varepsilon(x^\varepsilon, y^\varepsilon)$ is the equation of the bottom of the domain (supposed known), and $z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)$ is the equation of the free surface (unknown). We can also define $h^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) - H^\varepsilon(x^\varepsilon, y^\varepsilon)$ (water depth).

To make sure that Ω^ε is a shallow domain (remember that we are interested in a region where the depth is small when compared with the other dimensions), let us suppose that ε is small and

$$H^\varepsilon(x^\varepsilon, y^\varepsilon) = \varepsilon H(x, y), \quad s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \varepsilon s(t, x, y), \quad h^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = \varepsilon h(t, x, y) \tag{2}$$

(where $x = x^\varepsilon$, $y = y^\varepsilon$, $t = t^\varepsilon$ are independent of ε). We are assuming that H^ε, h^ε and s^ε are of order ε, so they are small when ε is small. We can then interpret ε as a small parameter of the same order as the quotient of the characteristic depth and the diameter of the domain.

Let us assume that the flow obeys the three-dimensional anisotropic Navier-Stokes equations, that is

$$\frac{\partial \vec{\mathbf{u}}^\varepsilon}{\partial t^\varepsilon} + \left(\vec{\mathbf{u}}^\varepsilon \cdot \nabla^\varepsilon\right) \vec{\mathbf{u}}^\varepsilon = -\frac{1}{\rho_0} \nabla^\varepsilon p^\varepsilon + \operatorname{div} \mathbf{\Sigma}^\varepsilon + \vec{\mathbf{F}}_e^\varepsilon \tag{3}$$

$$\operatorname{div} \vec{\mathbf{u}}^\varepsilon = 0 \tag{4}$$

where $\vec{\mathbf{u}}^\varepsilon = \vec{\mathbf{u}}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon)$ is the velocity vector, $p^\varepsilon = p^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon)$ is the pressure, $\nabla^\varepsilon = \left(\frac{\partial}{\partial x^\varepsilon}, \frac{\partial}{\partial y^\varepsilon}, \frac{\partial}{\partial z^\varepsilon}\right)$, ρ_0 denotes the density of the fluid,

$$\Sigma_{ij}^\varepsilon = \nu_j^\varepsilon \frac{\partial u_i^\varepsilon}{\partial x_j^\varepsilon} + \nu_i^\varepsilon \frac{\partial u_j^\varepsilon}{\partial x_i^\varepsilon} \tag{5}$$

with $(\nu_1^\varepsilon, \nu_2^\varepsilon, \nu_3^\varepsilon)$ denoting the viscosity vector, and $\vec{\mathbf{F}}_e^\varepsilon$ the volume force per unit mass. Let us suppose that

$$\vec{\mathbf{F}}_e^\varepsilon = -g\vec{k} - 2\vec{\phi} \times \vec{\mathbf{u}}^\varepsilon \tag{6}$$

where g is the gravity acceleration (assumed constant) and $-2\vec{\phi} \times \vec{\mathbf{u}}^\varepsilon$ is the Coriolis acceleration (where the angular velocity of rotation of the Earth is $\vec{\phi} = \phi\left(\sin\varphi^\varepsilon \vec{k} + \cos\varphi^\varepsilon \vec{\jmath}\right)$ with $\phi = 7.29 \times 10^{-5}$ rad/s; $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ denote the unit vectors pointing East, North and vertically upward (respectively); φ^ε is the North latitude, that we consider either constant or depending on y^ε).

We must also impose the boundary conditions. We shall assume that the pressure p^ε coincides with the atmospheric pressure at the surface

$$p^\varepsilon = p_s^\varepsilon \quad \text{at } z^\varepsilon = s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) \tag{7}$$

(where the atmospheric pressure at the surface, $p_s^\varepsilon = p_s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)$, is supposed to be known and independent of ε, that is, $p_s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) = p_s(t, x, y)$). The fluid satisfies the non-penetration condition at the bottom so

$$\vec{\mathbf{u}}^\varepsilon \cdot \vec{\mathbf{n}}^\varepsilon = 0 \quad \text{at } z^\varepsilon = H^\varepsilon(x^\varepsilon, y^\varepsilon) \tag{8}$$

where $\vec{\mathbf{n}}^\varepsilon$ denotes the outer unit normal to the boundary of the domain.

Rather than assuming that the velocity vanishes at the bottom, we are going to take friction into account, which translates into a condition on the stresses at the bottom:

$$\left[\mathbf{T}^\varepsilon \vec{\mathbf{n}}^\varepsilon\right]_\tau = -\vec{\mathbf{f}}_R^\varepsilon \quad \text{at} \quad z^\varepsilon = H^\varepsilon \tag{9}$$

where $[\cdot]_\tau$ denotes the projection onto the plane tangent to the bottom at each point and, typically, the friction force is of the form $\vec{\mathbf{f}}_R^\varepsilon = \rho_0 C_R^\varepsilon |\vec{\mathbf{u}}^\varepsilon| \vec{\mathbf{u}}^\varepsilon$ (C_R^ε is small).

Similarly we give a condition on the stresses at the surface, so if $\vec{\mathbf{f}}_W^\varepsilon$ is the force of the wind:

$$\left[\mathbf{T}^\varepsilon \vec{\mathbf{n}}^\varepsilon\right]_\tau = \left[\vec{\mathbf{f}}_W^\varepsilon\right]_\tau \quad \text{at} \quad z^\varepsilon = s^\varepsilon \tag{10}$$

We also suppose that the incoming and outcoming flows are known at each instant. Other kind of boundary conditions may be easily considered.

We shall need to introduce the vorticity

$$\vec{\gamma}^\varepsilon = \nabla^\varepsilon \times \vec{\mathbf{u}}^\varepsilon \tag{11}$$

that verifies the following equation (see, for example, Temam and Miranville[3]):

$$\frac{\partial \vec{\gamma}^\varepsilon}{\partial t^\varepsilon} + (\vec{\mathbf{u}}^\varepsilon \cdot \nabla^\varepsilon)\, \vec{\gamma}^\varepsilon - (\vec{\gamma}^\varepsilon \cdot \nabla^\varepsilon)\, \vec{\mathbf{u}}^\varepsilon = \nabla^\varepsilon \times \left(\operatorname{div} \mathbf{\Sigma}^\varepsilon + \vec{\mathbf{F}}_e^\varepsilon \right) \tag{12}$$

Finally, we must include a free surface condition at $z^\varepsilon = s^\varepsilon$, that we have replaced with the following equivalent condition derived from the law of conservation of mass:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \frac{\partial}{\partial x^\varepsilon} \int_{H^\varepsilon}^{s^\varepsilon} u_1^\varepsilon \, dz^\varepsilon + \frac{\partial}{\partial y^\varepsilon} \int_{H^\varepsilon}^{s^\varepsilon} u_2^\varepsilon \, dz^\varepsilon = 0 \tag{13}$$

Initial conditions must be imposed too.

2 ASYMPTOTIC ANALYSIS

We are going to study now problem (3)–(13) using asymptotic analysis (see our previous works[1,2] for a similar example). Let us consider $\Omega = D \times (0,1)$ as reference domain, related to Ω^ε through the following change of variable:

$$t^\varepsilon = t, \quad x^\varepsilon = x, \quad y^\varepsilon = y, \quad z^\varepsilon = \varepsilon[H(x,y) + zh(t,x,y)] \tag{14}$$

Using this change of variable, we can associate to each function F^ε defined on $[0,T] \times \overline{\Omega}^\varepsilon$, another function $F(\varepsilon)$ defined on $[0,T] \times \overline{\Omega}$ in this way: $F(\varepsilon)(t,x,y,z) = F^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon)$.

Under this change of variable, problem (3)–(13) becomes an equivalent system of equations posed in the reference domain. For example, equation (4) now reads:

$$\frac{\partial u_1(\varepsilon)}{\partial x} - \frac{\frac{\partial H}{\partial x} + z \frac{\partial h}{\partial x}}{h} \frac{\partial u_1(\varepsilon)}{\partial z} + \frac{\partial u_2(\varepsilon)}{\partial y} - \frac{\frac{\partial H}{\partial y} + z \frac{\partial h}{\partial y}}{h} \frac{\partial u_2(\varepsilon)}{\partial z} + \frac{1}{\varepsilon h} \frac{\partial u_3(\varepsilon)}{\partial z} = 0 \tag{15}$$

In order to apply the formal asymptotic analysis method, let us suppose that each function in the equivalent system posed in the reference domain ($\vec{\mathbf{u}}(\varepsilon)$, $p(\varepsilon)$, ...) allows a representation in powers of ε. More precisely, if F^ε is an unknown of system (3)–(13) we assume that, once in the reference domain, can be written as follows:

$$F(\varepsilon) = F^0 + \varepsilon F^1 + \varepsilon^2 F^2 + \cdots \tag{16}$$

The next step is to substitute the expansion in powers of ε of each unknown into the system of equations in the reference domain and then, to group the terms multiplied by the same power of ε. This allows us to identify the first terms of the expansion of the unknowns ($\vec{\mathbf{u}}^0$, p^0, ...). For example, if we apply this methodology to equation (15), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[u_1^0+\varepsilon u_1^1+\varepsilon^2 u_1^2+\cdots\right]-\frac{1}{h}\left(\frac{\partial H}{\partial x}+z\frac{\partial h}{\partial x}\right)\frac{\partial}{\partial z}\left[u_1^0+\varepsilon u_1^1+\varepsilon^2 u_1^2+\cdots\right]\\
&+\frac{\partial}{\partial y}\left[u_2^0+\varepsilon u_2^1+\varepsilon^2 u_2^2+\cdots\right]-\frac{1}{h}\left(\frac{\partial H}{\partial y}+z\frac{\partial h}{\partial y}\right)\frac{\partial}{\partial z}\left[u_2^0+\varepsilon u_2^1+\varepsilon^2 u_2^2+\cdots\right]\\
&+\frac{1}{\varepsilon h}\frac{\partial}{\partial z}\left[u_3^0+\varepsilon u_3^1+\varepsilon^2 u_3^2+\cdots\right]=0 \qquad (17)
\end{aligned}$$

Grouping now the terms multiplied by the same power of ε,

$$\begin{aligned}
&\varepsilon^{-1}\frac{1}{h}\frac{\partial u_3^0}{\partial z}\\
&\quad+\frac{\partial u_1^0}{\partial x}-\frac{1}{h}\left(\frac{\partial H}{\partial x}+z\frac{\partial h}{\partial x}\right)\frac{\partial u_1^0}{\partial z}+\frac{\partial u_2^0}{\partial y}-\frac{1}{h}\left(\frac{\partial H}{\partial y}+z\frac{\partial h}{\partial y}\right)\frac{\partial u_2^0}{\partial z}+\frac{1}{h}\frac{\partial u_3^1}{\partial z}\\
&\quad+\varepsilon\left[\frac{\partial u_1^1}{\partial x}-\frac{1}{h}\left(\frac{\partial H}{\partial x}+z\frac{\partial h}{\partial x}\right)\frac{\partial u_1^1}{\partial z}\right.\\
&\qquad\left.+\frac{\partial u_2^1}{\partial y}-\frac{1}{h}\left(\frac{\partial H}{\partial y}+z\frac{\partial h}{\partial y}\right)\frac{\partial u_2^1}{\partial z}+\frac{1}{h}\frac{\partial u_3^2}{\partial z}\right]+\cdots=0 \qquad (18)
\end{aligned}$$

from where we deduce

$$\frac{\partial u_3^0}{\partial z}=0 \qquad (19)$$

$$\frac{\partial u_1^0}{\partial x}-\frac{1}{h}\left(\frac{\partial H}{\partial x}+z\frac{\partial h}{\partial x}\right)\frac{\partial u_1^0}{\partial z}+\frac{\partial u_2^0}{\partial y}-\frac{1}{h}\left(\frac{\partial H}{\partial y}+z\frac{\partial h}{\partial y}\right)\frac{\partial u_2^0}{\partial z}+\frac{1}{h}\frac{\partial u_3^1}{\partial z}=0 \qquad (20)$$

$$\frac{\partial u_1^1}{\partial x}-\frac{1}{h}\left(\frac{\partial H}{\partial x}+z\frac{\partial h}{\partial x}\right)\frac{\partial u_1^1}{\partial z}+\frac{\partial u_2^1}{\partial y}-\frac{1}{h}\left(\frac{\partial H}{\partial y}+z\frac{\partial h}{\partial y}\right)\frac{\partial u_2^1}{\partial z}+\frac{1}{h}\frac{\partial u_3^2}{\partial z}=0 \qquad (21)$$

$$\vdots$$

In our previous paper[2] this kind of calculus is done with detail for a similar example.

To apply this method to equation (3) we need to make an hypothesis on the asymptotic behavior of the viscosity vector. When doing the asymptotic approximation it is pointed out by the computations that the two first components of the viscosity vector must be of order of ε, while the third component must be or order of ε^2, so let us suppose that

$$\nu_1^\varepsilon=\nu_2^\varepsilon=\nu^\varepsilon=\varepsilon\bar{\nu}, \qquad \nu_3^\varepsilon=\varepsilon^2\bar{\nu}_3 \qquad (22)$$

There is another hypothesis we need to make. Let us assume that the two first components of the term of order zero of the asymptotic expansion of $\vec{\gamma}(\varepsilon)$ are polynomial in

z, that is,

$$\gamma_i^0 = \sum_{j=0}^{k} \left(z^j \gamma_i^{0,j} \right) \qquad (i = 1, 2) \tag{23}$$

for some k. This hypothesis will make us easier to integrate with respect to z some expressions and can be considered approximately true if k is large enough (by the Weierstrass Approximation Theorem).

Finally, we can build an approximation of the solution using only the first terms of the asymptotic expansion. For each unknown F^ε defined on the original domain, we can consider the following approximation of $F(\varepsilon)$ in the reference domain:

$$\tilde{F}(\varepsilon) = F^0 + \varepsilon F^1 \tag{24}$$

and then, an approximation of F^ε in the original domain:

$$\tilde{F}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) = \tilde{F}(\varepsilon)(t, x, y, z) \tag{25}$$

3 THE PROPOSED SHALLOW WATER MODEL

We are able now to propose a new shallow water model. Once we have built an approximation of each of the unknowns of Navier-Stokes system (3)–(13) using the asymptotic expansion method, we can find the equations they verify and, after neglecting the higher order terms, write them. For sake of clarity, we shall drop the $\sim$ symbol in what follows.

Previously to present the new model we have deduced, we shall introduce some notation: $\vec{\bar{\mathbf{u}}}^\varepsilon = (\bar{u}_1^\varepsilon, \bar{u}_2^\varepsilon)$ will denote the horizontal components of the average velocity (averaged in depth, so independent of z^ε); $\vec{\breve{\mathbf{u}}}^\varepsilon = (\breve{u}_1^\varepsilon, \breve{u}_2^\varepsilon)$ are the horizontal components of the velocity at the bottom (that is, at $z^\varepsilon = H^\varepsilon$); the gradient operator ∇^ε will represent here only the horizontal gradient, that is, $\nabla^\varepsilon = (\partial/\partial x^\varepsilon, \partial/\partial y^\varepsilon)$; and we shall denote $\vec{\gamma}^{0,\varepsilon}$ the zeroth order approximation of the two first components of the vorticity, $\vec{\gamma}^{0,\varepsilon} = (\gamma_1^{0,\varepsilon}, \gamma_2^{0,\varepsilon})$. From (23) we deduce that

$$\vec{\gamma}^{0,\varepsilon} = \sum_{j=0}^{k} \left(\frac{z^\varepsilon - H^\varepsilon}{h^\varepsilon} \right)^j \vec{\gamma}^{0,j,\varepsilon} \tag{26}$$

where $\vec{\gamma}^{0,j,\varepsilon} = (\gamma_1^{0,j,\varepsilon}, \gamma_2^{0,j,\varepsilon})$.

With this notation, the new shallow water model obtained by asymptotic analysis yields:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}\left(h^\varepsilon \vec{\bar{\mathbf{u}}}^\varepsilon \right) = 0 \tag{27}$$

$$\begin{aligned} &\frac{\partial \vec{\bar{\mathbf{u}}}^\varepsilon}{\partial t^\varepsilon} + \nabla^\varepsilon \vec{\bar{\mathbf{u}}}^\varepsilon \cdot \vec{\bar{\mathbf{u}}}^\varepsilon - \vec{\mathbf{F}}_D^\varepsilon + g \nabla^\varepsilon h^\varepsilon \\ &\quad = -\frac{1}{\rho_0} \nabla^\varepsilon p_s^\varepsilon - g \nabla^\varepsilon H^\varepsilon + \frac{1}{\rho_0 h^\varepsilon} \left(\vec{\mathbf{f}}_W^\varepsilon - \vec{\mathbf{f}}_R^\varepsilon \right) + \vec{\mathbf{F}}_C^\varepsilon \end{aligned} \tag{28}$$

$$\frac{\partial \vec{\gamma}^{0,j,\varepsilon}}{\partial t^\varepsilon} + \nabla^\varepsilon \vec{\gamma}^{0,j,\varepsilon} \cdot \vec{\bar{\mathbf{u}}}^\varepsilon - (\nabla^\varepsilon \vec{\bar{\mathbf{u}}}^\varepsilon)^T \cdot \vec{\gamma}^{0,j,\varepsilon} = \vec{\mathbf{F}}_V^{j,\varepsilon} \quad (j = 0, 1, \ldots, k) \tag{29}$$

where

$$\vec{\mathbf{F}}^\varepsilon_D = \nu^\varepsilon \left\{ \Delta^\varepsilon \vec{\mathbf{u}}^\varepsilon + \nabla^\varepsilon (\operatorname{div} \vec{\mathbf{u}}^\varepsilon) + \frac{1}{h^\varepsilon} \left[(\nabla^\varepsilon \vec{\mathbf{u}}^\varepsilon)^T + (\nabla^\varepsilon \vec{\mathbf{u}}^\varepsilon) \right] \nabla^\varepsilon h^\varepsilon \right\} \tag{30}$$

$$\vec{\mathbf{F}}^\varepsilon_C = 2\phi \begin{pmatrix} (\sin\varphi^\varepsilon)\bar{u}^\varepsilon_2 + (\cos\varphi^\varepsilon)\left(\frac{\partial (h^\varepsilon \bar{u}^\varepsilon_1)}{\partial x^\varepsilon} + \frac{h^\varepsilon}{2}\frac{\partial \bar{u}^\varepsilon_2}{\partial y^\varepsilon} - \bar{u}^\varepsilon_2 \frac{\partial H^\varepsilon}{\partial y^\varepsilon} \right) \\ -(\sin\varphi^\varepsilon)\bar{u}^\varepsilon_1 + \frac{h^\varepsilon}{2}\frac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon_1] + \frac{\partial s^\varepsilon}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon_1] \end{pmatrix} \tag{31}$$

$$\vec{\mathbf{F}}^{0,\varepsilon}_V = 2\phi \begin{pmatrix} (\sin\varphi^\varepsilon)\gamma_2^{0,0,\varepsilon} + \frac{\partial}{\partial y^\varepsilon}[(\cos\varphi^\varepsilon)\bar{u}^\varepsilon_1] \\ -(\sin\varphi^\varepsilon)\gamma_1^{0,0,\varepsilon} + (\cos\varphi^\varepsilon)\frac{\partial \bar{u}^\varepsilon_2}{\partial y^\varepsilon} \end{pmatrix} + \frac{2\nu_3^\varepsilon}{(h^\varepsilon)^2} \begin{pmatrix} \gamma_1^{0,2,\varepsilon} \\ \gamma_2^{0,2,\varepsilon} \end{pmatrix} \tag{32}$$

$$\begin{aligned} \vec{\mathbf{F}}^{j,\varepsilon}_V &= 2\phi(\sin\varphi^\varepsilon) \begin{pmatrix} \gamma_2^{0,j,\varepsilon} \\ -\gamma_1^{0,j,\varepsilon} \end{pmatrix} \\ &\quad + \nu_3^\varepsilon \frac{(j+2)(j+1)}{(h^\varepsilon)^2} \begin{pmatrix} \gamma_1^{0,j+2,\varepsilon} \\ \gamma_2^{0,j+2,\varepsilon} \end{pmatrix} \qquad (j = 1, \ldots, k-2) \end{aligned} \tag{33}$$

$$\vec{\mathbf{F}}^{j,\varepsilon}_V = 2\phi(\sin\varphi^\varepsilon) \begin{pmatrix} \gamma_2^{0,j,\varepsilon} \\ -\gamma_1^{0,j,\varepsilon} \end{pmatrix} \qquad (j = k-1, k) \tag{34}$$

The three components of the velocity and the pressure can be now computed for all z^ε:

$$u^\varepsilon_1 = \bar{u}^\varepsilon_1 + h^\varepsilon \sum_{j=0}^{k} \left[\frac{\gamma_2^{0,j,\varepsilon}}{j+1} \left(\left(\frac{z^\varepsilon - H^\varepsilon}{h^\varepsilon} \right)^{j+1} - \frac{1}{j+2} \right) \right] \tag{35}$$

$$u^\varepsilon_2 = \bar{u}^\varepsilon_2 - h^\varepsilon \sum_{j=0}^{k} \left[\frac{\gamma_1^{0,j,\varepsilon}}{j+1} \left(\left(\frac{z^\varepsilon - H^\varepsilon}{h^\varepsilon} \right)^{j+1} - \frac{1}{j+2} \right) \right] \tag{36}$$

$$\begin{aligned} u^\varepsilon_3 &= \breve{u}^\varepsilon_1 \frac{\partial H^\varepsilon}{\partial x^\varepsilon} + \breve{u}^\varepsilon_2 \frac{\partial H^\varepsilon}{\partial y^\varepsilon} + (H^\varepsilon - z^\varepsilon)\left(\frac{\partial \breve{u}^\varepsilon_1}{\partial x^\varepsilon} + \frac{\partial \breve{u}^\varepsilon_2}{\partial y^\varepsilon} \right) \\ &\quad + \sum_{j=0}^{k} \left[\frac{(z^\varepsilon - H^\varepsilon)^{j+1}}{(j+1)(h^\varepsilon)^j} \left(\frac{\partial H^\varepsilon}{\partial x^\varepsilon}\gamma_2^{0,j,\varepsilon} - \frac{\partial H^\varepsilon}{\partial y^\varepsilon}\gamma_1^{0,j,\varepsilon} \right) \right] \\ &\quad + \sum_{j=0}^{k} \left[\frac{j}{(j+1)(j+2)} \frac{(z^\varepsilon - H^\varepsilon)^{j+2}}{(h^\varepsilon)^{j+1}} \left(\frac{\partial h^\varepsilon}{\partial x^\varepsilon}\gamma_2^{0,j,\varepsilon} - \frac{\partial h^\varepsilon}{\partial y^\varepsilon}\gamma_1^{0,j,\varepsilon} \right) \right] \\ &\quad - \sum_{j=0}^{k} \left[\frac{(z^\varepsilon - H^\varepsilon)^{j+2}}{(j+1)(j+2)(h^\varepsilon)^j} \left(\frac{\partial \gamma_2^{0,j,\varepsilon}}{\partial x^\varepsilon} - \frac{\partial \gamma_1^{0,j,\varepsilon}}{\partial y^\varepsilon} \right) \right] \end{aligned} \tag{37}$$

$$p^\varepsilon = p^\varepsilon_s + \rho_0 (s^\varepsilon - z^\varepsilon)[g - 2\phi(\cos\varphi^\varepsilon)\bar{u}^\varepsilon_1] \tag{38}$$

Remark.- Equations (27)–(28) can be written in terms of the flow $\vec{\mathbf{Q}}^\varepsilon = h^\varepsilon \vec{\mathbf{u}}^\varepsilon$ as follows:

$$\frac{\partial h^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}\vec{\mathbf{Q}}^\varepsilon = 0 \tag{39}$$

$$\begin{aligned}&\frac{\partial \vec{\mathbf{Q}}^\varepsilon}{\partial t^\varepsilon} + \operatorname{div}\left(\vec{\mathbf{u}}^\varepsilon \otimes \vec{\mathbf{Q}}^\varepsilon\right) - h^\varepsilon \vec{\mathbf{F}}_D^\varepsilon + g h^\varepsilon \nabla^\varepsilon h^\varepsilon \\ &\quad = -\frac{h^\varepsilon}{\rho_0}\nabla^\varepsilon p_s^\varepsilon - g h^\varepsilon \nabla^\varepsilon H^\varepsilon + \frac{1}{\rho_0}\left(\vec{\mathbf{f}}_W^\varepsilon - \vec{\mathbf{f}}_R^\varepsilon\right) + h^\varepsilon \vec{\mathbf{F}}_C^\varepsilon\end{aligned} \tag{40}$$

Remark.- There are two major novelties in the new shallow water model (26)–(38). The first one is the new viscosity terms given by (30). If we compare with other shallow water models in literature[4,5], we find that all the diffusion terms that appear are different from the viscosity terms we have deduced. In a previous article[6] we have obtained a similar model but without dependence on depth, and precisely this dependence on depth of the velocities (35)–(37) is the second major innovation of this model.

4 NUMERICAL RESULTS

We have already compared, in our previous work[6], the new viscosity terms (30) with other that can be found in literature, so let us study in this section the improvement achieved by using equation (29) and formulae (35)–(36). In order to do that we shall approximate some analytical solutions of Navier-Stokes equations (3)–(13) solving numerically the new model (26)–(38) and the classical shallow water model (27)–(28). In the tables we shall present below to compare the errors committed in this numerical simulations, we shall denote by **CM** (Classical Model) the model given by system (27)–(28) and by **NM**-k (New Model of order k) the model given by (27)–(29) with the polynomial approximation of vorticity of order k. MacCormack[7] scheme has been implemented for the numerical resolution of the different models. This numerical method has good stability properties and has been applied successfully to the resolution of similar problems.

Let us consider now the following exact solution of Navier-Stokes equations (3)–(13):

$$\begin{aligned}&u_1^\varepsilon = -(3a_2(z^\varepsilon)^2 + 2b_2 z^\varepsilon + c_2), \quad u_2^\varepsilon = 3a_1(z^\varepsilon)^2 + 2b_1 z^\varepsilon + c_1, \quad u_3^\varepsilon = 0,\\ &\phi = 0, \quad H^\varepsilon = A_1 x^\varepsilon + A_2 y^\varepsilon + A_3, \quad h^\varepsilon = B_1 x^\varepsilon + B_2 y^\varepsilon + B_3,\\ &p^\varepsilon = p_s^\varepsilon + \rho_0(s^\varepsilon - z^\varepsilon), \quad p_s^\varepsilon = \rho_0[-6a_2\nu_3^\varepsilon x^\varepsilon + 6a_1\nu_3^\varepsilon y^\varepsilon - g s^\varepsilon + E],\\ &x^\varepsilon \in [0,10], y^\varepsilon \in [0,2], z^\varepsilon \in [H^\varepsilon(x^\varepsilon,y^\varepsilon), s^\varepsilon(x^\varepsilon,y^\varepsilon)], t^\varepsilon \in [0,10],\end{aligned} \tag{41}$$

with

$$\begin{aligned}&A_1 = 0.01, A_2 = 0.05, A_3 = 0, B_1 = 0.05, B_2 = B_1A_2/A_1, B_3 = 0.1,\\ &a_2 = 3/5, a_1 = A_1a_2/A_2, b_2 = -17/20, b_1 = A_1b_2/A_2, c2 = -0.6, c_1 = A_1c_2/A_2,\\ &g = 9.8, \rho_0 = 998.2, \nu_1^\varepsilon = \nu_2^\varepsilon = \nu_3^\varepsilon = 1\text{e-}6,\end{aligned}$$

and where $\vec{\mathbf{f}}_W^\varepsilon$ and $\vec{\mathbf{f}}_R^\varepsilon$ are chosen to satisfy (9)–(10). The discretization step is chosen to be $\Delta x^\varepsilon = \Delta y^\varepsilon = \Delta z^\varepsilon = 0.1$ and $\Delta t^\varepsilon = 0.01$.

In table 1 we can see the error committed when solving numerically system (27)–(28) (that is, when computing only the depth and the average velocities). The error is shown in the infinity norm, so we are calculating the maximum absolute difference between the exact depth and the exact average velocities and their numerical approximation.

Error h^ε	Error $\bar{u}_1^\varepsilon$	Error $\bar{u}_2^\varepsilon$
4.4e-5	4.2e-4	2.1e-4

Table 1: Depth and average velocities absolute errors for example (41)

These are the values usually computed by shallow water models. Now, we are able to obtain the velocities taking into account their dependence on z^ε, so in table 2 we show the errors committed when approximating the exact velocities (which depend on z^ε) by their numerical estimations. We can observe that when we solve the classical model (**CM**) the errors are relatively large, since we just have the average velocities to approximate the velocities at any depth, but when we solve the new model (**NM**) the errors decrease as we increase the order of the polynomial approximation of the vorticity. This is because we improve the approximation on z^ε achieving an error that is only slightly worst than the error obtained when approximating the average velocities, which is the best we could expect.

	Error u_1^ε	Error u_2^ε
CM	9.1e-1	1.8e-1
NM-0	3.6e-1	7.3e-2
NM-1	4.6e-4	2.5e-4

Table 2: Velocity absolute errors for example (41)

Let us show another example. We introduce this analytical solution of Navier-Stokes equations (3)–(13):

$$\begin{aligned}
&u_1^\varepsilon = \exp(\nu_2^\varepsilon A^2 t^\varepsilon + A y^\varepsilon + z^\varepsilon), \quad u_2^\varepsilon = u_3^\varepsilon = 0,\\
&\phi = 0, \quad H^\varepsilon = 0, \quad h^\varepsilon = a y^\varepsilon + b,\\
&p^\varepsilon = p_s^\varepsilon + \rho_0(h^\varepsilon - z^\varepsilon), \quad p_s^\varepsilon = P_0 - \rho_0 g h^\varepsilon,\\
&x^\varepsilon \in [0,10], y^\varepsilon \in [0,2], z^\varepsilon \in [0, h^\varepsilon(y^\varepsilon)], t^\varepsilon \in [0,10],
\end{aligned} \tag{42}$$

with $A = 0.2$, $a = -0.5$, $b = 1.5$, $g = 9.8$, $\rho_0 = 998.2$, $\nu_1^\varepsilon = \nu_2^\varepsilon =$ 1e-6, $\nu_3^\varepsilon = 0$, and where $\vec{\mathbf{f}}_W^\varepsilon$ and $\vec{\mathbf{f}}_R^\varepsilon$ are again chosen to satisfy (9)–(10).

In table 3 we present the absolute errors for the depth and the average velocities, and in table 4 we show the absolute errors for velocities. We can see that the errors decrease as the order of the polynomial approximation of the vorticity increases, almost reaching the best result we can expect, that is, the same approximation as for the average velocities.

Error h^ε	Error $\bar{u}_1^\varepsilon$	Error $\bar{u}_2^\varepsilon$
3.4e-7	2.5e-6	1.3e-6

Table 3: Depth and average velocities absolute errors for example (42)

	Error u_1^ε	Error u_2^ε
CM	2.2e0	1.4e-6
NM-0	5.7e-1	1.9e-6
NM-1	1.8e-1	1.9e-6
NM-2	2.7e-2	1.9e-6
NM-3	4.7e-3	1.9e-6
NM-4	5.1e-4	1.9e-6
NM-5	6.1e-5	1.9e-6
NM-6	5.5e-6	1.9e-6
NM-7	4.1e-6	1.9e-6

Table 4: Velocity absolute errors for example (42)

5 CONCLUSIONS

In classical shallow water models only depth-averaged velocities are computed. Here we present a new shallow water model (with viscosity) that is able to compute a polynomial approximation on depth of the velocities. In two previous papers[2,6] we have obtained a shallow water model with polynomial dependence on depth (but without viscosity) and a shallow water model with new viscosity terms (but constant on depth). Now we have derived a new model that combines both good qualities.

Numerical resolution of some examples where we know the exact solution of Navier-Stokes equations (3)–(13), proves that the new model proposed is able to obtain similar precision at any depth than the classical models obtain only for average velocities, what we think is the best result we could expect. Consequently we recommend using our model instead of the classical shallow water models.

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