Numerical simulation of one-dimensional pulsatile flow with a combined Fourier-Adomian method

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Abstract

The aim of this paper is to present an approximate solution to an initial boundary value problem related to blood flow by using an hybrid method. It is assumed that the fluid is homogeneous, incompressible and follows the Newton law of viscosity. This procedure does not use any kind of linearization of the equations or discretization of the variables.

1 Introduction

The Navier-Stokes equations are the equations governing the motion of Newtonian fluids such as water, oil, air, etc. Blood is a complex Non-Newtonian fluid but can be regarded as Newtonian during the flow in large blood vessels, [1]. This paper deals with an initial boundary value problem related to blood flow. Namely, it is assumed that the fluid is homogeneous, incompressible and follows the Newton law of viscosity. An expression for the pressure gradient is provided in order to simulate the systole and diastole movements. With these considerations we have for the momentum equation

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \,\mathbf{u} = \varepsilon^2 \Delta \mathbf{u} - \frac{1}{\rho} \nabla p, \tag{1}$$

where ε^2 is the kinematic viscosity, ρ is the fluid density, **u** is the fluid velocity and p is the pressure. Imposing an oscillating pressure gradient to the flow given by

$$-\frac{1}{\rho}\nabla p = \alpha_0 + \alpha\cos\left(\omega t\right),\tag{2}$$

where α and ω are related, respectively, to the amplitude and frequency of the cardiac movement and $\alpha_0 > \alpha$ represents a static favorable pressure gradient that ensures no reverse flow, equation (1) can represent the blood flow in large blood vessels. The center-line flow velocity in a 2–D blood vessel or channel is approximated by the x component of equation (1) with $u_y = 0$, e.g. the one-dimensional differential equation

$$u_t + uu_x = \varepsilon^2 u_{xx} + \alpha_0 + \alpha \cos\left(\omega t\right) \tag{3}$$

with the boundary conditions

$$u(t,0) = u(t,L) = \beta_0 + \beta \sin(\omega t)$$
(4)

and with the initial condition

$$u\left(0,x\right) = \beta_0. \tag{5}$$

The Fourier Method is combined with the Adomian Decomposition Method in order to provide a solution satisfying both initial and boundary conditions.

The proposed exact solution is of the form

$$u(t,x) = h(t,x) + \sum_{k=1}^{\infty} \theta_k(t) \sin\left(\frac{k\pi x}{L}\right).$$
(6)

where h(t, x) is a function that depends on the boundary conditions and $\theta_k(t)$ and $\varphi_k(t)$, for $k \in \mathbb{N}$, are the solutions of a nonlinear system of ordinary differential equations. The initial conditions for this system are given by the projection on the functions $\theta_0(x)$ and $\varphi_0(x)$ in a suitable function space. An approximate solution to the system is obtained by using the Adomian Decomposition Method, described in the next section. The solution in the form (6) satisfies the boundary conditions. Pulsating flows of newtonian and non-newtonian fluids have been studied in detail, theoretically and numerically with a finite volume method in [2].

2 The Adomian Decomposition Method

In [3], G. Adomian developed a decomposition method for solving nonlinear (stochastic) differential equations using special polynomials A_n , usually called Adomian polynomials. The A_n 's are generated for each nonlinearity.

One of the main advantages of the Adomian's polynomials is that they depend only on the known function $u_0(x)$. Another great advantage of this method is that the algorithm is of simple implementation.

Unfortunately the solutions provided by the standard Adomian Decomposition Method usually do not satisfy the boundary conditions. In [4] and [5] the authors present a method that allows the standard Adomian Decomposition Method to solve initial boundary valued problems for Partial Differential Equations.

The convergence of the Adomian Decomposition series has been investigated by several authors. In [6] and [7], the authors showed that the method does not always converge, particularly, when the method is applied to linear operator equations. The theoretical analysis of convergence and speed of convergence of the decomposition method was considered in [8], [9], [10], [11] and [12].

Let us now consider the differential equation

$$\mathcal{L}u = Ru + \Phi u,\tag{7}$$

where \mathcal{L} (linear) and R are differential operators and Φ is a nonlinear operator.

The Adomian polynomials decompose a given function u(t, x) in a series

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x), \qquad (8)$$

and for a nonlinear operator Φ we have the following decomposition

$$\Phi\left(u\left(t,x\right)\right) = \sum_{n=0}^{\infty} A_n,\tag{9}$$

,

where the A_n , usually called the Adomian's Polynomials, are given by the recurrence formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\Phi\left(\sum_{n=0}^{\infty} \lambda^n u_n\right) \right] \Big|_{\lambda=0}, \quad n \ge 0.$$
(10)

The Adomian polynomials can be constructed as follows:

$$A_{0} = \Phi(u_{0}),$$

$$A_{1} = u_{1}\Phi(u_{0}),$$

$$A_{2} = u_{2}\Phi'(u_{0}) + \frac{1}{2}u_{1}^{2}\Phi''(u_{0}),$$

$$A_{3} = u_{3}\Phi'(u_{0}) + u_{1}u_{2}\Phi''(u_{0}) + \frac{1}{3!}u_{1}^{3}\Phi'''(u_{0})$$

$$\vdots$$

Algorithms for formulating Adomian polynomials where investigated in [13] and [14].

We also suppose the existence of the inverse operator \mathcal{L}^{-1} . Thus, applying \mathcal{L}^{-1} to (7), we obtain the recurrence relation,

$$u_{n+1} = \mathcal{L}^{-1} R u_n + \mathcal{L}^{-1} \Phi u_n, \ u_0 = u_0(x), \ n \in \mathbb{N} \cup \{0\},$$
(11)

that provides a reliable approach to the solution of the problem.

Now we briefly describe how to apply the Adomian Decomposition Method to systems of ordinary differential equations. Let us consider a system of ordinary differential equations in the form

$$\begin{cases}
\mathcal{L}u_{1} = f_{1}(t, u_{1}, u_{2}, \dots, u_{n}) \\
\mathcal{L}u_{2} = f_{2}(t, u_{1}, u_{2}, \dots, u_{n}) \\
\vdots \\
\mathcal{L}u_{n} = f_{n}(t, u_{1}, u_{2}, \dots, u_{n})
\end{cases},$$
(12)

with initial conditions $u_i(0)$, for $1 \le i \le n$, where $\mathcal{L}u = \dot{u} \equiv \frac{du}{dt}$ with inverse $\mathcal{L}^{-1}(\cdot) = \int_0^t (\cdot) dt$.

Applying the inverse operator \mathcal{L}^{-1} to (12) we obtain the following canonical form

$$\begin{cases}
 u_1 = u_{1,0} + \mathcal{L}_t^{-1} \left[f_1 \left(t, u_1, u_2, \dots, u_n \right) \right] \\
 u_2 = u_{2,0} + \mathcal{L}_t^{-1} \left[f_2 \left(t, u_1, u_2, \dots, u_n \right) \right] \\
 \vdots \\
 u_n = u_{n,0} + \mathcal{L}_t^{-1} \left[f_n \left(t, u_1, u_2, \dots, u_n \right) \right]
\end{cases}$$
(13)

where $u_{i,0} = u_i(0)$ for $1 \le i \le n$.

Applying the Adomian Decomposition Method, each component of the solution of (12) can be expressed as a series of the form

$$u_j = \sum_{i=0}^{\infty} f_{i,j} \tag{14}$$

and the integrands on the right side of (13), using (10), are expressed as

$$f_i(t, u_1, u_2, \dots, u_n) = \sum_{j=0}^{\infty} A_{i,j}(f_{i,0}, f_{i,1}, f_{i,2}, \dots, f_{i,j}), \ 1 \le i \le n,$$
(15)

where the $A_{i,j}$ are the Adomian Polynomials corresponding to the nonlinear part f_i .

We should note that in order to solve system (12), we obtain a system of Volterra integral equations of the second kind, (13).

Therefore (6) is actually of the form

$$u(t,x) = h(t,x) + \sum_{k=1}^{\infty} u_k(t) \sin\left(\frac{k\pi x}{L}\right) = h(t,x) + \sum_{k=1}^{\infty} \left(\sum_{i=0}^{\infty} u_{k,i}(t)\right) \sin\left(\frac{k\pi x}{L}\right).$$
(16)

In order to accelerate the convergence of the method when applied to nonlinear systems of Volterra integral equations of second kind, we will proceed as in [15]. For considerations related to the convergence of the Adomian Decomposition Method when applied to nonlinear systems of Volterra integral equations of second kind, we refer the reader to [16], where the problem of convergence is studied.

As an example of application of this procedure to obtain an approximate solution to the initial boundary valued problem for higher order nonlinear parabolic equations we refer the reader to [17].

3 The parameters of the model

Under certain simplifications, the differential equations (3) and (6) can represent a model for blood flow in large blood vessels. For large shear rates, blood behaves like a suspension of particles in a Newtonian fluid. Einstein derived an equation for spherical particles at a low volumetric concentration, [18]. This equation applied to blood reads,

$$\eta = \eta_p \left(\frac{1}{1 - \alpha \phi}\right)$$

where η is the blood viscosity, η_p the plasma viscosity and α is a parameter related to the shape of the particles. For high values of haematocrit, $H = 100\phi$, the following empirical relation is available,

$$\alpha = 0.076 exp\left[2.49\phi + \frac{1107}{T}e^{-1.69\phi}\right] \quad \text{for} \quad 0.05 \le \phi \le 0.6$$

with T (Kelvin) the human body temperature. Considering $T = 37^{\circ} = 310K$, the plasma viscosity is $\eta_p \approx 1.24 \times 10^{-2} Pa.s$. The healthy haematocrit is H = 45 and then $\phi = 0.45$. These conditions lead to $\alpha = 1.237$ and $\eta = 0.028 Pa.s$. The kinematic viscosity is $\varepsilon^2 = \frac{\eta}{\rho}$ where ρ is the blood density. For the healthy value of $\rho = 1060 Kg/m^3$ we have $\varepsilon^2 = 0.265 \times 10^{-4}m^2/s$. The steady pressure gradient parameter, α_0 , was estimated considering a decrease of $\Delta p = -10mmHg$ along the aorta lenght L = 0.5m. This leads to $\alpha_0 = -\frac{1}{\rho}\frac{\Delta p}{L} = 2.5m/s^2$. The oscillating parameter α is considered to be 20% of the mean steady pressure, $\alpha = 0.2\alpha_0 = 0.5m/s^2$, denoting the amplitude of the systolic-diastolic pressure, usually 12 - 8mmHg. For the boundary conditions parameters, β_0 and β , an average aortic blood velocity was considered, $\beta_0 = 0.5m/s$, and an oscillation of 20% is assumed, $\beta = 0.2\beta_0 = 0.1m/s$. The frequency of the cardiac movement, $f = \frac{75}{60}$ beats per second, is assumed for the calculation of $\omega = 2\pi f = 7.854rad/s$.

Pulsating flows of newtonian and non-newtonian elastic fluid was studied theoretically using the method of separation of variables and numerically using the finite-volume method in [19], [2], [20] and [21] respectively.

4 The approximate solution for the nonlinear problem

Let us now consider the problem

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$$\begin{cases}
 u_t + uu_x = \varepsilon^2 u_{xx} + f(t, x) \\
 u(t = 0, x) = u_0(x) \\
 u(t, 0) = u(t, L) = g(t)
\end{cases}$$
(17)

In order to obtain an approximate solution to this problem, it is necessary to do some mathematical manipulation to get homogeneous boundary conditions. Let us consider that the exact solution, u(t, x) can be expressed as a sum of two functions, an unknown function w(t, x) and

$$S(t,x) = A(t)\left(1 - \frac{x}{L}\right) + B(t)\left(\frac{x}{L}\right)$$

being function that satisfies the boundary conditions.

It is easy to see that

$$S\left(t,x\right) = g\left(t\right),$$

and therefore,

$$u(t,x) = w(t,x) + g(t).$$
 (18)

Using (18) in (17), we obtain the nonlinear homogeneous problem (for w(t, x)),

$$\begin{cases} w_t + ww_x + g(t) w_x = \varepsilon^2 w_{xx} + f(t, x) - \dot{g}(t) \\ w(t = 0, x) = u_0(x) - g(0) \\ w(t, 0) = w(t, L) = 0 \end{cases}$$
(19)

Let us now consider that the exact solution of (19) is of the form

$$w(t,x) = \sum_{k=1}^{\infty} w_k(t) \sin\left(\frac{k\pi x}{L}\right).$$
 (20)

Using an approximation of (20) in (19) we obtain the relation

$$\sum_{k=1}^{n} \dot{w}_{k}(t) \sin\left(\frac{k\pi x}{L}\right) = -\frac{\pi}{L} \sum_{j,k=1}^{n} kw_{k}(t) w_{j}(t) \cos\left(\frac{k\pi x}{L}\right) \sin\left(\frac{j\pi x}{L}\right) -g(t) \frac{\pi}{L} \sum_{k=1}^{n} kw_{k}(t) \cos\left(\frac{k\pi x}{L}\right) -\left(\frac{\varepsilon\pi}{L}\right)^{2} \sum_{k=1}^{n} k^{2} w_{k}(t) \sin\left(\frac{k\pi x}{L}\right) +\sum_{k=1}^{n} c_{k}(t) \sin\left(\frac{k\pi x}{L}\right),$$
(21)

where

$$c_k(t) = \frac{2}{L} \int_0^L \left(f(t, x) - \dot{g}(t) \right) \sin\left(\frac{k\pi x}{L}\right) dx.$$
(22)

Multiplying (21) by $\frac{2}{L}\sin\left(\frac{i\pi x}{L}\right)$, for $1 \le i \le n$ and integrating in order to the variable x we obtain the following nonlinear system of ordinary differential equations:

$$\dot{w}_{i}(t) = -\frac{2\pi}{L^{2}} \sum_{j,k=1}^{n} k w_{k}(t) w_{j}(t) \int_{0}^{L} \cos\left(\frac{k\pi x}{L}\right) \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{i\pi x}{L}\right) dx$$
$$-\frac{\pi}{L} \sum_{k=1}^{n} k g(t) w_{k}(t) \int_{0}^{L} \cos\left(\frac{k\pi x}{L}\right) \sin\left(\frac{i\pi x}{L}\right) dx$$
$$-\left(\frac{\varepsilon i\pi}{L}\right)^{2} w_{i}(t) + \frac{2}{L} \int_{0}^{L} (f(t,x) - \dot{g}(t)) \sin\left(\frac{i\pi x}{L}\right) dx, \qquad (23)$$

for $1 \leq i \leq n$.

Integrating in order to the variable t, we obtain the following recurrence scheme:

$$w_{i}(t) = w_{i}(0) - \frac{2\pi}{L^{2}} \sum_{j,k=1}^{n} k \mathcal{I}_{kji} \int_{0}^{t} w_{k}(t) w_{j}(t) dt - \frac{\pi}{L} \sum_{k=1}^{n} k \left[\int_{0}^{L} \cos\left(\frac{k\pi x}{L}\right) \sin\left(\frac{i\pi x}{L}\right) dx \right] \int_{0}^{t} g(t) w_{k}(t) dt - \left(\frac{\varepsilon i\pi}{L}\right)^{2} \int_{0}^{t} w_{i}(t) dt + \frac{2}{L} \int_{0}^{t} \left[\int_{0}^{L} (f(t,x) - \dot{g}(t)) \sin\left(\frac{i\pi x}{L}\right) dx \right] dt, (24)$$

for $1 \leq i \leq n$ and \mathcal{I}_{kji} is given by

$$\mathcal{I}_{kji} = \int_0^L \cos\left(\frac{k\pi x}{L}\right) \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{i\pi x}{L}\right) dx.$$
 (25)

Thus, we obtain the following recurrence scheme

$$w_{i,m+1}(t) = -\frac{2\pi}{L^2} \sum_{j,k=1}^n k \left[\mathcal{I}_{kji} \right] \int_0^t w_{k,m}(t) w_{j,m}(t) dt - \left(\frac{\varepsilon i\pi}{L}\right)^2 \int_0^t w_{i,m}(t) dt - \frac{\pi}{L} \sum_{k=1}^n k \left[\int_0^L \cos\left(\frac{k\pi x}{L}\right) \sin\left(\frac{i\pi x}{L}\right) dx \right] \int_0^t g(t) w_{k,m}(t) dt + \frac{2}{L} \int_0^t \left[\int_0^L (f(t,x) - \dot{g}(t)) \sin\left(\frac{i\pi x}{L}\right) dx \right] dt,$$
(26)

for $1 \leq i \leq n$.

The initial conditions for this system are given by

$$w_{i}(0) = \frac{2}{L} \int_{0}^{L} \left(u_{0}(x) - g(0) \right) \sin\left(\frac{i\pi x}{L}\right) dx.$$
(27)

The nonlinearities in (26) (or in (23) and (24)) are of the form $\Psi(u, v) = uv$. Thus, using (10) we have for $\Psi(u, v)$

$$A_{0} = u_{0}v_{0}$$

$$A_{1} = u_{1}v_{0} + u_{0}v_{1}$$

$$A_{2} = u_{2}v_{0} + u_{1}v_{1} + u_{0}v_{2}$$

$$A_{3} = u_{3}v_{0} + u_{2}v_{1} + u_{1}v_{2} + u_{0}v_{3}$$

$$A_{4} = u_{4}v_{0} + u_{3}v_{1} + u_{2}v_{2} + u_{1}v_{3} + u_{0}v_{4}$$

$$\vdots$$

In the next section an example of application is presented.

5 Numerical application

The method previously described was applied to the non-linear blood flow model

$$\begin{cases} u_t + uu_x = \varepsilon^2 u_{xx} + \alpha_0 + \alpha \cos(\omega t) \\ u(t = 0, x) = \beta_0, \ 0 \le x \le L, \\ u(t, 0) = u(t, L) = \beta_0 + \beta \sin(\omega t), \ t > 0 \end{cases}$$
(28)

The numerical simulation was achieved with only 5 terms of the series (6).

The model parameters have been established in 3 and the value L = 0.5m is assumed to be the aorta length



Figure 1: Solution for $0 \le t \le 0.2$

Figure 2: Solution for $0 \le t \le 0.4$

It is observed, from the figures, that the initial steady velocity is significantly perturbed after 0.1 seconds. For the interval of time 0 < t < 0.2 the pulsating flow is not evident and the velocity begins to be affected by the pulsating pressure gradient at the end of the channel. The pulsating flow begins after 0.5 seconds and is completely established after 0.8 second. This



Figure 3: Solution for $0 \le t \le 0.6$

Figure 4: Solution for $0 \le t \le 0.8$

response to the oscillating pressure gradient gives an idea of the inertia of the flow. Blood flow inertia can represent an important role in some diseases. The results of the numerical simulation agree with physical considerations and showed that the method is consistent and convergent when applied to this flow.

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