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HYBRIDIZATION OF NUMERICAL SCHEMES IN TIME DOMAIN TO SOLVE THE VLASOV-MAXWELL EQUATIONS

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Abstract. *In this paper, we present a hybrid method to solve the 2D Maxwell-Vlasov system. The idea is to use a domain computational decomposition method with buffer zone's presence [1]-[2]. The solution of the Maxwell equations on the global domain is obtained by the sum of the solutions on each subdomain. These equations are solved on the global domain by introducing an artificial connecting function. Contrary to these equations, the Vlasov equation is solved on the global domain to take into account the solution of the Maxwell equations on each subdomain. Some numerical examples have been added to validate the method.*

1 INTRODUCTION

For the modelling of high power microwave (HPM) sources, we are developping efficient methods for the Vlasov-Maxwell equations. Concerning the Maxwell equations, to take account as precisely as possible for curved geometrical structures, it is interesting to couple different numerical schemes (Finite-Difference Time-Domain (FDTD), Finite-Volume Time-Domain (FVTD) and Discontinuous Galerkin Time-Domain (DGTD) methods). In this context, several solutions have been proposed by using conforming or non-conforming strategies. The conforming strategies correspond to hybrid methods where the meshes used in the numerical schemes match perfectly at the nodes located at the boundaries of the different subdomains. Unlike these methods, the non-conforming strategies do not ensure this matching. In fact, the meshes of the different subdomains are totally independent. Therefore, these non-conforming hybrid methods present some advantages like the meshing. However, the implementation of the evolution scheme is more difficult. Nevertheless, it remains an interesting option to obtain stable processes.

Section 1 describes the Vlasov-Maxwell system with a Particle-In-Cell (PIC) method [3]. This problem is solved in a 2D configuration. We also propose a hybridization strategy based on the partition of unity method. In Section 2, we introduce this hybrid method in the Maxwell equations and a numerical example is given to test the method's capacities. Then, in Section 3, we apply this same hybridization strategy in the Vlasov equation. Section 4 presents a physical model (Larmor radius) about plasma simulation by including this hybrid method of the Maxwell-Vlasov system. Finally, a conclusion is given in Section 5.

2 DEFINITION OF 2D PROBLEM

2.1 Maxwell-Vlasov system

We consider a 2D domain Ω. We split it into two subdomains Ω_1 and Ω_2 with an overlapping zone, named buffer zone. We define meshes independent one of the other. We define two cartesian grids $(x_{1,i}, y_{1,i})_{i=1,\cdots,N_1+1}$ and $(x_{2,i}, y_{2,i})_{i=1,\cdots,N_2+1}$ such as $x_{1,i} = i\Delta x_1$ and $y_{1,i} = i\Delta y_1, i \in [0, N_1+1]$ on Ω_1 and $x_{2,i} = i\Delta x_2$ and $y_{2,i} = i\Delta y_2, i \in [0, N_2+1]$ on Ω_2. N_1 and N_2 represent respectively the number of interior points on Ω_1 and on Ω_2. We suppose that the spatial discretization steps Δx_1 and Δx_2 are constant on Ω_1 and on Ω_2. We focus on a 2D particular case with the same discretization on the y-axis such as $(y_{1,i})_{i=1,\cdots,N_1+1} = (y_{2,j})_{j=1,\cdots,N_2+1}$, the discretization step Δy being constant on the two subdomains.

On this global domain, we solve the Maxwell-Valsov system. The kinetic model of collisionless and weak density plasma is described by the evolution of the distribution function for each particles species s. In the no-relativistic case, this function satisfies the Vlasov equation :

$$\frac{\partial \mathbf{f}_s}{\partial t} + \mathbf{V}.\frac{\partial \mathbf{f}_s}{\partial \mathbf{X}} + \frac{q}{m}(\mathbf{E} + \mathbf{V} \times \mathbf{B}).\frac{\partial \mathbf{f}_s}{\partial \mathbf{V}} = 0 \tag{1}$$

where $\mathbf{f}_s = \mathbf{f}_s(\mathbf{V}, \mathbf{X}, t) = \mathbf{f}_s(v_x, v_y, x, y, t)$ is the distribution function for each particles species s with a charge q and a mass m. It corresponds to the statistical average of the particle distribution in phase-space. $\mathbf{B}$ and $\mathbf{E}$ represent the magnetic induction and the electric field acting on particles. This function is coupled with the TE Maxwell equations evaluating the electromagnetic fields $(\mathbf{E}, \mathbf{H})$ and given, on a computational domain Ω, by :

$$\begin{aligned} &\mu_0 \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0 \\ &\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} + \mathbf{J} = 0 \\ &\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \end{aligned} \tag{2}$$

with $\mathbf{H} = H_z$ $(= \mu_0^{-1}\mathbf{B}$) and $\mathbf{E} = (E_x, E_y)$. μ_0 and ε_0 are the magnetic permeability and the electric permittivity of the medium. $\mathbf{J} = (J_x, J_y)$ and ρ are respectively current and charge densities generated by the particles motion. On the boundary $\partial\Omega$ of the domain, we impose Silver-Muller boundary conditions which are absorbing boundary conditions. Moreover, we put a metallic surface and we send an electromagnetic plane wave which propagates in all the domain. Then, we evaluate the electromagnetic field diffracted by the metallic surface.

These current and charge densities verify the following charge conservation law :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{3}$$

and they are defined by :

$$\begin{aligned} \rho(\mathbf{X}, t) &= \sum_s q_s \int_{\mathbb{R}^2} f_s(\mathbf{X}, \mathbf{V}, t) d\mathbf{V} \\ \mathbf{J}_s(\mathbf{X}, t) &= \sum_s q_s \int_{\mathbb{R}^2} \mathbf{V} f_s(\mathbf{X}, \mathbf{V}, t) d\mathbf{V} \end{aligned} \tag{4}$$

2.2 Particle-In-Cell method

The distribution function $\mathbf{f}_s$ is conserved along the particle trajectories which are determined by their motion (positions and velocities). Positions (x, y) and velocities (v_x, v_y) of particles are solutions of the characteristic equations (motion equations) defined by :

$$\begin{aligned} \frac{d\mathbf{X}}{dt} &= \mathbf{V} \\ \frac{d\mathbf{V}}{dt} &= \frac{q}{m}(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \end{aligned} \tag{5}$$

In the equations (5) , we observe the interaction between the electromagnetic fields and the particles. The fields are solutions of the Maxwell equations (2) which are evaluated by using a Finite-Difference Time-Domain (FDTD) numerical scheme. Then, the fields are defined on the mesh of physical space while the particles have a position in this same mesh. It is necessary to do interpolations between the positions of the particles and the fields (mesh) in order to evaluate the coupling terms. This method is named *Particle-In-Cell* method.

2.3 Principles of hybrid method

Let Ω be a computational domain where we evaluate U as solution of the problem $\frac{\partial U}{\partial t} = \mathcal{A}U$. We split the domain Ω into two subdomains Ω_1 and Ω_2 with a buffer zone $\Omega_{12} = \Omega_1 \cap \Omega_2 \neq \emptyset$ and we define meshes independent one of the other. We introduce on Ω an artificial transition function χ independent of time such that :

$$\begin{aligned} &\chi = 1 \text{ on } \Omega_1 - \Omega_{12} \\ &\chi = 0 \text{ on } \Omega_2 - \Omega_{12} \\ &\chi \in [0,1] \text{ on } \Omega_{12} \end{aligned} \tag{6}$$

We define $(U_1) = \chi U$ and $U_2 = (1-\chi)U$. We obtain $U_1 + U_2 = U$ and we observe that U_1 and U_2 are solutions of the initial problem with second member. These solutions can be restricted on the subdomains Ω_1 and Ω_2. By introducing χ in the initial problem, we obtain the following coupling system :

$$\begin{aligned} \frac{\partial U_1}{\partial t} &= \chi \mathcal{A} U_1 + \chi \mathcal{A} U_2 \text{ on } \Omega_1 \\ \frac{\partial U_2}{\partial t} &= (1-\chi)\, \mathcal{A} U_2 + (1-\chi)\, \mathcal{A} U_1 \text{ on } \Omega_2 \end{aligned} \tag{7}$$

We have two equations on Ω_1 and Ω_2 coupled by "source" terms which are solutions of equations (7) which can be restricted respectively on Ω_2 and Ω_1. Moreover, we denote that the boundary condition of the solution on each subdomain at the buffer zone can be $\mathbf{n} \times \mathbf{E} = 0$.

On these two hybrid zones, we apply a numerical scheme to solve the equations. The main difficulty of this coupling system (7) is the interpolation of solutions from a mesh to the other.

3 HYBRIDIZATION OF THE MAXWELL EQUATIONS

We apply this hybrid method on the studied 2D TE Maxwell problem (2) without taking into account the current density $\mathbf{J}$. We obtain on Ω_1 :

$$\begin{aligned} \varepsilon_0 \frac{\partial E_{x_1}}{\partial t} &= \chi \frac{\partial H_{z_1}}{\partial y_1} + \chi \frac{\partial H_{z_2}}{\partial y_1} \\ \varepsilon_0 \frac{\partial E_{y_1}}{\partial t} &= -\chi \frac{\partial H_{z_1}}{\partial x_1} - \chi \frac{\partial H_{z_2}}{\partial x_1} \end{aligned} \tag{8}$$

$$\mu_0 \frac{\partial H_{z_1}}{\partial t} = \chi \left(\frac{\partial E_{x_1}}{\partial y_1} - \frac{\partial E_{y_1}}{\partial x_1} \right) + \chi \left(\frac{\partial E_{x_2}}{\partial y_1} - \frac{\partial E_{y_2}}{\partial x_1} \right)$$

And on Ω_2 :

$$\begin{aligned}
\varepsilon_0 \frac{\partial E_{x_2}}{\partial t} &= (1-\chi) \frac{\partial H_{z_2}}{\partial y_2} + (1-\chi) \frac{\partial H_{z_1}}{\partial y_2} \\
\varepsilon_0 \frac{\partial E_{y_2}}{\partial t} &= -(1-\chi) \frac{\partial H_{z_2}}{\partial x_2} - (1-\chi) \frac{\partial H_{z_1}}{\partial x_2} \\
\mu_0 \frac{\partial H_{z_2}}{\partial t} &= (1-\chi) \left(\frac{\partial E_{x_2}}{\partial y_2} - \frac{\partial E_{y_2}}{\partial x_2} \right) + (1-\chi) \left(\frac{\partial E_{x_1}}{\partial y_2} - \frac{\partial E_{y_1}}{\partial x_2} \right)
\end{aligned} \tag{9}$$

On each subdomain, we use a FDTD numerical method [4] to solve the modified 2D TE Maxwell equations. To write the FDTD formalism, we use the well known Yee scheme where the electric fields are evaluated at the time $n\Delta t$ and the magnetic fields at the time $(n+\frac{1}{2})\Delta t$, with Δt the time step and n the current iteration. For the space discretization, we evaluate the electric fields at the grid points of cells and the magnetic fields at the center of cells.

In this 2D configuration, similar to a 1D configuration, we have studied different forms of the function χ in the overlapping zone [1]. Numerical results have shown that the polynomial function of odd degree was the most interesting and that a high polynomial degree improved solutions. So, here, we consider χ as the polynomial function of odd degree n :

$$\begin{aligned}
&\chi(x) = 1 \ \ \forall y, \ \ \text{on } \Omega_1 - \Omega_1 \cap \Omega_2 \\
&\chi(x) = a_0\, x^n + a_1\, x^{n-1} x + \cdots + a_n \ \ \forall y, \ \ \text{on } \Omega_1 \cap \Omega_2 \\
&\chi(x) = 0 \ \ \forall y, \ \ \text{on } \Omega_2 - \Omega_1 \cap \Omega_2
\end{aligned} \tag{10}$$

where n is a non-negative integer and a_0, a_1, a_2, $\cdots$, a_n are constant coefficients.

This choice allows us to ensure the continuity of the function and its $n/2$-order derivatives at the boundaries of the buffer zone.

To quantify the advantages of the hybrid method, we consider two subdomains $\Omega_1 = [0, 1.08] \times [0, 1]$ and $\Omega_2 = [0.72, 1.8] \times [0, 1]$. In the case of an incidente plane wave, we compute the electric and magnetic fields at two test-points, one of them located at $x = 0.5m$, $y = 0.5m$ in Ω_1 and the other at $x = 1.5m$, $y = 0.5m$ in the subdomain Ω_2.

This incidente plane wave is given by a Gaussian pulse :

$$E_{inc} = exp\left(- \left(\frac{t - x/3e8 - t_0}{\sigma} \right)^2 \right) \tag{11}$$

with $t_0 = 1.e-08s$ and $\sigma = 1.e-09s$.

The figures represent the comparison between the analytic solution and the numerical solutions with and without hybridization in Ω_1 (figure 1) and Ω_2 (figure 2). We observe a good behaviour of our results and an improvement of the solution with hybridization.

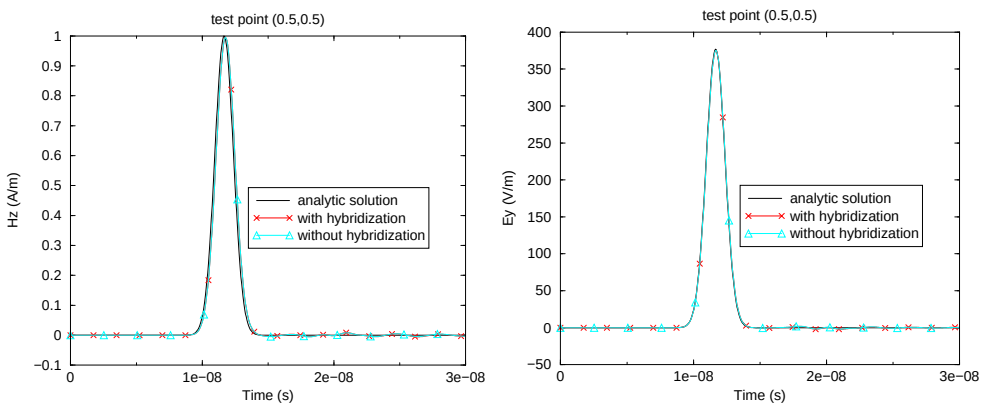

Figure 1: Evaluated electromagnetic fields Hz (left) Ey (right) with and without hybridization at the test point $(0.5, 0.5)$

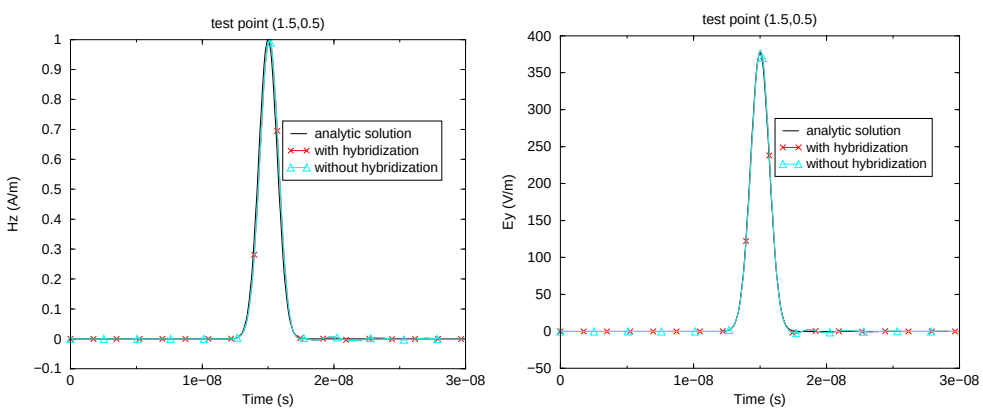

Figure 2: Evaluated electromagnetic fields Hz (left) Ey (right) with and without hybridization at the test point $(1.5, 0.5)$.

4 HYBRIDIZATION OF THE VLASOV EQUATION - BORIS CORRECTION

In this section, we consider the motion equations (5) describing the particles trajectory in plasma. By using the hybrid method, this particles trajectory is given by positions and velocities as follows :

$$\begin{aligned}
\frac{dx}{dt} &= v_x \\
\frac{dy}{dt} &= v_y \\
\frac{dv_x}{dt} &= \frac{q}{m}\left[(E_{x_1} + E_{x_2}) + v_y.\,(B_{z_1} + B_{z_2})\right] \\
\frac{dv_y}{dt} &= \frac{q}{m}\left[(E_{y_1} + E_{y_2}) - v_x.\,(B_{z_1} + B_{z_2})\right]
\end{aligned} \tag{12}$$

with E_{x_1}, E_{y_1} and B_{z_1} (respectively E_{x_2}, E_{y_2} and B_{z_2}) are the fields on Ω_1 (respectively Ω_2). We use a Leap-Frog scheme to solve these motion equations (12). The positions of particles are evaluated at the time $n\Delta t$ and their velocities at the time $(n+\frac{1}{2})\Delta t$, with Δt the time step and n the current iteration.

By considering charge density ρ generated by particles mouvement, we define two values $\rho_1 = \chi\rho$ and $\rho_2 = (1-\chi)\,\rho$, not equal to zero on the subdomains Ω_1 and Ω_2, such as $\rho = \rho_1 + \rho_2$. So, the modified Maxwell equations on Ω_1 are rewritten :

$$\begin{aligned} \varepsilon_0 \frac{\partial \mathbf{E}_1}{\partial t} &= \chi \nabla \times \mathbf{H}_1 + \chi \nabla \times \mathbf{H}_2 - \chi \mathbf{J} \\ \mu_0 \frac{\partial \mathbf{H}_1}{\partial t} &= -\chi \nabla \times \mathbf{E}_1 - \chi \nabla \times \mathbf{E}_2 \\ \chi\left(\nabla \cdot \mathbf{E}_1 + \nabla \cdot \mathbf{E}_2\right) &= \frac{\rho_1}{\varepsilon_0} \end{aligned} \tag{13}$$

And on Ω_2 :

$$\begin{aligned} \varepsilon_0 \frac{\partial \mathbf{E}_2}{\partial t} &= (1-\chi)\,\nabla \times \mathbf{H}_2 + (1-\chi)\,\nabla \times \mathbf{H}_1 - (1-\chi)\,\mathbf{J} \\ \mu_0 \frac{\partial \mathbf{H}_2}{\partial t} &= -(1-\chi)\,\nabla \times \mathbf{E}_2 - (1-\chi)\,\nabla \times \mathbf{E}_1 \\ (1-\chi)\left(\nabla \cdot \mathbf{E}_2 + \nabla \cdot \mathbf{E}_1\right) &= \frac{\rho_2}{\varepsilon_0} \end{aligned} \tag{14}$$

In the PIC method, there is a problem of charge conservation due to the interpolation between fields and particles. The charge conservation equation (3) is not verified and the constraint on the discrete divergence of the electric field is not satisfied.

To guarantee the continuity equation or $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$, we focus on one of the corrections the most often used in PIC codes named Boris correction [5]-[6]. This method consists in modifying the irrotational part of the electric field $\mathbf{E}$ by :

$$\mathbf{E}_{\text{corr}} = \mathbf{E} - \nabla\phi \tag{15}$$

where $\mathbf{E}_{\text{corr}}$ is the corrected electric field and the potential ϕ is given by :

$$\nabla \cdot \mathbf{E}_{\text{corr}} = \frac{\rho}{\varepsilon_0} \Longleftrightarrow \Delta\phi = \text{div}\mathbf{E} - \frac{\rho}{\varepsilon_0} \tag{16}$$

and

$$\phi|_{\partial\Omega} = 0, \tag{17}$$

with $\partial\Omega$ is the boundary of computational domain Ω. So, in the Boris correction, it is necessary to solve a Laplacian at each iteration.

To take into account this correction in the hybrid method, we also need to define $\phi_1 = \chi\phi$ and $\phi_2 = (1-\chi)\,\phi$ on Ω_1 and Ω_2. We deduce on Ω_1 :

$$\Delta\phi_1 = \nabla\cdot\mathbf{E}_1 + \nabla\cdot\mathbf{E}_2 - \frac{\rho_1}{\chi} - \Delta\phi_2 \tag{18}$$

and the following corrected electric field :

$$\mathbf{E}_{1,\text{corr}} = \mathbf{E}_1 - \chi\left(\nabla\phi_1 + \nabla\phi_2\right) \tag{19}$$

On Ω_2, we have :

$$\Delta\phi_2 = \nabla\cdot\mathbf{E}_1 + \nabla\cdot\mathbf{E}_2 - \frac{\rho_2}{1-\chi} - \Delta\phi_1 \tag{20}$$

$$\mathbf{E}_{2,\text{corr}} = \mathbf{E}_2 - (1-\chi)\left(\nabla\phi_1 + \nabla\phi_2\right) \tag{21}$$

5 NUMERICAL EXPERIMENTS

To validate this hybrid method, we study the motion of one charged particle in an uniform magnetic field equal to $1000A.m^{-1}$. First, we consider only one domain $\Omega_1 = [0, 1.8]\times[0,1]$ without hybridization in the simulation. Secondly, we take two subdomains $\Omega_1 = [0, 1.08]\times[0,1]$ and $\Omega_2 = [0.72, 1.8]\times[0,1]$ such as $\Omega = \Omega_1\cup\Omega_2$ and $\Omega_1\cap\Omega_2 \neq \emptyset$. By using the presented hybrid method, we evaluate also the particle trajectory. This charged particle is an electron with an initial position $(0.5, 0.5)$ and an initial velocity $(1.e7, O)$.

The figure 3 represents the particle trajectory obtained by using or not the hybrid method. In the case of a hybridization, different degrees of polynomial function χ are given. We note that the highest degree gives the best solution.

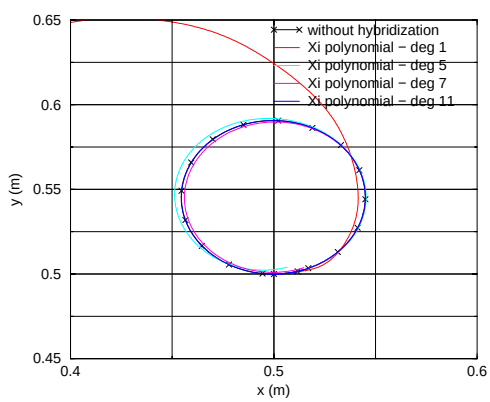

Figure 3: Particle trajectory

6 CONCLUSION

In this paper, we have proposed a hybrid approach for solving a 2D Maxwell-Vlasov problem with a PIC model. Numerical examples show that the method seems also to have a stable behaviour and to improve the solutions accuracy. These first results are encouraging and future works will bring improvements to our model.

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