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INFLUENCE OF BOUNDARY CONDITIONS ON THE ORDER OF ACCURACY OF A MIXED HIGH-ORDER NUMERICAL CODE

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Abstract. *This paper presents results on a verification test of a Direct Numerical Simulation code of mixed high-order of accuracy using the Method of Manufactured Solutions (MMS). The present numerical code was aimed at simulating the temporal evolution of instability waves in a plane Poiseuille flow. The governing equations were solved in a vorticity-velocity formulation for a two-dimensional incompressible flow. For the flow normal direction, the code employed mixed high-order compact and non-compact finite-differences from fourth-order to sixth-order of accuracy. Pseudospectral methods were used for the streamwise direction, which was periodic. Attention was paid to the boundary conditions of the physical problem of interest. Therefore, a manufactured solution that satisfied such boundary conditions was generated. The current manufactured solution also evaluated the nonlinear terms. In this work, special attention was paid to the possible influence of the order of accuracy of the boundary condition numerical errors on the accuracy of the other points of the domain.*

1 INTRODUCTION

Verification test using the Method of Manufactured Solutions (MMS) is very useful to reach confidence that the code is free of programming errors. In fact, the MMS is generally considered more complete than other tests discussed in the literature. One of the reasons is that this methods allows the verification all the terms of the equations, including the nonlinear terms. Also, it allows studying the code without modifications of the boundary conditions used for the physical problem being studied. This test is based on the formulation of an exact solution for the Navier-Stokes equations modified by the addition of a source term [1,2]. A mesh refinement test gives the order of accuracy of the calculations.

The current verification test using MMS was performed on a Direct Numerical Simulation code of mixed high-order of accuracy. In general, simulations of laminar-turbulent transition, turbulence and aeroacustics require the use of numerical methods of high-order of accuracy. This is necessary because the simulation of small spatial and temporal scales at small amplitudes is of fundamental importance for the reproduction of this physical problems. When nonperiodic boundary conditions are implemented, such as wall bounded flow, the use of mixed high-order approximations is often required near or at the boundary. In fact at these points an asymmetric stencil is often used. It is a good practice to ensure that the error at the boundaries is of diffusive nature, which for asymmetric stencils require an odd order of accuracy, while the symmetric schemes used far from the boundaries are even.

For codes of mixed high order of accuracy, much care must be exercised both in choosing the manufactured solutions and in interpreting the results. However, it was possible to verify the numerical error in a grid refinement test and the complete behavior of the order of accuracy of the calculations was reached [3]. The results in [3] also showed that, for high-order codes, it can be very difficult to reach the asymptotic range of accuracy. In fact, for some manufactured solutions, even when the numerical error was reduced to the level of the round off error the asymptotic range was not reached. On the other hand, by judiciously modifying the manufactured solution, the asymptotic order of accuracy asymptotic was reached. The width of the asymptotic range also depended on the test case. Moreover, for the results that were outside the asymptotic range, the observed order was found to be consistent with the other discretization order of accuracy employed in the code.

Another interesting fact that deserves to be explored is that, depending on the region of the domain and on the variable, the numerical error can be very different in magnitude and order. This was indicated by the analysis of the local discretization error for the vorticity at different points at the domain. The results indicated a dominant numerical error from the calculation of the vorticity at the wall. This occurred because in the vorticity-velocity formulation it is necessary to calculate the wall boundary condition rather than specify it directly. This calculation used approximation less accurate than those used at other regions of the domain. Nevertheless, little contamination from this dominant error was observed in other regions. The results indicated that the local order of accuracy at other points was not affected by wall vorticity calculation. At the moment the analysis is restricted to steady state simulation, but it is expected that this analysis will be extended to unsteady conditions. The code was developed for three-dimensional simulations, but for the current discussion two-dimensional simulations are enough and make the analysis clearer.

2 GEOMETRY AND EQUATIONS

The current numerical code is aimed at simulating the temporal evolution of instability waves in a plane Poiseuille flow [3]. Figure 1 shows the geometry of the problem. In a

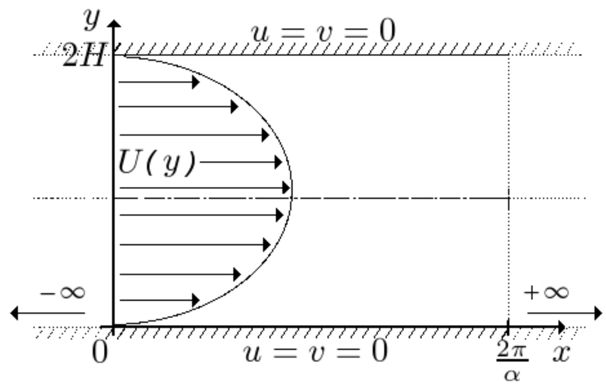

Figure 1: Schematic Illustration of the domain.

two-dimensional vorticity-velocity formulation the governing equations are

$$\frac{\partial \omega}{\partial t} + \frac{\partial u\omega}{\partial x} + \frac{\partial v\omega}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right), \tag{1}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial \omega}{\partial x}, \tag{2}$$

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial \omega}{\partial y}, \tag{3}$$

where ω indicates the vorticity, u and v, the velocity components and Re, the Reynolds number $\frac{U_{max}H}{\nu}$. ν denotes the kinematic viscosity, U_{max}, the velocity at the centerline of the channel and, H, half the channel height.

In the current test, particular attention was paid to both the boundary conditions of the physical problem of interest and the nonlinear calculations. Therefore the following exact solution was manufactured that imitated a Tollmien-Schlichting instability wave in a nonlinear stage:

$$u(x,y) = A\mathrm{e}^{y}y(y+1-\sqrt{5})(y+1+\sqrt{5})(y-2)\mathrm{cos}(\alpha x), \tag{4}$$
$$v(x,y) = A\mathrm{sin}(\alpha x)\alpha\mathrm{e}^{y}y^2(y^2-4y+4), \tag{5}$$
$$\omega(x,y) = -A\mathrm{e}^{y}\mathrm{cos}(\alpha x)(8y^2-8+4\alpha^2y^2+\alpha^2y^4-4\alpha^2y^3-4y^3+8y-y^4). \tag{6}$$

The manufactured solution described above is the analytical solution of the following fictitious problem:

$$\frac{\partial \omega}{\partial t} + \frac{\partial u\omega}{\partial x} + \frac{\partial v\omega}{\partial y} = \frac{1}{Re}(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}) + F(x,y), \tag{7}$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial \omega}{\partial x}, \tag{8}$$
$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial \omega}{\partial y}, \tag{9}$$

where $F(x,y)$ indicates the forcing term that corresponds to the manufactured solution. More details of F, choose numerical schemes and procedures using the MMS, can be found in [4].

3 RESULTS AND FINAL REMARKS

Figure 2 shows the behavior of the local discretization error for vorticity. The results show that while asymptotic error at interior points is of 6^{th} order, the error at the wall was the largest and of 5^{th}. Note that it was necessary to use quadruple precision to avoid the influence of the round off error. In fact, the discretization error on the fourth mesh was around 10^{-12} which prevented the use double precision on the calculations. Note also that the numerical error at the wall is very large as compared to other points of the domain. However, no contamination of the lower accuracy at the wall was observed for points at the interior of the domain.

Lele [5] verified that using numerical methods with reduced order near and at the boundary conditions did not affect the results of a compressible Navier/Stokes simulation. Nevertheless, his boundary conditions were placed in the free-stream, where the derivatives tend to zero. Via a different route, Gustaffson [6] shows that for hyperbolic equations the overall accuracy is not affected by the lower order of accuracy at the boundaries. However, in the current test the equations are not hyperbolic. In fact, the whole system is coupled via a elliptical Poisson equation. Moreover, the low accurate boundary condition was placed at the wall, where the derivatives are high.

The perhaps unexpected but interesting result requires as yet further investigation. This maybe be a feature that is restricted to steady state solutions. It may also be that the manufactured solution is somewhat biased and prevents the solution from reaching

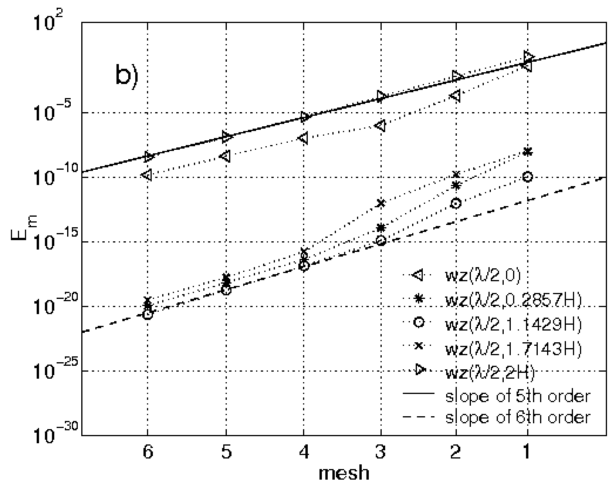

Figure 2: Behavior of the local discretization error for the vorticity with $\alpha = 5$. It was extracted from [3].

the asymptotic regime. Yet, the result does not mean that the same features would hold for a channel flow instability calculation because the phenomenon there is governed by the vorticity shed from the wall, and it would be expected that, in this situation, the overall accuracy of the phenomenon should be affected by the accuracy at the wall.

Other tests, including the analysis of unsteady calculations will be performed in the current initiative. Further, the theoretical study on the peculiarity of mathematical model also will be discussed. It is expected that these results will be available for presentation at the conference.

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