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OPTIMAL CONTROL FOR INCOMPRESSIBLE STEADY MHD FLOWS VIA CONSTRAINED EXTENDED BOUNDARY APPROACH

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Abstract. *Optimal boundary control problems associated with the Magnetohydrodynamic (MHD) equations have a wide and important range of applications. If one desires to increase or decrease the velocity inside a channel filled with a conductive fluid or to impose a desired velocity profile near the channel outflow, the control of the magnetic field may represent the best way to reach the objective. The magnetic field inside the domain obeys to the MHD equations and the control by an external field can be achieved only through the control of the boundary conditions. In this work, we study a class of stationary boundary MHD optimal velocity and optimal flow control problems. Standard boundary control approach is not very straightforward to implement numerically in comparison to distributed control and often leads to unnecessary smooth controls. In this paper we present a very different approach from the standard one where the optimal boundary control problem is transformed into an extended distributed problem. This can be achieved by considering boundary controls in the form of lifting functions which extend from the boundary into the inner domain. The optimal solution is then searched by exploring all possible extended functions. This approach gives robustness to the boundary control algorithm which can be solved by standard distributed control techniques over the interior part of the domain. Boundary controls obtained by extended functions have several advantages. The extended function can easily take into account several possible boundary conditions and both Dirichlet and Neumann controls. We can seek these boundary controls in their natural functional spaces differently from the standard approach where the control must be, for feasibility reasons, in smoother spaces. Also in this approach integral constraints on the boundary magnetic field may be implicitly taken into account. Some theoretical aspects of this optimal control approach are investigated and numerical examples of boundary controls in channels are presented in order to show the performance of this approach on conductive flows.*

1 INTRODUCTION

In this paper we propose an optimal control method that, by using the boundary condition of the magnetic field $\mathbf{B}$, can achieve control on the velocity field $\mathbf{u}$ and solve the steady MHD equations

$$
\begin{aligned}
& -\frac{1}{Re}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - S_1(\nabla\times\mathbf{B})\times\mathbf{B} - \mathbf{f} = \mathbf{0} \\
& \nabla\cdot\mathbf{u} = 0 \\
& \frac{1}{Re_m}\nabla\times(\nabla\times\mathbf{B}) - \nabla\times(\mathbf{u}\times\mathbf{B}) + \nabla\sigma = \mathbf{0} \\
& \nabla\cdot\mathbf{B} = 0\,.
\end{aligned} \tag{1}
$$

The MHD system is completed with appropriate boundary conditions over velocity and magnetic fields. The coefficient S_1 is $H_m^2/Re\,Re_m$ where $Re = UL/\nu$, $Re_m = \mu_0\sigma UL$ and $H_m = BL\sqrt{\sigma/\mu}$ are the viscous Reynolds, the magnetic Reynolds and the Hartmann number respectively. The quantities $U, B, L, \nu, \mu_0, \sigma$ indicate the reference values for velocity, magnetic field, length, kinematic viscosity, magnetic permeability and electrical conductivity of the fluid.

The interest in these equations arises from a lot of applications in science and engineering, such as fusion technology, fission nuclear reactors with liquid metal coolant and submarine propulsion devices.[3,12] Numerous formulations of the MHD system have been proposed and analyzed in literature, based on different physical assumptions on the MHD model.[7,13–16] This has lead to the adoption of different sets of state variables for the description of the electromagnetic phenomena, which consist of a combination of quantities such as magnetic field, current density, electric field and electric potential.[7] It is well-known that, while Navier-Stokes equations are valid over the region occupied by the fluid, Maxwell equations extend to three-dimensional space;[14] therefore, the adoption of specific boundary and interface conditions for a certain physical model affects its mathematical weak formulation along with the choice of the functional spaces associated with the problem.

Different approaches have been studied in literature for the optimal control of MHD equations.[7,9,11] In this paper we present a new boundary control approach where the optimal boundary control problem is transformed into an extended distributed one. Standard boundary control approach is not very straightforward to implement numerically in comparison to distributed control and often leads to unnecessary smooth controls.[10,14] We choose to split the magnetic field into two pieces: a lifting function $\mathbf{B}_e$, which matches the boundary conditions, and an auxiliary field $\mathbf{b}$ with homogeneous conditions such that $\mathbf{B} = \mathbf{B}_e + \mathbf{b}$. The magnetic field $\mathbf{B}_e$ can be considered as an extension of the function from the boundary to the interior of the domain. The optimal solution is then searched by exploring all possible extended functions. This approach gives robustness to the boundary control algorithm which can be solved by standard distributed control techniques over the inner part of the domain.

We consider the following optimal control functional

$$\mathcal{J}(\mathbf{u},\mathbf{B}_e) = \int_\Omega \left(\frac{\alpha}{2}\|\mathbf{u}-\mathbf{u}_d\|^2 + \frac{\beta}{2}\|\mathbf{B}_e\|^2 + \frac{\gamma}{2}\|\nabla \mathbf{B}_e\|^2 \right) d\mathbf{x}\,, \tag{2}$$

where α, β and γ are positive constants and $\mathbf{B}_e$ is the extended function or boundary lifting function introduced above. The objective is to track the velocity to a desired field $\mathbf{u}_d$ and to limit the boundary control by limiting the $\mathbf{H}^1$-norm of the extended function $\mathbf{B}_e$. In Section 2 we state the variational formulation of the optimal control problem. The finite element approximation is introduced in Section 3 and finally we present the results of some numerical computations in Section 4.

2 THE OPTIMAL CONTROL PROBLEM

In order to formulate the optimal control problem in the discrete finite element form we introduce some functional spaces. Let $\Omega \subset \mathbb{R}^3$ be a bounded connected open Lipschitz domain. We denote by $\mathbf{H}^m(\Omega)$ the usual vector-valued Sobolev spaces endowed with the standard norm $\|\cdot\|_m$ and by $\mathbf{H}^0(\Omega)$ the space of square-integrable functions $(\mathbf{L}^2(\Omega))$ with norm $\|\cdot\| = \|\cdot\|_0$. Let $\mathbf{H}_0^m(\Omega)$ denote the closure of $\mathbf{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_m$ and $\mathbf{H}^{-m}(\Omega)$ denote the dual space of $\mathbf{H}_0^m(\Omega)$. The dual space of $\mathbf{H}^1(\Omega)$ is denoted by $\mathbf{H}^1(\Omega)^*$. The spaces $\mathbf{V}(\Omega)$ and $L_0^2(\Omega)$ are defined as[1,6]

$$\begin{aligned} \mathbf{V}(\Omega) &= \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \nabla\cdot\mathbf{u} = 0\} \\ L_0^2(\Omega) &= \{p \in L^2(\Omega) : \int_\Omega p\, d\mathbf{x} = 0\}\,. \end{aligned}$$

The trace operator, which restricts the function to its boundary, is denoted by γ_0, i.e. $\gamma_0\mathbf{f} := \mathbf{f}|_\Gamma$. The trace space of $\mathbf{H}^1(\Omega)$ is denoted by $\mathbf{H}^{1/2}(\Gamma)$ and its dual by $\mathbf{H}^{-1/2}(\Gamma)$.

We introduce the continuous bilinear forms

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\mathbf{x} \qquad \forall\, \mathbf{u},\, \mathbf{v} \in \mathbf{H}^1(\Omega)\,, \tag{3}$$

$$a_m(\mathbf{u},\mathbf{v}) = \int_\Omega (\nabla\times\mathbf{u})\cdot(\nabla\times\mathbf{v})\, d\mathbf{x} \quad \forall\, \mathbf{u},\, \mathbf{v} \in \mathbf{H}^1(\Omega)\,, \tag{4}$$

$$d(\mathbf{v},q) = -\int_\Omega q\nabla\cdot\mathbf{v}\, d\mathbf{x} \qquad \forall\, q \in L_0^2(\Omega), \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega) \tag{5}$$

and the continuous trilinear forms

$$c(\mathbf{u};\mathbf{v},\mathbf{w}) = \frac{1}{2}\int_\Omega \mathbf{w}\cdot\Big(\mathbf{u}\cdot\nabla\Big)\mathbf{v}\, d\mathbf{x} - \frac{1}{2}\int_\Omega \mathbf{w}\cdot\Big(\mathbf{v}\cdot\nabla\Big)\mathbf{u}\, d\mathbf{x} \qquad \forall\, \mathbf{u},\, \mathbf{v},\, \mathbf{w} \in \mathbf{H}^1(\Omega)\,, \tag{6}$$

$$c_m(\mathbf{u};\mathbf{v},\mathbf{w}) = \int_\Omega \Big(\nabla\times\mathbf{u}\Big)\cdot\Big(\mathbf{v}\times\mathbf{w}\Big) d\mathbf{x} \qquad \forall\, \mathbf{v},\, \mathbf{u},\, \mathbf{w} \in \mathbf{H}^1(\Omega)\,. \tag{7}$$

The form c is the usual anti-symmetrized form for the non-linear term of the Navier-Stokes equation.[16,17] Since all the arguments of the trilinear form $c(\mathbf{u};\mathbf{v},\mathbf{w})$ belong to $\mathbf{H}^1(\Omega)$, integration by parts yields a boundary integral which does not vanish and therefore must be included in the boundary stress term of the equation.

In this framework $(\mathbf{u}, p, \boldsymbol{\tau}, \mathbf{B}, \sigma, \boldsymbol{\tau}_m) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ is called a *weak solution for the MHD equations* if it satisfies the weak form of the steady MHD system (1) given by

$$\begin{aligned}
&\frac{1}{Re} a(\mathbf{u}, \mathbf{v}_1) + c(\mathbf{u}; \mathbf{u}, \mathbf{v}_1) - S_1\, c_m(\mathbf{B}; \mathbf{B}, \mathbf{v}_1) + \\
&\qquad\qquad d(\mathbf{v}_1, p) + <\boldsymbol{\tau}, \mathbf{v}_1>_\Gamma = <\mathbf{f}, \mathbf{v}_1> \qquad \forall\, \mathbf{v}_1 \in \mathbf{H}^1(\Omega) \\
&d(\mathbf{u}, q_1) = 0 \quad \forall\, q_1 \in L^2_0(\Omega) \\
&\frac{1}{Re_m} a_m(\mathbf{B}, \mathbf{v}_2) - c_m(\mathbf{v}_2; \mathbf{u}, \mathbf{B}) + d(\mathbf{v}_2, \sigma) + <\boldsymbol{\tau}_m, \mathbf{v}_2>_\Gamma = 0 \quad \forall\, \mathbf{v}_2 \in \mathbf{H}^1(\Omega) \\
&d(\mathbf{B}, q_2) = 0 \quad \forall\, q_2 \in L^2_0(\Omega)
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\boldsymbol{\tau} &= -\frac{1}{Re}\frac{\partial \mathbf{u}}{\partial n} + p\mathbf{n} + \frac{1}{2}\mathbf{u}\mathbf{u}\cdot\mathbf{n} \\
\boldsymbol{\tau}_m &= -\mathbf{E}\times\mathbf{n} + \sigma\mathbf{n} \\
\mathbf{E} &= \frac{1}{Re_m}(\nabla\times\mathbf{B}) - (\mathbf{u}\times\mathbf{B})\,,
\end{aligned} \tag{9}$$

with appropriate boundary conditions. It is important to remark that $\boldsymbol{\tau}$ contains the term $\mathbf{u}\mathbf{u}\cdot\mathbf{n}/2$ which is zero only if the normal velocity vanishes. Clearly, if $\mathbf{u}$ is a solution of (1), then it is also a solution of the weak formulation (8), while the viceversa may or may not take place.

The boundary conditions for the velocity field can be defined by $\mathbf{g}$ and $\mathbf{t}$ in different combinations as

$$\begin{aligned}
&\mathbf{u} = \mathbf{g} \quad \text{or} \\
&\mathbf{u}\cdot\mathbf{n} = \mathbf{g}\cdot\mathbf{n} \qquad \boldsymbol{\tau}\times\mathbf{n} = \mathbf{t}\times\mathbf{n} \quad \text{or} \\
&\mathbf{u}\times\mathbf{n} = \mathbf{g}\times\mathbf{n} \qquad \boldsymbol{\tau}\cdot\mathbf{n} = \mathbf{t}\cdot\mathbf{n}\,.
\end{aligned} \tag{10}$$

In a similar way the boundary conditions for the magnetic field could be defined by $\mathbf{q}$ and $\mathbf{k}$ as

$$\begin{aligned}
&\mathbf{B} = \mathbf{q} \quad \text{or} \\
&\mathbf{B}\cdot\mathbf{n} = \mathbf{q}\cdot\mathbf{n} \qquad \boldsymbol{\tau}_m\times\mathbf{n} = \mathbf{k}\times\mathbf{n} \quad \text{or} \\
&\mathbf{B}\times\mathbf{n} = \mathbf{q}\times\mathbf{n} \qquad \boldsymbol{\tau}_m\cdot\mathbf{n} = \mathbf{k}\cdot\mathbf{n}\,.
\end{aligned} \tag{11}$$

The boundary functions $\mathbf{g}$ and $\mathbf{q}$ define the Dirichlet boundary conditions. The natural boundary conditions are defined by $\mathbf{k}$ and $\mathbf{t}$. If Dirichlet boundary conditions are specified by $\mathbf{g}$ and $\mathbf{q}$ in (8) then $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_m$ are the corresponding unknown Lagrangian multipliers. On the other hand, if $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_m$ are given, the boundary fields $\mathbf{u}$ and $\mathbf{B}$ are unknown and must be computed.[5,6,10] The existence of solutions in (8) with boundary conditions taken from (10-11) is still an open issue and the hypothesis of small data for $\mathbf{g}$, $\mathbf{t}$, $\mathbf{q}$ and $\mathbf{k}$ is required.[11,13,16]

In order to transform the boundary control into a distributed control we extend the function which defines the boundary conditions $\mathbf{q}$ inside the domain and consider the corresponding lifting functions. This brings several theoretical and numerical advantages. In fact in this case boundary controls can be sought in their natural space $\mathbf{H}^{1/2}(\Gamma)$, while standard boundary control approaches require to search for boundary controls in smoother spaces such as $\mathbf{H}^1(\Gamma)$.[10] The numerical implementation of distributed controls is more straightforward than a standard boundary control approach, where compatibility conditions on the control variables have to be satisfied. With the use of a lifting function, one can find the distributed control $\mathbf{B}_e$ as a solution of the optimality system and then obtain the boundary control $\mathbf{q}$ by using the trace operator

$$\begin{aligned} &\mathbf{B}\cdot\mathbf{n} = \mathbf{q}\cdot\mathbf{n} = \gamma_0\mathbf{B}_e\cdot\mathbf{n} \quad \text{over} \quad \Gamma \\ &\mathbf{B}\times\mathbf{n} = \mathbf{q}\times\mathbf{n} = \gamma_0\mathbf{B}_e\times\mathbf{n} \quad \text{over} \quad \Gamma. \end{aligned} \tag{12}$$

Thus, one can decompose the magnetic field $\mathbf{B}$ as

$$\mathbf{B} = \mathbf{b} + \mathbf{B}_e\,, \tag{13}$$

where $\mathbf{B}_e$ is the lifting function that retains the information of the boundary conditions, so that $\mathbf{b}$ assumes homogeneous values on the boundary. Moreover, the lifting function $\mathbf{B}_e$ is such that

$$\nabla\cdot\mathbf{B}_e = 0 \tag{14}$$

and thus, the compatibility condition for $\mathbf{q}$

$$\int_\Omega \nabla\cdot\mathbf{B}d\Omega = \int_\Omega \nabla\cdot\mathbf{B}_e d\Omega = \int_\Gamma \gamma_0\mathbf{B}_e\cdot\mathbf{n}d\Gamma = \int_\Gamma \mathbf{q}\cdot\mathbf{n}d\Gamma = 0 \tag{15}$$

is automatically satisfied.

With the decomposition of $\mathbf{B}$ we can reformulate the weak form of the MHD equations as follows. Given the lifting function $\mathbf{B}_e$ we seek $(\mathbf{u}, p, \boldsymbol{\tau}, \mathbf{b}, \sigma, \boldsymbol{\tau}_m) \in \mathbf{H}^1(\Omega)\times L^2_0(\Omega)\times \mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^1(\Omega)\times L^2_0(\Omega)\times\mathbf{H}^{-1/2}(\Gamma)$ such that

$$\begin{aligned} &\frac{1}{Re}a(\mathbf{u},\mathbf{v}_1) + c(\mathbf{u};\mathbf{u},\mathbf{v}_1) - S_1\, c_m(\mathbf{b}+\mathbf{B}_e;\mathbf{b}+\mathbf{B}_e,\mathbf{v}_1) + \\ &\qquad\qquad d(\mathbf{v}_1,p) + <\boldsymbol{\tau},\mathbf{v}_1>_\Gamma = <\mathbf{f},\mathbf{v}_1> \qquad \forall\,\mathbf{v}_1\in\mathbf{H}^1(\Omega) \\ &d(\mathbf{u},q_1) = 0 \quad \forall\,q_1\in L^2_0(\Omega) \\ &\frac{1}{Re_m}a_m(\mathbf{b}+\mathbf{B}_e,\mathbf{v}_2) - c_m(\mathbf{v}_2;\mathbf{u},\mathbf{b}+\mathbf{B}_e) + d(\mathbf{v}_2,\sigma) + <\boldsymbol{\tau}_m,\mathbf{v}_2>_\Gamma = 0 \quad \forall\,\mathbf{v}_2\in\mathbf{H}^1(\Omega) \\ &d(\mathbf{b}+\mathbf{B}_e,q_2) = 0 \quad \forall\,q_2\in L^2_0(\Omega) \end{aligned} \tag{16}$$

with appropriate boundary conditions for $\mathbf{b}$. For any choice of the lifting function $\mathbf{B}_e$ satisfying the divergence-free constraint (14) one recovers a different solution $(\mathbf{u}, p, \boldsymbol{\tau}, \mathbf{b}, \sigma, \boldsymbol{\tau}_m)$ of the system (16). Different lifting functions $\mathbf{B}_e$ lead to different solutions $\mathbf{b}$, but the solution $\mathbf{B} = \mathbf{b} + \mathbf{B}_e$ of the original system (8) only depends on the values assumed by $\mathbf{B}_e$ at the boundary.

In order to define the optimal control problem let $\Gamma_s \subset \Gamma$ be the portion of the boundary over which we set Dirichlet boundary conditions on both the velocity and the magnetic field. The functions $\mathbf{g}$ and $\mathbf{q}$ are fixed over Γ_s and the corresponding Lagrangian multipliers $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_m$ are unknown. The tangential component of the velocity field $\mathbf{u}$, the normal component of the multipliers $\boldsymbol{\tau}$, $\boldsymbol{\tau}_m$ and the controlled tangential magnetic field $\mathbf{B}_e$ are enforced over the rest of the boundary $\Gamma_c = \Gamma \backslash \Gamma_s$. Different sets of boundary conditions and controls can be taken into account in a similar way. With these notations the optimal problem is to find the control $\mathbf{B}_e \in \mathbf{V}(\Omega)$ such that the functional (2) attains a local minimum and the solution $(\mathbf{u}, p, \boldsymbol{\tau}, \mathbf{b}, \sigma, \boldsymbol{\tau}_m) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ satisfies the system (16) together with the prescribed boundary conditions.

In order to obtain the first order necessary conditions and the optimality system for the optimal control problem we introduce the operator $M : \mathbf{B}_1 \to \mathbf{B}_2$ which is defined between the spaces $\mathbf{B}_1 = \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^1(\Omega)$ and $\mathbf{B}_2 = \mathbf{H}^{1*}(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{1*}(\Omega) \times L^2_0(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma_s) \times \mathbf{H}^{1/2}(\Gamma_s) \times \mathbf{H}^{1/2}(\Gamma_s) \times \mathbf{H}^{1/2}(\Gamma_c) \times \mathbf{H}^{1/2}(\Gamma_c)$ so that $M(\mathbf{u}, p, \boldsymbol{\tau}, \mathbf{b}, \sigma, \boldsymbol{\tau}_m, \mathbf{B}_e) = (\mathbf{f}_1, q_1, \mathbf{f}_2, q_2, q_3, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_5)$, where

$$
\begin{aligned}
& < \mathbf{f}_1, \mathbf{v}_1 > := \frac{1}{Re}\, a(\mathbf{u}, \mathbf{v}_1) + c(\mathbf{u}; \mathbf{u}, \mathbf{v}_1) + d(\mathbf{v}_1, p) \qquad\qquad \forall\, \mathbf{v}_1 \in \mathbf{H}^1(\Omega) \\
& \qquad - S_1 c_m(\mathbf{b} + \mathbf{B}_e; \mathbf{b} + \mathbf{B}_e, \mathbf{v}_1) + < \boldsymbol{\tau}, \mathbf{v}_1 >_\Gamma - < \mathbf{f}, \mathbf{v}_1 > \\
& (q_1, z_1) := d(\mathbf{u}, z_1) \qquad\qquad \forall\, z_1 \in L^2_0(\Omega) \\
& < \mathbf{f}_2, \mathbf{v}_2 > := \frac{1}{Re_m} a(\mathbf{b} + \mathbf{B}_e, \mathbf{v}_2) - c_m(\mathbf{v}_2; \mathbf{u}, \mathbf{b} + \mathbf{B}_e) + \\
& \qquad d(\mathbf{v}_2, \sigma) + < \boldsymbol{\tau}_m, \mathbf{v}_2 >_\Gamma \qquad\qquad \forall\, \mathbf{v}_2 \in \mathbf{H}^1(\Omega) \\
& (q_2, z_2) := d(\mathbf{b}, z_2) \qquad\qquad \forall\, z_2 \in L^2_0(\Omega) \\
& (q_3, z_3) := d(\mathbf{B}_e, z_3) \qquad\qquad \forall\, z_3 \in L^2_0(\Omega) \\
& (\mathbf{r}_1, \mathbf{s}_1) := < \mathbf{u} - \mathbf{g}, \mathbf{s}_1 >_{\Gamma_s} \qquad\qquad \forall\, \mathbf{s}_1 \in \mathbf{H}^{-1/2}(\Gamma_s) \\
& (\mathbf{r}_2, \mathbf{s}_2) := < \mathbf{B}_e - \mathbf{q}, \mathbf{s}_2 >_{\Gamma_s} \qquad\qquad \forall\, \mathbf{s}_2 \in \mathbf{H}^{-1/2}(\Gamma_s) \\
& (\mathbf{r}_3, \mathbf{s}_3) := < \mathbf{b}, \mathbf{s}_3 >_{\Gamma_s} \qquad\qquad \forall\, \mathbf{s}_3 \in \mathbf{H}^{-1/2}(\Gamma_s) \\
& (\mathbf{r}_4, \mathbf{s}_4) := < \mathbf{u} \times \mathbf{n}, \mathbf{s}_4 \times \mathbf{n} >_{\Gamma_c} \qquad\qquad \forall\, \mathbf{s}_4 \in \mathbf{H}^{-1/2}(\Gamma_c) \\
& (\mathbf{r}_5, \mathbf{s}_5) := < \mathbf{b} \times \mathbf{n}, \mathbf{s}_5 \times \mathbf{n} >_{\Gamma_c} \qquad\qquad \forall\, \mathbf{s}_5 \in \mathbf{H}^{-1/2}(\Gamma_c)\,.
\end{aligned}
\tag{17}
$$

The operator (17) is completed by the appropriate values of $\boldsymbol{\tau} \cdot \mathbf{n}$ and $\boldsymbol{\tau}_m \cdot \mathbf{n}$ over Γ_c. Through the usual method of Lagrangian multipliers, we turn the constrained minimization problem into an unconstrained one. The new problem is then to find $(\mathbf{u}, p, \boldsymbol{\tau}, \mathbf{b}, \sigma, \boldsymbol{\tau}_m,$

$\mathbf{B}_e, \boldsymbol{\lambda}, \pi_1, \boldsymbol{\xi}, \pi_2, \pi_3, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \boldsymbol{\chi}_3, \boldsymbol{\chi}_4, \boldsymbol{\chi}_5)$ such that the augmented functional

$$\begin{aligned}
&\mathcal{J}_{aug}(\mathbf{u}, p, \boldsymbol{\tau}, \mathbf{b}, \sigma, \boldsymbol{\tau}_m, \mathbf{B}_e, \boldsymbol{\lambda}, \pi_1, \boldsymbol{\xi}, \pi_2, \pi_3, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \boldsymbol{\chi}_3, \boldsymbol{\chi}_4, \boldsymbol{\chi}_5) = \\
&\qquad \mathcal{J}(\mathbf{u}, \mathbf{B}_e) + < M(\mathbf{u}, p, \boldsymbol{\tau}, \mathbf{b}, \sigma, \boldsymbol{\tau}_m, \mathbf{B}_e), (\boldsymbol{\lambda}, \pi_1, \boldsymbol{\xi}, \pi_2, \pi_3, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \boldsymbol{\chi}_3, \boldsymbol{\chi}_4, \boldsymbol{\chi}_5) >
\end{aligned} \tag{18}$$

is minimized in the set of admissible states, costates and controls.

Now, we can formulate the optimal control problem as a non-trivial solution to a system of differential equations. It suffices to compute the Fréchet differentials of the augmented functional $\mathcal{J}_{aug}$ with respect to the state, adjoint and control variables. Clearly, the variations $(\delta\boldsymbol{\lambda}, \delta\pi_1, \delta\boldsymbol{\xi}, \delta\pi_2, \delta\pi_3, \delta\boldsymbol{\chi}_1, \delta\boldsymbol{\chi}_2, \delta\boldsymbol{\chi}_3, \delta\boldsymbol{\chi}_4, \delta\boldsymbol{\chi}_5)$ with respect to the Lagrangian multipliers yield the state equations (8) and the constraints defined by $M(\mathbf{u}, p, \boldsymbol{\tau}, \mathbf{b}, \sigma, \boldsymbol{\tau}_m, \mathbf{B}_e) = \mathbf{0}$. For the other variables we can proceed in a standard way and obtain the following corresponding Euler equations.[7,8,10,11,13] The variables $(\mathbf{B}_e, \pi_3, \boldsymbol{\chi}_2)$ satisfy the system

$$\begin{aligned}
&\beta(\mathbf{B}_e, \delta\mathbf{B}_e) + \gamma a(\mathbf{B}_e, \delta\mathbf{B}_e) - S_1\, c_m(\delta\mathbf{B}_e, \mathbf{b} + \mathbf{B}_e, \boldsymbol{\lambda}) - S_1\, c_m(\mathbf{b} + \mathbf{B}_e; \delta\mathbf{B}_e, \boldsymbol{\lambda}) + \\
&\qquad \frac{1}{Re_m} a_m(\delta\mathbf{B}_e, \boldsymbol{\xi}) - c_m(\boldsymbol{\xi}; \mathbf{u}, \delta\mathbf{B}_e) + d(\delta\mathbf{B}_e, \pi_3) + < \boldsymbol{\chi}_2, \delta\mathbf{B}_e >= 0 \quad \forall \delta\mathbf{B}_e \in \mathbf{H}^1(\Omega) \\
&d(\mathbf{B}_e, \delta\pi_3) = 0 \quad \forall \delta\pi_3 \in L^2_0(\Omega) \\
&< \mathbf{B}_e, \delta\boldsymbol{\chi}_2 >_{\Gamma_s} = < \mathbf{q}, \delta\boldsymbol{\chi}_2 >_{\Gamma_s} \qquad \forall\, \delta\boldsymbol{\chi}_2 \in \mathbf{H}^{-1/2}(\Gamma_s)\,.
\end{aligned} \tag{19}$$

The variables $(\boldsymbol{\lambda}, \pi_1, \boldsymbol{\chi}_1, \boldsymbol{\chi}_4)$ in $\mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma_s) \times \mathbf{H}^{1/2}(\Gamma_c)$ are defined by

$$\begin{aligned}
&\frac{1}{Re} a(\delta\mathbf{u}, \boldsymbol{\lambda}) + c(\delta\mathbf{u}; \mathbf{u}, \boldsymbol{\lambda}) + c(\mathbf{u}, \delta\mathbf{u}, \boldsymbol{\lambda}) + d(\delta\mathbf{u}, \pi_1) - c_m(\boldsymbol{\xi}; \delta\mathbf{u}, \mathbf{b} + \mathbf{B}_e) + \\
&\qquad < \chi_1, \delta\mathbf{u} >_{\Gamma_s} + < \chi_4 \times \mathbf{n}, \delta\mathbf{u} \times \mathbf{n} >_{\Gamma_c} + \alpha(\mathbf{u} - \mathbf{u}_d, \delta\mathbf{u}) = 0 \quad \forall \delta\mathbf{u} \in \mathbf{H}^1(\Omega) \\
&d(\boldsymbol{\lambda}, \delta p) = 0 \qquad \forall \delta p \in L^2_0(\Omega) \\
&< \mathbf{u}, \delta\boldsymbol{\chi}_1 >_{\Gamma_s} = < \mathbf{g}, \delta\boldsymbol{\chi}_1 >_{\Gamma_s} \qquad \forall\, \delta\boldsymbol{\chi}_1 \in \mathbf{H}^{-1/2}(\Gamma_s) \\
&< \mathbf{u} \times \mathbf{n}, \delta\boldsymbol{\chi}_4 \times \mathbf{n} >_{\Gamma_c} = 0 \qquad \forall\, \delta\boldsymbol{\chi}_4 \in \mathbf{H}^{-1/2}(\Gamma_c)
\end{aligned} \tag{20}$$

and $(\boldsymbol{\xi}, \pi_2, \boldsymbol{\chi}_3, \boldsymbol{\chi}_5)$ in $\mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma_s) \times \mathbf{H}^{1/2}(\Gamma_c)$ by

$$\begin{aligned}
&\frac{1}{Re_m} a_m(\delta\mathbf{b}, \boldsymbol{\xi}) - c_m(\boldsymbol{\xi}; \mathbf{u}, \delta\mathbf{b}) - S_1\, c_m(\delta\mathbf{b}; \mathbf{b} + \mathbf{B}_e, \boldsymbol{\lambda}) - S_1\, c_m(\mathbf{b} + \mathbf{B}_e, \delta\mathbf{b}, \boldsymbol{\lambda}) + \\
&\qquad d(\delta\mathbf{b}, \pi_2) + < \boldsymbol{\chi}_3, \delta\mathbf{b} >_{\Gamma_s} + < \boldsymbol{\chi}_5 \times \mathbf{n}, \delta\mathbf{b} \times \mathbf{n} >_{\Gamma_c} = 0 \quad \forall \delta\mathbf{b} \in \mathbf{H}^1(\Omega) \\
&d(\boldsymbol{\xi}, \delta\sigma) = 0 \qquad \forall \delta\sigma \in L^2_0(\Omega) \\
&< \mathbf{b}, \delta\boldsymbol{\chi}_3 >_{\Gamma_s} = 0 \qquad \forall\, \delta\boldsymbol{\chi}_3 \in \mathbf{H}^{-1/2}(\Gamma_s) \\
&< \mathbf{b} \times \mathbf{n}, \delta\boldsymbol{\chi}_5 \times \mathbf{n} >_{\Gamma_c} = 0 \qquad \forall\, \delta\boldsymbol{\chi}_5 \in \mathbf{H}^{-1/2}(\Gamma_c)\,.
\end{aligned} \tag{21}$$

The optimality system is a very complex system and the numerical solution is a difficult and expensive task.

3 FINITE ELEMENT APPROXIMATION

In this section we approximate the problem by using the finite element method. We consider only conforming finite element approximations. Let $\mathbf{X}_h \subset \mathbf{H}^1(\Omega)$ and $S_h \subset L^2(\Omega)$ be two families of finite dimensional subspaces parameterized by h that tends to zero. We denote $S_{h0} = S_h \cap L^2_0(\Omega)$ and make the following assumptions on $\mathbf{X}_h$ and S_h :
i) the approximation hypotheses: there exist an integer l and a constant C, independent of h, $\mathbf{u}$, and p, such that for $1 \leq k \leq l$ we have

$$\inf_{\mathbf{u}_h \in \mathbf{X}_h} \|\mathbf{u}_h - \mathbf{u}\|_1 \leq C h^k \|\mathbf{u}\|_{k+1} \quad \forall\, \mathbf{u} \in \mathbf{H}^{k+1}(\Omega) \cap \mathbf{H}^1(\Omega) \tag{22}$$

$$\inf_{p_h \in S_h} \|p - p_h\| \leq C h^k \|p\|_k \qquad \forall\, p \in \mathbf{H}^k(\Omega) \cap L^2_0(\Omega)\,. \tag{23}$$

ii) the inf-sup condition or LBB condition: there exists a constant C' , independent of h, such that[1,6,17]

$$\inf_{0 \neq q_h \in S_{h0}} \sup_{0 \neq \mathbf{u}_h \in \mathbf{X}_h} \frac{\int_\Omega q_h \,\nabla \cdot \mathbf{u}_h \, d\mathbf{x}}{\|\mathbf{u}_h\|_1 \, \|q_h\|} \geq C' > 0\,. \tag{24}$$

Next, let $\mathbf{P}_h = \mathbf{X}_h|_\Gamma$, i.e. $\mathbf{P}_h$ consists of the restriction, to the boundary Γ, of functions $\mathbf{u} \in \mathbf{X}_h$. For all choices of conforming finite element space $\mathbf{X}_h$ we then have that $\mathbf{P}_h \subset \mathbf{H}^{-1/2}(\Gamma)$. For the subspaces $\mathbf{P}_h = \mathbf{X}_h|_\Gamma$, we assume the *boundary approximation property*: there exists an integer l and a constant C, independent of $h, \mathbf{s}$ such that for $1 \leq k \leq l$ we have[2]

$$\inf_{\mathbf{s}_h \in \mathbf{P}_h} \|\mathbf{s}_h - \mathbf{s}\,\|_{-1/2,\Gamma} \leq C h^k \|\mathbf{u}\|_{k-1/2} \quad \forall \mathbf{s} \in H^{k-1/2}(\Gamma)\,. \tag{25}$$

In order to solve the optimal control problem we must solve the optimality system in the variables $(\mathbf{u}_h, p_h, \mathbf{b}_h, \sigma_h, \boldsymbol{\lambda}_h, \pi_{1h}, \boldsymbol{\xi}_h, \pi_{2h}, \mathbf{B}_{eh}, \pi_{3h})$. We can divide the discrete optimality system into three parts: the MHD system, the adjoint system and the control equation. The discrete MHD system (8) for the state variables $(\mathbf{u}_h, p_h, \boldsymbol{\tau}_h, \mathbf{b}_h, \sigma_h, \boldsymbol{\tau}_{mh})$ can be written in operator form as

$$\begin{cases} \dfrac{1}{Re} a(\mathbf{u}_h, \mathbf{v}_{1h}) + c(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_{1h}) - S_1\, c_m(\mathbf{b}_h + \mathbf{B}_{eh}; \mathbf{b}_h + \mathbf{B}_{eh}, \mathbf{v}_{1h}) + \\ \qquad d(\mathbf{v}_{1h}, p_h) + < \boldsymbol{\tau}_h, \mathbf{v}_{1h} >_\Gamma = < \mathbf{f}_h, \mathbf{v}_{1h} > \quad \forall \mathbf{v}_{1h} \in \mathbf{X}_h(\Omega_h) \\ d(\mathbf{u}_h, q_{1h}) = 0 \quad \forall q_{1h} \in S_{h0}(\Omega_h) \\ \dfrac{1}{Re_m} a_m(\mathbf{b}_h + \mathbf{B}_{eh}, \mathbf{v}_{2h}) - c_m(\mathbf{v}_{2h}; \mathbf{u}_h, \mathbf{b}_h + \mathbf{B}_{eh}) + \\ \qquad d(\mathbf{v}_{2h}, \sigma_h) + < \boldsymbol{\tau}_{mh}, \mathbf{v}_{2h} >_\Gamma = 0 \quad \forall \mathbf{v}_{2h} \in \mathbf{X}_h(\Omega_h) \\ d(\mathbf{b}_h + \mathbf{B}_{eh}, q_{2h}) = 0 \quad \forall q_{2h} \in S_{h0}(\Omega_h) \\ < \mathbf{u}_h - \mathbf{g}, \mathbf{s}_{1h} >_{\Gamma_{sh}} = 0 \qquad \forall\, \mathbf{s}_{1h} \in \mathbf{P}_h(\Gamma_{sh}) \\ < \mathbf{b}_h, \mathbf{s}_{3h} >_{\Gamma_{sh}} = 0 \qquad \forall\, \mathbf{s}_{3h} \in \mathbf{P}_h(\Gamma_{sh}) \\ < \mathbf{u}_h \times \mathbf{n}, \mathbf{s}_{4h} \times \mathbf{n} >_{\Gamma_{ch}} = 0 \qquad \forall\, \mathbf{s}_{4h} \in \mathbf{P}_h(\Gamma_{ch}) \\ < \mathbf{b}_h \times \mathbf{n}, \mathbf{s}_{5h} \times \mathbf{n} >_{\Gamma_{ch}} = 0 \qquad \forall\, \mathbf{s}_{5h} \in \mathbf{P}_h(\Gamma_{ch})\,. \end{cases} \tag{26}$$

The adjoint system, in $(\boldsymbol{\lambda}_h, \pi_{1h}, \boldsymbol{\xi}, \pi_{2h}, \boldsymbol{\chi}_{1h}, \boldsymbol{\chi}_{4h})$, can be written as

$$\begin{cases} \frac{1}{Re} a(\delta \mathbf{u}_h, \boldsymbol{\lambda}_h) + c(\delta \mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\lambda}_h) + c(\mathbf{u}_h, \delta \mathbf{u}_h, \boldsymbol{\lambda}_h) + d(\delta \mathbf{u}_h, \pi_{1h}) - c_m(\boldsymbol{\xi}_h; \delta \mathbf{u}_h, \mathbf{b}_h + \mathbf{B}_{eh}) + \\ \quad < \chi_{1h}, \delta \mathbf{u}_h >_{\Gamma_{sh}} + < \chi_{4h} \times \mathbf{n}, \delta \mathbf{u}_h \times \mathbf{n} >_{\Gamma_{ch}} + \alpha(\mathbf{u}_h - \mathbf{u}_d, \delta \mathbf{u}_h) = 0 \quad \forall \delta \mathbf{u}_h \in \mathbf{X}_h(\Omega_h) \\ d(\boldsymbol{\lambda}_h, \delta p_h) = 0 \qquad \forall \delta p_h \in S_{h0}(\Omega_h) \\ < \mathbf{u}_h, \delta \boldsymbol{\chi}_{1h} >_{\Gamma_{sh}} = < \mathbf{g}, \delta \boldsymbol{\chi}_{1h} >_{\Gamma_{sh}} \qquad \forall\, \delta \boldsymbol{\chi}_{1h} \in \mathbf{P}_h(\Gamma_{sh}) \\ < \mathbf{u}_h \times \mathbf{n}, \delta \boldsymbol{\chi}_{4h} \times \mathbf{n} >_{\Gamma_{ch}} = 0 \qquad \forall\, \delta \boldsymbol{\chi}_{4h} \in \mathbf{P}_h(\Gamma_{ch}) \\ \frac{1}{Re_m} a_m(\delta \mathbf{b}_h, \boldsymbol{\xi}_h) - c_m(\boldsymbol{\xi}_h; \mathbf{u}_h, \delta \mathbf{b}_h) + < \boldsymbol{\chi}_{3h}, \delta \mathbf{b}_h >_{\Gamma_{sh}} + < \boldsymbol{\chi}_{5h} \times \mathbf{n}, \delta \mathbf{b}_h \times \mathbf{n} >_{\Gamma_{ch}} + \\ \quad d(\delta \mathbf{b}_h, \pi_{2h}) = S_1\, c_m(\delta \mathbf{b}_h; \mathbf{b}_h + \mathbf{B}_{eh}, \boldsymbol{\lambda}_h) + S_1\, c_m(\mathbf{b}_h + \mathbf{B}_{eh}, \delta \mathbf{b}_h, \boldsymbol{\lambda}_h) \quad \forall \delta \mathbf{b}_h \in \mathbf{X}_h(\Omega_h) \\ d(\boldsymbol{\xi}_h, \delta \sigma_h) = 0 \qquad \forall \delta \sigma_h \in S_{h0}(\Omega_h) \\ < \mathbf{b}_h, \delta \boldsymbol{\chi}_{3h} >_{\Gamma_{sh}} = 0 \qquad \forall\, \delta \boldsymbol{\chi}_{3h} \in \mathbf{P}_h(\Gamma_{sh}) \\ < \mathbf{b}_h \times \mathbf{n}, \delta \boldsymbol{\chi}_{5h} \times \mathbf{n} >_{\Gamma_{ch}} = 0 \qquad \forall\, \delta \boldsymbol{\chi}_{5h} \in \mathbf{P}_h(\Gamma_{ch}) \end{cases} \tag{27}$$

and the control equations, for the variables $(\mathbf{B}_{eh}, \pi_{3h}, \boldsymbol{\chi}_{2h})$, take the form

$$\begin{cases} \beta(\mathbf{B}_{eh}, \delta \mathbf{B}_{eh}) + \gamma a(\mathbf{B}_{eh}, \delta \mathbf{B}_{eh}) + \frac{1}{Re_m} a_m(\delta \mathbf{B}_{eh}, \boldsymbol{\xi}_h) - c_m(\boldsymbol{\xi}_h; \mathbf{u}_h, \delta \mathbf{B}_{eh}) - \\ \quad S_1\, c_m(\delta \mathbf{B}_{eh}, \mathbf{b}_h + \mathbf{B}_{eh}, \boldsymbol{\lambda}_h) - S_1\, c_m(\mathbf{b}_h + \mathbf{B}_{eh}; \delta \mathbf{B}_{eh}, \boldsymbol{\lambda}_h) + \\ \qquad d(\delta \mathbf{B}_{eh}, \pi_{3h}) + < \boldsymbol{\chi}_{2h}, \delta \mathbf{B}_{eh} > = 0 \qquad \forall \delta \mathbf{B}_{eh} \in \mathbf{X}_h(\Omega_h) \\ d(\mathbf{B}_{eh}, \delta \pi_{3h}) = 0 \quad \forall \delta \pi_{3h} \in S_{h0}(\Omega_h) \\ < \mathbf{B}_{eh}, \delta \boldsymbol{\chi}_{2h} >_{\Gamma_{sh}} = < \mathbf{q}, \delta \boldsymbol{\chi}_{2h} >_{\Gamma_{sh}} \qquad \forall\, \delta \boldsymbol{\chi}_{2h} \in \mathbf{P}_h(\Gamma_{sh})\,. \end{cases} \tag{28}$$

The optimal boundary control $\mathbf{q}_h$ for the magnetic field is then extracted directly as

$$\mathbf{q} = \gamma_0\, (\mathbf{b}_h + \mathbf{B}_{eh}) \tag{29}$$

where γ_0 is the trace operator.

4 NUMERICAL RESULTS

4.1 Hartmann case and lifting function test

In order to test the boundary control and the lifting method we would like to reproduce well known MHD physical situations. We refer to the Hartmann flow as a basic configuration. Let us consider a two-dimensional channel with square domain $[0, 1] \times [0, 1]$ as in Figure 4.1. Let Γ_c be the bottom and top sides of the square which are assumed as inlet and outlet of the channel. At the wall Γ_s we specify no slip boundary conditions $\mathbf{u}_h = \mathbf{g}_h = \mathbf{0}$ for the velocity field and $\mathbf{B}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} = B_{0x}$, $\mathbf{B}_h \times \mathbf{n} = 0$ for the magnetic field. On Γ_c we assume $\mathbf{u}_h \times \mathbf{n} = u_h = 0$ and uniform pressure p on each side with pressure gradient $P = p_1 - p_0 > 0$ reproducing an infinite channel flow. Since u_h is uniformly zero

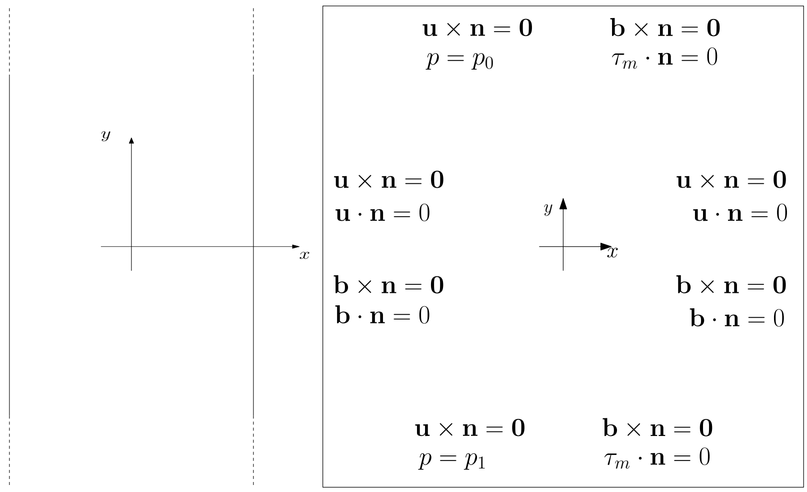

Figure 1: Domain and boundary conditions

then $\partial u_h/\partial x = 0$ and from the divergence-free constraint one has $\partial v_h/\partial y = 0$. In a similar way we set $\mathbf{B}_h \times \mathbf{n} = 0$ and $\sigma_h = 0$ uniformly on the bottom and top of the domain which implies again $\partial B_{xh}/\partial x = 0$ and $\partial B_{yh}/\partial y = 0$.

Since we use lifting functions it is important that the boundary control is not affected by the particular function used. The total magnetic field $\mathbf{B}_h$ can be split as the sum of two contributions $\mathbf{B}_h = \mathbf{b}_h + \mathbf{B}_{eh}$ and one can write

$$\begin{aligned}
&\frac{1}{Re_m}\nabla \times \Big(\nabla \times \mathbf{b}_h\Big) - \nabla \times (\mathbf{u}_h \times \mathbf{b}_h) + \nabla \sigma_h = \\
&\qquad \nabla \times (\mathbf{u}_h \times \mathbf{B}_{eh}) - \frac{1}{Re_m}\nabla \times \Big(\nabla \times \mathbf{B}_{eh}\Big) \\
&\nabla \cdot \mathbf{b}_h = -\nabla \cdot \mathbf{B}_{eh} = 0 \\
&\mathbf{B}_{eh} \times \mathbf{n} = \mathbf{b}_h \times \mathbf{n} = 0 \quad \text{on } \Gamma_s \\
&\mathbf{B}_{eh} \cdot \mathbf{n} = B_{0x} \qquad \mathbf{b}_h \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_s \\
&\mathbf{B}_{eh} \times \mathbf{n} = \mathbf{b}_h \times \mathbf{n} = 0 \quad \text{on } \Gamma_c \\
&\boldsymbol{\tau}_{mh} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_c\,.
\end{aligned} \tag{30}$$

For any given $\mathbf{B}_{eh}$ and any velocity field $\mathbf{u}_h$ we can solve for (30) and obtain $\mathbf{b}_h = (b_{xh}, b_{yh})$. If we set $\mathbf{B}_{eh} = (B_{x0}, 0)$ as a lifting function it is possible to find the analytical solution as $v = v(x)$. This assumption implies that the pressure gradient P along the y direction is constant. We denote the non-dimensional pressure drop $\frac{PL}{\rho U^2}$ with P^* and the Hartmann number with $H_m = B_{x0}L\sqrt{\sigma/\mu}$. The Hartmann flow is the solution to the following MHD

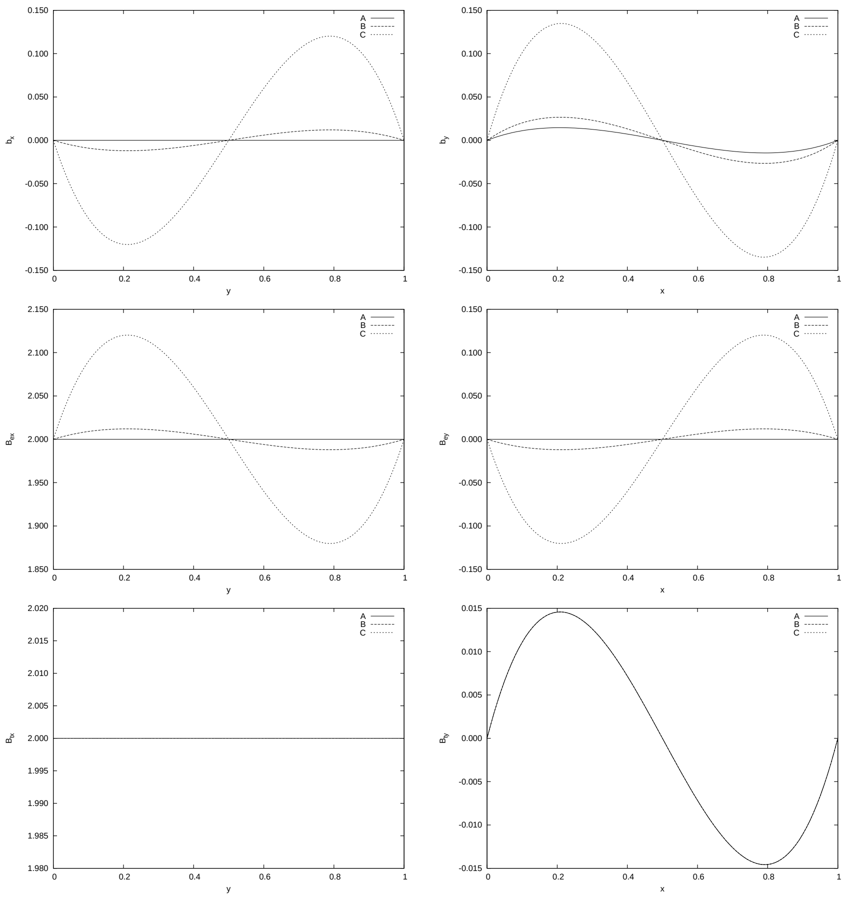

Figure 2: The magnetic fields b_{xh}, B_{exh}, B_{xh} along the line $x = 0.5$ (left) and the magnetic fields b_{xh}, B_{exh}, B_{yh} along the line $y = 0.5$ (right) for $K = 0$(A), 1 (B), 10 (C) in the divergence-free case.

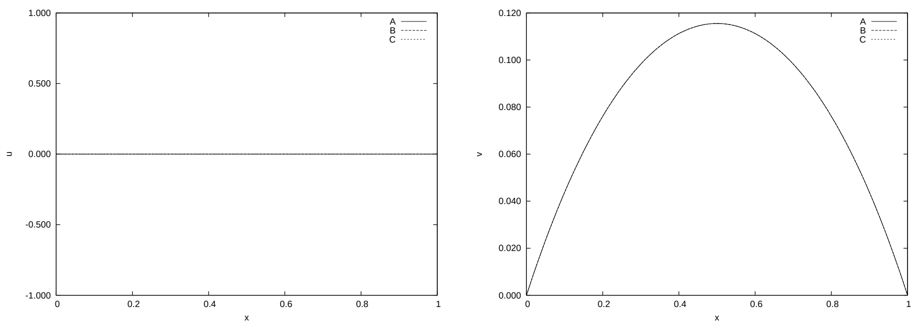

Figure 3: The velocity components u_h (left) and v_h (right) along the line $y = 0.5$ for $K = 0$(A), 1 (B), 10 (C) in the divergence-free case.

equations in strong form[4]

$$\begin{aligned} &\frac{d^2 v}{dx^2} - H_m^2 v = -P^* Re H_m \coth H_m \\ &\frac{db_y}{dz} = Re_m (P^* \frac{Re}{H_m^2}(H_m \coth H_m - 1) - v) \end{aligned} \tag{31}$$

with boundary conditions as stated above. The non-dimensional solutions for $v(x)$ and $b_y(x)$ are

$$\begin{aligned} v(x) &= P^* \frac{Re}{H_m} \frac{\cosh(H_m) - \cosh(H_m x)}{\sinh(H_m)} \\ b_y(x) &= P^* Re_m \frac{Re}{H_m^2} \frac{\sinh(H_m x) - x \sinh(H_m)}{\sinh(H_m)} \ . \end{aligned} \tag{32}$$

The choice of the lifting function $\mathbf{B}_{eh} = (B_{x0}, 0)$ results in a special case where $\mathbf{B}_{eh}$ and $\mathbf{b}_{eh}$ are orthogonal. It is important to see that for any choice of $\mathbf{B}_{eh}$ satisfying the same boundary conditions the MHD equation yields the same solution $(\mathbf{u}_h, p_h, \mathbf{B}_h, \sigma_h)$ provided that the lifting function is divergence-free. With the purpose of clarifying this condition we study two different cases: a divergence-free and a non divergence-free lifting function.

Let us first consider divergence-free lifting functions. Since we have a two-dimensional domain, a function in $\mathbf{V}(\Omega)$ can be derived from a potential $\phi(x, y)$. For instance, let us consider

$$\phi(x, y) = K[x(x-1)]^2[y(y-1)]^2 \tag{33}$$

and choose the lifting function as

$$\begin{aligned} B_{exh} &= B_{x0} + \frac{\partial \phi}{\partial y} = B_{x0} + K2y(y-1)(2y-1)[x(x-1)]^2 \\ B_{eyh} &= -\frac{\partial \phi}{\partial x} = -K2x(x-1)(2x-1)[y(y-1)]^2 \quad . \end{aligned} \tag{34}$$

In Figure 2 the magnetic field B_{exh}, b_{xh} and B_{xh} along the line $x = 0.5$ and the magnetic field B_{eyh}, b_{yh} and B_{yh} along the line $y = 0.5$ for $K = 0$ (A), 1 (B), 10 (C) are shown on the left and right column, respectively. It can be seen that for all the cases (A), (B) and (C) the sum $\mathbf{B}_{eh} + \mathbf{b}_h$ is the same. Also the velocity components u and v remain the same for the three cases as shown in Figure 3.

Now we consider a non divergence-free lifting function. We plot the results for

$$B_{exh} = B_{x0} + Kx(1-x)y(1-y) \qquad B_{eyh} = 0 \tag{35}$$

with $B_{x0} = 2$ and where K takes the values $K = \{0, 1, 10\}$. In Figure 4 the x and y components of the magnetic field $\mathbf{B}_h$ and its contributions $\mathbf{b}_h$ and $\mathbf{B}_{eh}$ are shown along the line $y = 0.5$ for $K = 0$ (A), 1 (B) and 10 (C). We see that the components of the magnetic field $\mathbf{B}_h$ depend strongly on the lifting function $\mathbf{B}_{eh}$. Since the divergence of the lifting $\mathbf{B}_{eh}$ is not zero, the total resulting magnetic field does not match. Therefore, the velocity component v_h, the pressure p_h and σ_h change, as shown in Figure 5.

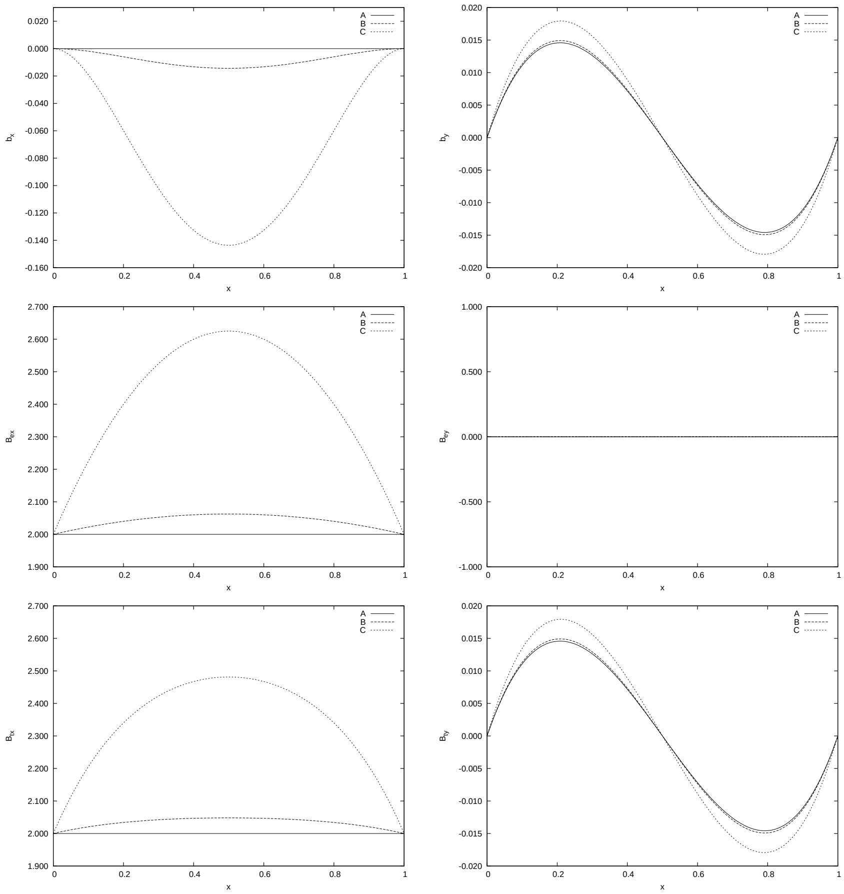

Figure 4: The x-components b_{xh}, B_{exh}, B_{xh} (left column) and the y-components b_{yh}, B_{eyh}, B_{yh} (right column) along the line $y = 0.5$ for $K = 0$ (A), 1 (B) and 10 (C) in the non divergence-free case.

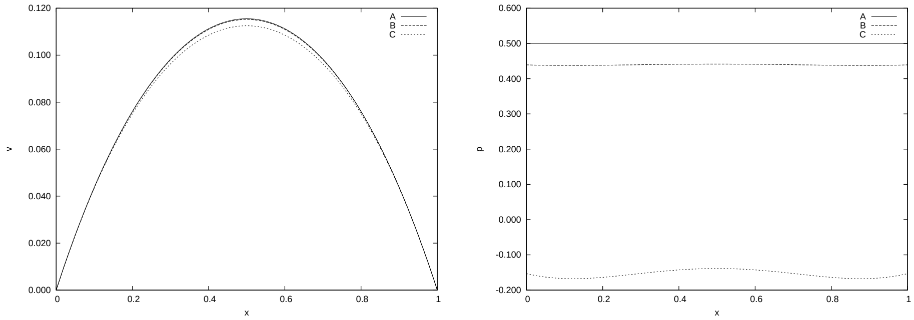

Figure 5: The y-component v_h of the velocity field and the pressure p_h along the line $y = 0.5$ for $K = 0$ (A), 1 (B) and 10 (C) in the non divergence-free case.

$(\bar{B}_{ex}, \bar{B}_{ey})$	α	$\int_{\Omega_1} \|\mathbf{u} - \mathbf{u}_d\|^2$	v_{max}	$(\bar{B}_{ex}, \bar{B}_{ey})$	α	$\int_{\Omega_1} \|\mathbf{u} - \mathbf{u}_d\|^2$	v_{max}
(6,0)	0	1.013e-5	0.0754	(2,0)	0	5.302e-4	0.115
(6,0)	1	9.768e-6	0.0755	(2,0)	10	4.209e-4	0.112
(6,0)	10	7.982e-6	0.0763	(2,0)	100	1.126e-4	0.0957
(6,0)	100	5.972e-6	0.0779	(2,0)	500	2.182e-5	0.0849
(6,0)	200	5.702e-6	0.0782	(2,0)	1000	1.502e-5	0.0828
(6,0)	500	5.236e-6	0.0784	(2,0)	1500	1.307e-5	0.0820
(6,0)	750	4.968e-6	0.0785	(2,0)	5000	1.043e-5	0.0808
(6,0)	1000	4.764e-6	0.0786	(2,0)	6000	1.005e-5	0.0809

Table 1: L^2-norm of the error $(\mathbf{u}_h - \mathbf{u}_d)$ and maximum velocity $v_{h,max}$ for various values of α and $(\bar{B}_{exh}, \bar{B}_{eyh})$

4.2 Optimal boundary control test

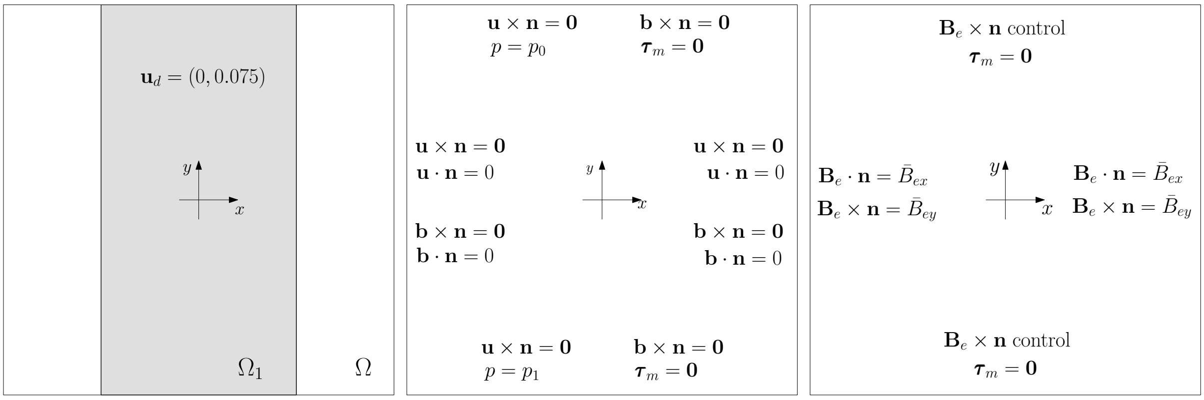

Figure 6: Target domain Ω_1 and boundary conditions for state and control variables

We consider an example of a numerical solution of the optimality system over the geometry of the previous section. The target is a desired constant velocity $\mathbf{u}_d$ on a central strip $\Omega_1 = \{(x, y) | x \in [0.25, 0.75], y \in [0, 1]\}$. We set $\beta = 0$ and $\gamma = 1$ so that the functional to be minimized becomes

$$\mathcal{J}(\mathbf{u}_h, \mathbf{B}_{eh}) = \frac{\alpha}{2} \int_{\Omega_1} \|\mathbf{u}_h - \mathbf{u}_d\|^2 d\mathbf{x} + \frac{1}{2} \int_{\Omega} \|\nabla \mathbf{B}_{eh}\|^2 d\mathbf{x} \,. \tag{36}$$

The control is the x-component of the magnetic field over Γ_c which consists of the top and bottom sides. The magnetic field $\mathbf{B}_{eh}$ is fixed on the left and right sides $x = 0$ and $x = 1$ (Γ_s) to a constant value $(\bar{B}_{exh}, \bar{B}_{eyh})$. In Figure 6 we show the subregion Ω_1 for the desired velocity and summarize the boundary conditions associated with the state $(\mathbf{u}_h, p_h, \mathbf{b}_h, \sigma_h)$ and the control $\mathbf{B}_{eh}$. In Figures 7 and 8 we plot the profiles of the variables of the optimality system along the lines $y = 0.5$ and $x = 0.5$ for $(\bar{B}_{exh}, \bar{B}_{eyh}) = (2, 0)$ and $\alpha = 6000$. In Table 4.2 we report the values of the L^2-norm error $\int_{\Omega_1} \|\mathbf{u}_h - \mathbf{u}_d\|^2 d\mathbf{x}$ together

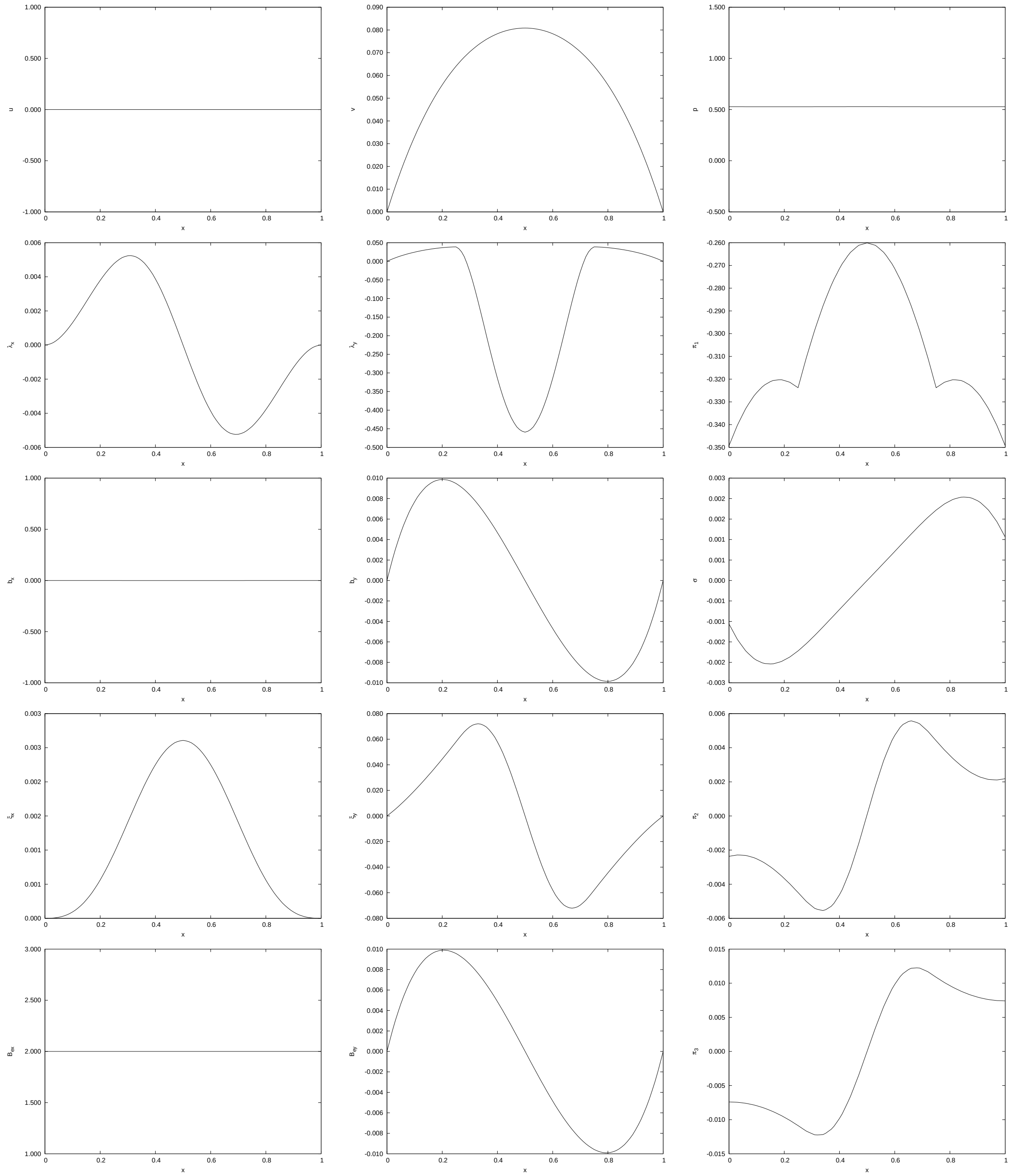

Figure 7: State, costate and control variables along the line $y = 0.5$ for the case $\bar{\mathbf{B}}_{eh} = (2, 0)$ and $\alpha = 6000$

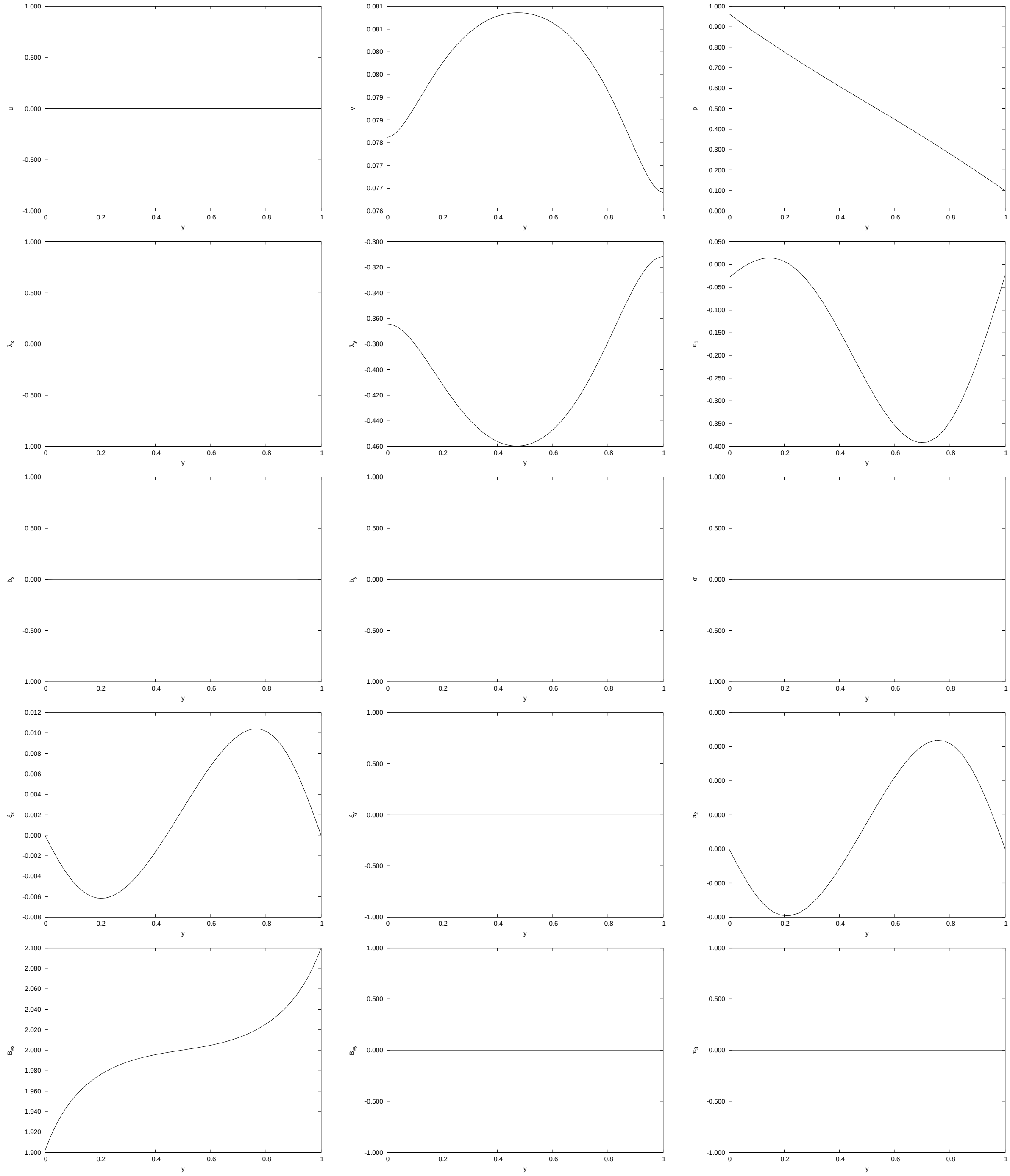

Figure 8: State, costate and control variables along the line $x = 0.5$ for the case $\bar{\mathbf{B}}_{eh} = (2, 0)$ and $\alpha = 6000$

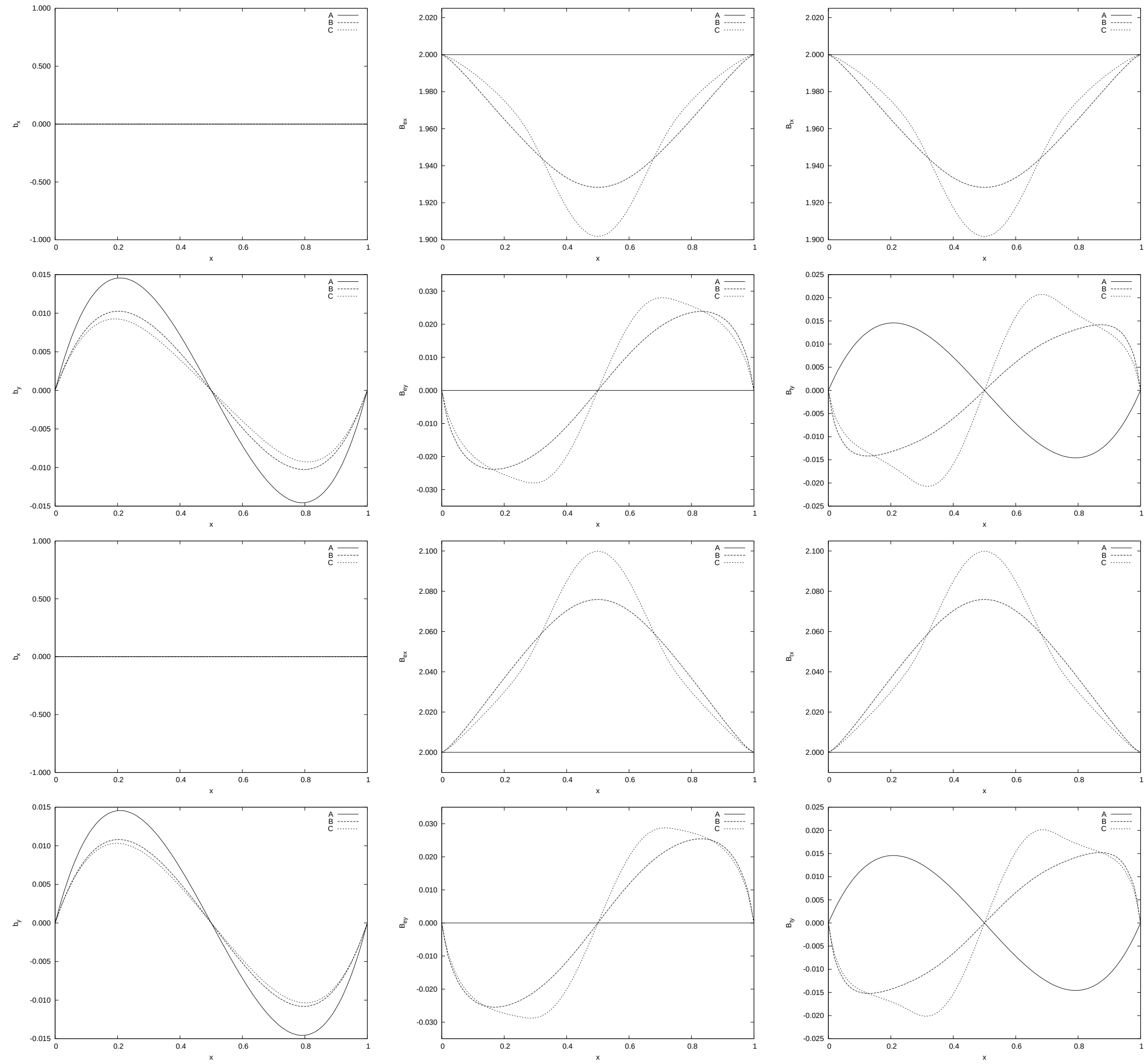

Figure 9: The components of the fields $\mathbf{b}_h$, $\mathbf{B}_{eh}$ and $\mathbf{B}_h$ on $y = 0$ (first and second row) and $y = 1$ (third and forth row)

with the maximum velocity $v_{h,max}$ for different values of α and magnetic field $(\bar{B}_{exh}, \bar{B}_{eyh})$. We remark that for increasing α the solution of the optimality system yields a more and more accurate result, in the sense that the L^2-norm of the distance between the computed and the desired velocity decreases. For the case $(\bar{B}_{exh}, \bar{B}_{eyh}) = (2, 0)$ the optimization results in decreasing the velocity $v_{h,max}$ with increasing α, while $v_{h,max}$ increases with increasing α for $(\bar{B}_{exh}, \bar{B}_{eyh}) = (6, 0)$. Even if the distance $|v_{h,max} - v_d|$ decreases in the first case and increases in the second case, in both circumstances this is a consequence of the minimization of the L^2-norm of the error toward the desired velocity profile. In order to study the influence of the parameter α on the tracking problem, we show in Figure 9 the

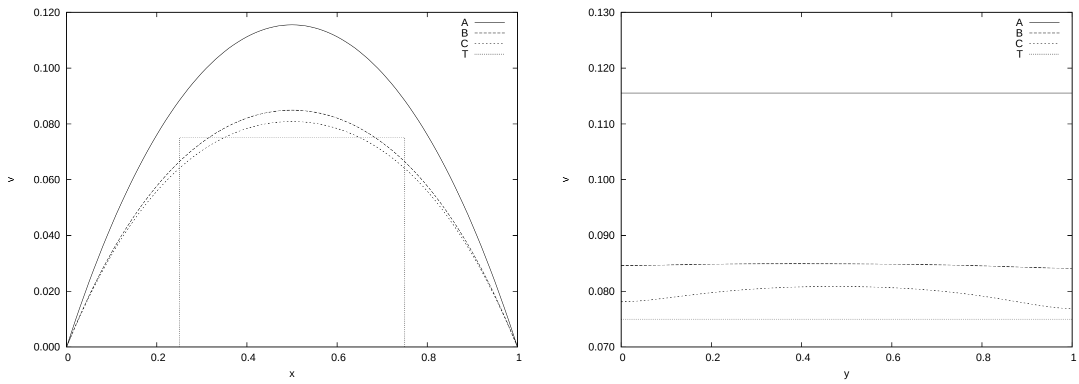

Figure 10: Velocity component v_h for $\alpha = 0$ (A), 500 (B) and 6000 (C) and target velocity v_d (T) along the lines $y = 0.5$ (left) and $x = 0.5$ (right)

components of the fields $\mathbf{b}_h$, $\mathbf{B}_{eh}$ and $\mathbf{B}_h$ at the boundaries $y = 0$ and $y = 1$ for the case $\mathbf{B}_{eh} = (2, 0)$ with $\alpha = \{0, 500, 6000\}$. We also plot in Figure 10 the velocity component v_h for the case $\mathbf{B}_{eh} = (2, 0)$ with $\alpha = \{0, 500, 6000\}$ and compare the computed velocity with the target, along the lines $y = 0.5$ and $x = 0.5$.

5 CONCLUSIONS

In this paper we have introduced a new approach for the boundary optimal control problem of stationary MHD equations. The boundary control problem is transformed into an extended distributed one by considering boundary controls as restrictions of lifting functions. This approach leads to a simple formulation that takes into account all possible Dirichlet and Neumann boundary conditions with a single lifting function. Numerical results using finite element approximation have been discussed.

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