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THE PROBLEM OF BOUNDARY CONDITION ON THE OUTFLOW FOR AN INCOMPRESSIBLE FLOW THROUGH A CASCADE OF PROFILES

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Abstract. *We are concerned with the theoretical analysis of the model of incompressible, viscous, stationary flow through a plane cascade of profiles. The boundary value problem for the Navier-Stokes system is formulated in a domain representing the exterior to an infinite row of profiles, periodically spaced in one direction. Then the problem is reformulated in a bounded domain of the form of one space period and completed by the Dirichlet boundary condition on the inlet and the profile, a suitable natural boundary condition on the outlet and periodic boundary conditions on artificial cuts. Specially, we derive and study the question of existence and uniqueness of the weak solution of this problem for linear separated "do nothing" type boundary condition (which we derive) and for nonlinear modification of the "do nothing" type of boundary condition which was proposed by C. H. Bruneau, F. Fabrie in ([1]). The problems of existence and uniqueness for these two cases are discussed and compared.*

1 GEOMETRY OF THE PROBLEM

We study the steady flow through a simplified plane cascade of profiles. The model of cascade of profiles describes e.g. the flow through a turbine or thorough a general blade machine. If we consider the intersection of the real 3D region filled by the moving fluid with a surface defined along the streamlines of the flow, and expand the surface in the x_1, x_2–plane we will naturally arrive at a 2D domain. The obtained two dimensional domain is unbounded, however periodic in the x_2–direction. Its complement in R^2 consists of the infinite number of profiles, numbered from $-\infty$ to $+\infty$.

The following assumptions are naturally fulfilled. We suppose that the boundary of the profile No. 0 is a simple closed curve C_0 in R^2, piecewise of the class C^2, whose interior and exterior are domains with a Lipschitz–continuous boundary. We put $C_k = \{(x_1, x_2 + k\tau);\ (x_1, x_2) \in C_0\}$ (for $k \in Z$), where τ is a positive constant. We assume that τ is so large that the curves C_k are mutually disjoint. The set $M := \bigcup_{k=-\infty}^{+\infty} \overline{\mathrm{Int}\, C_k}$ is called a *cascade of profiles*. ($\mathrm{Int}\, C_k$ denotes the interior of curve C_k.) Number τ is called the *period* of the cascade.

From the definition of the domain it is reasonable to assume that the flow through the cascade is periodic in the x_2–direction with the period τ. Consequently, we can study the flow just in one spatial period of the whole domain. The chosen period is denoted Ω. Its boundary consists of the curves Γ_i, Γ_o, Γ_+, Γ_- and Γ_w.

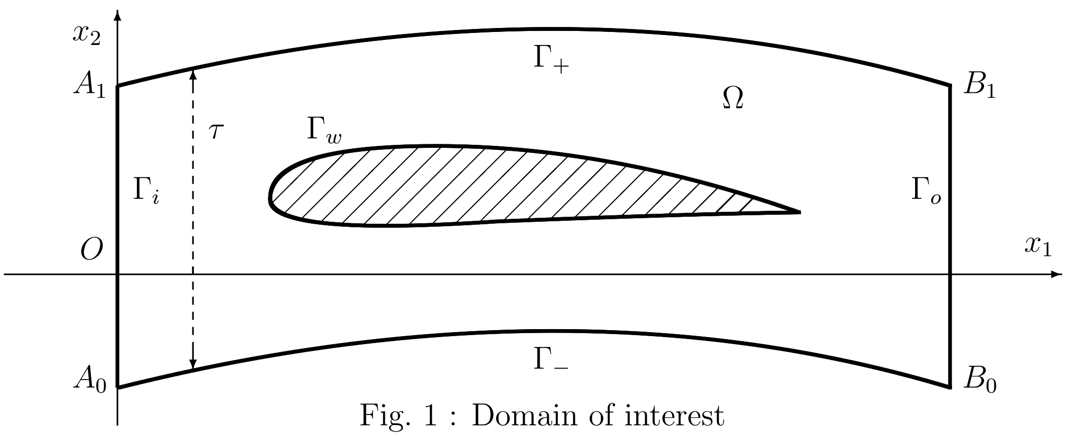

Fig. 1 : Domain of interest

2 AUXILIARY RESULTS

We shall work with the following function spaces.

By $H^1(\Omega)$ we denote the usual Sobolev space and by $H^1(\Omega)^2 := [H^1(\Omega)]^2$ we denote functions with two components, both in $H^1(\Omega)$. Space X is a space of test functions constructed for the deriving of the weak solution.

$$\begin{aligned} X \;=\; & \{\boldsymbol{v} \in H^1(\Omega)^2;\ \boldsymbol{v} = \boldsymbol{0} \ \text{ a.e. in } \Gamma_i \cup \Gamma_w,\ \boldsymbol{v}(x_1, x_2 + \tau) = \boldsymbol{v}(x_1, x_2) \\ & \text{for a.a. } \ (x_1, x_2) \in \Gamma_-\} \ . \end{aligned}$$

The boundary conditions on the curves Γ_i, Γ_w and Γ_- are interpreted in the sense of traces. Let

$$V \;=\; \Big\{\boldsymbol{v} \in X;\ \mathrm{div}\, \boldsymbol{v} = 0 \ \text{ a.e. in } \Omega\Big\} \ .$$

The norm in X, defined in this way

$$|||\boldsymbol{v}||| \;=\; \left(\int_\Omega \sum_{i,j=1}^{2} \left(\frac{\partial v_i}{\partial x_j}\right)^2 \mathrm{d}\boldsymbol{x}\right)^{1/2} \tag{1}$$

is an equivalent norm in $H^1(\Omega)^2$.

2.1 Function $\boldsymbol{g}^*$ - the realization of the boundary condition on Γ_i

Lemma Let $s \in (\frac{1}{2}, 1\rangle$ and let function $\boldsymbol{g}$ belongs to the Sobolev–Slobodetskii space $H^s(\Gamma_i)^2$. Then there exists a constant $c_g > 0$ independent of $\boldsymbol{g}$ and a divergence–free extension $\boldsymbol{g}^* \in H^1(\Omega)^2$ of function $\boldsymbol{g}$ from Γ_i onto Ω such that $\boldsymbol{g}^* = \mathbf{0}$ on Γ_w, $\boldsymbol{g}^*$ satisfies the condition of periodicity

$$\boldsymbol{g}^*(x_1, x_2 + \tau) \;=\; \boldsymbol{g}^*(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_- \tag{2}$$

and the estimate

$$\|\boldsymbol{g}^*\|_1 \;\leq\; c_g \, \|\boldsymbol{g}\|_{s;\,\Gamma_i} \quad . \tag{3}$$

We will seek the weak solution $\boldsymbol{u}$ in the form $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z} \in V$ will be a new unknown function. This form of $\boldsymbol{u}$ guarantees that $\boldsymbol{u}$ will satisfy the prescribed velocity profile on the boundary Γ_i.

3 THE PROBLEM WITH THE NONLINEAR BOUNDARY CONDITION ON THE OUTLET

3.1 Used Equations

We assume that the fluid is viscous, stationary, incompressible and newtonian. For simplicity we suppose that the unit system is chosen in such a way that the constant density of the fluid is one. The conservation of momentum is described by the Navier-Stokes equations in the form

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \;=\; \boldsymbol{f} \;-\; \nabla p \;+\; \nu\, \Delta \boldsymbol{u}. \tag{4}$$

where $\boldsymbol{u}$ $(= (u_1, u_2))$ is the velocity of the fluid and p the pressure in the fluid, $\boldsymbol{f}$ $(= (f_1, f_2))$ is the density of the volume force and constant $\nu > 0$ is the kinematic viscosity. The conservation of mass is described by the equation of continuity

$$\operatorname{div} \boldsymbol{u} \;=\; 0. \tag{5}$$

3.2 Boundary conditions

We prescribe the inhomogeneous Dirichlet boundary condition on the inlet:

$$\boldsymbol{u}\,|_{\Gamma_i} \;=\; \boldsymbol{g}. \tag{6}$$

We assume that the fluid satisfies the no slip Dirichlet boundary condition on the profile:

$$\boldsymbol{u}\,|_{\Gamma_w} \;=\; \mathbf{0}. \tag{7}$$

According to the definition of the model we suppose that the following conditions of periodicity are fulfilled on the artificial boundaries Γ_+ and Γ_-:

$$\boldsymbol{u}(x_1, x_2+\tau) \;=\; \boldsymbol{u}(x_1, x_2) \qquad \text{for } (x_1,x_2)\in\Gamma_-, \tag{8}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2+\tau) \;=\; -\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2). \qquad \text{for } (x_1,x_2)\in\Gamma_-, \tag{9}$$

$$p(x_1, x_2+\tau) \;=\; p(x_1, x_2) \qquad \text{for } (x_1,x_2)\in\Gamma_-. \tag{10}$$

We use the the nonlinear form of the do-nothing type of boundary condition proposed C. H. Bruneau, F. Fabrie in [1].

$$-\nu\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\cdot\boldsymbol{n} - \frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\,\boldsymbol{u} \;=\; \boldsymbol{h} \qquad \text{on } \Gamma_o \tag{11}$$

where $\boldsymbol{n}$ is the outer normal vector and $\boldsymbol{h}$ is a given function. For $a\in R$ we set $a^+ = (|a|+a)/2$ and $a^- = (|a|-a)/2$.

3.3 Weak formulation

In order to derive formally the weak formulation of the problem, we multiply equation (4) by an arbitrary test function $\boldsymbol{v}=(v_1,v_2)\in V$ and integrate in Ω. We obtain

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \;&=\; \int_\Omega\Big(-\nu\,\Delta\boldsymbol{u} + \sum_{j=1}^{2} u_j\,\frac{\partial \boldsymbol{u}}{\partial x_j} + \nabla p\Big)\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ &=\; \nu\int_\Omega \sum_{i,j=1}^{2}\frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\boldsymbol{x} \,-\, \nu\int_{\partial\Omega}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\cdot\boldsymbol{v}\,\mathrm{d}S \,+\, \int_\Omega\sum_{i,j=1}^{2} u_j\,\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}\boldsymbol{x} \\ &\quad - \int_\Omega p\,\operatorname{div}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \,+\, \int_{\partial\Omega} p\;\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S. \end{aligned} \tag{12}$$

If we apply Green's theorem and use all the boundary conditions (6)–(11), we arrive at the equation

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \;=\; &\nu\int_\Omega \sum_{i,j=1}^{2}\frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\boldsymbol{x} \,+\, \int_\Omega\sum_{i,j=1}^{2} u_j\,\frac{\partial u_i}{\partial x_j}\,v_i\,\mathrm{d}\boldsymbol{x} \\ &+ \int_{\Gamma_o}\frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}S \,+\, \int_{\Gamma_o}\boldsymbol{h}\cdot\boldsymbol{v}\,\mathrm{d}S, \quad \boldsymbol{v}\in V. \end{aligned} \tag{13}$$

In order to simplify its form, we introduce the following notation: for $\boldsymbol{u}=(u_1,u_2)$, $\boldsymbol{v}=(v_1,v_2)$, $\boldsymbol{w}=(w_1,w_2)\in H^1(\Omega)^2$, we put

$$a_1(\boldsymbol{u},\boldsymbol{v}) \;:=\; \nu\int_\Omega\sum_{i,j=1}^{2}\frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\boldsymbol{x},$$

$$\begin{aligned}
a_2(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &:= \int_\Omega \sum_{i,j=1}^{2} u_j\,\frac{\partial v_i}{\partial x_j}\,w_i\,\mathrm{d}\boldsymbol{x},\\
a_3(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) &:= \int_{\Gamma_o} \frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{n})^-\,\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}S,\\
a(\boldsymbol{u},\boldsymbol{v}) &:= a_1(\boldsymbol{u},\boldsymbol{v}) + a_2(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + a_3(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}),\\
(\boldsymbol{f},\boldsymbol{v}) &:= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\\
b(\boldsymbol{h},\boldsymbol{v}) &:= -\int_{\Gamma_o} \boldsymbol{h}\cdot\boldsymbol{v}\,\mathrm{d}S.
\end{aligned}$$

Obviously, all these forms are well defined for $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in H^1(\Omega)^2$, $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. Now the identity (13) can shortly be written as

$$a(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) + b(\boldsymbol{h},\boldsymbol{v}), \qquad \boldsymbol{v}\in V. \tag{14}$$

Suppose that the function $\boldsymbol{g}$, appearing in the boundary condition (6), belongs to $H^s(\Gamma_i)^2$ for $s \in (\frac{1}{2}, 1\rangle$ and $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$. (Let us recall that A_0 and A_1 are the end–points of Γ_i.) Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$ be given functions. We seek a vector function $\boldsymbol{u} \in H^1(\Omega)^2$ which satisfies the equation of continuity (5) a.e. in Ω, the boundary conditions (6) (respectively (7)) in the sense of traces on Γ_i (respectively on Γ_w), the condition of periodicity (8) a.e. on Γ_- and such that identity (14) holds for all test functions $\boldsymbol{v} \in V$.

The solution $\boldsymbol{u}$ of this problem is called a weak solution in the domain Ω.

3.4 Existence of a weak solution

Now we shall seek for the weak solution $\boldsymbol{u}$ in the form $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z} \in V$ is a new unknown function. This guarantees that $\boldsymbol{u}$ satisfies all the boundary and periodicity conditions (6)–(11). Substituting this form of $\boldsymbol{u}$ into the equation (14), we derive the following problem: Find a function $\boldsymbol{z} \in V$ such that it satisfies the equation

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) + b(\boldsymbol{h},\boldsymbol{v}) \tag{15}$$

for all $\boldsymbol{v} \in V$. The following theorem can be proved.

Theorem (on the existence of a weak solution). There exists $\varepsilon > 0$ such that if $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} < \varepsilon$ then there exists a solution $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ of the problem. Moreover $\boldsymbol{z}$ satisfies the estimate

$$|\!|\!|\boldsymbol{z}|\!|\!| \le R_1\ . \tag{16}$$

Consequently, the weak problem (15) has a solution $\boldsymbol{u}$ $(= \boldsymbol{z} + \boldsymbol{g}^*)$ that satisfies

$$\|\nabla\boldsymbol{u}\|_{L^2(\Omega)^2} \le R_1 + \|\nabla\boldsymbol{g}^*\|_{L^2(\Omega)^2} \le R_1 + c\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} := R_2\ . \tag{17}$$

Here R_1 and c are constants based on the construction of the weak solution in the proof of the theorem.

The proof of this theorem is carried out by using the method of Galerkin approximations. We need to prove the coercivity of the form a and construct the weak solution. The value of the constant ε comes from the proof to ensure coercivity. The complete proof can be found in [2] and in [4].

Remark It is not possible to prove the existence of a weak solution for the basic do nothing boundary condition in the form $-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p \cdot \boldsymbol{n} = \boldsymbol{h}$ which was proposed by J. Heywood, R. Rannacher R. and S. Turek. As you can see it is possible to prove the coercivity of the form a and the existence of the weak solution for the nonlinear modification of this condition. The restrictive condition on the function $\boldsymbol{g}$ in the theorem is in agreement with the literature. The case for general inlet is so far unsolved. However the nonlinear condition cause difficulties in the numerical computation and in the proof of the uniqueness of the solution. In the fourth section we will show linear type of do nothing boundary condition which enables us to prove existence of a weak solution of the flow problem.

3.5 Uniqueness of a weak solution

Theorem (on the uniqueness of a weak solution). There exists $R > 0$ such that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the problem (15) such that $\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \leq R$ and $\|\nabla \boldsymbol{u}_2\|_{L^2(\Omega)^4} \leq R$ then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.

Proof. Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the solutions of the problem (15), they fulfil the equations

$$\begin{aligned} a(\boldsymbol{u}_1, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}). \\ a(\boldsymbol{u}_2, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}) \end{aligned}$$

for all $\boldsymbol{v} \in V$. Subtracting these equations, we get

$$a(\boldsymbol{u}_1, \boldsymbol{v}) - a(\boldsymbol{u}_2, \boldsymbol{v}) = 0.$$

Expressing the bilinear form a by means of the forms a_1, a_2 and a_3, we obtain

$$\begin{aligned} a_1(\boldsymbol{u}_1, \boldsymbol{v}) - a_1(\boldsymbol{u}_2, \boldsymbol{v}) + a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{v}) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{v}) \\ + a_3(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{v}) - a_3(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{v}) = 0. \end{aligned}$$

This holds for all $\boldsymbol{v} \in V$. If we choose $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ then this identity yields

$$\begin{aligned} a_1(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) + a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \\ + a_3(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_3(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) = 0. \end{aligned} \tag{18}$$

If we denote

$$I_1 \;:=\; a_1(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2) \;=\; \nu\int_\Omega |\nabla(\boldsymbol{u}_1-\boldsymbol{u}_2)|^2\,\mathrm{d}\boldsymbol{x} \;=\; \nu\,|||\boldsymbol{u}_1-\boldsymbol{u}_2|||^2,$$

$$I_2 \;:=\; a_2(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u}_1-\boldsymbol{u}_2) - a_2(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2),$$

$$I_3 \;:=\; a_3(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{u}_1-\boldsymbol{u}_2) - a_3(\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2)$$

then (18) takes the form

$$I_1 \;=\; -I_2 \;-\; I_3. \tag{19}$$

For the terms on the right hand side of (19) we can prove

$$|I_2| \;\le\; c\,R\,|||\boldsymbol{u}_1-\boldsymbol{u}_2|||^2, \qquad\qquad |I_3| \;\le\; c\,R\,|||\boldsymbol{u}_1-\boldsymbol{u}_2|||^2$$

Substituting from this result into (19), we obtain

$$\nu\,|||\boldsymbol{u}_1-\boldsymbol{u}_2|||^2 \;\le\; c\,R\,|||\boldsymbol{u}_1-\boldsymbol{u}_2|||^2.$$

Now it is seen that if R is small enough (according to the constant c which comes from the more detailed proof. The complete proof can be seen for example in [4]) then $\boldsymbol{u}_1=\boldsymbol{u}_2$. The theorem is proved.

4 THE PROBLEM WITH THE LINEAR BOUNDARY CONDITION ON THE OUTLET

Now let us consider the flow with a linear boundary condition on the outlet of the domain Ω.

4.1 Used Equations

We study the flow described by Navier–Stokes equation in the form (different from the third section)

$$\omega(\boldsymbol{u})\,\boldsymbol{u}^\perp \;=\; -\nabla q + \nu\,(-\partial_2,\partial_1)\,\omega(\boldsymbol{u}) + \boldsymbol{f} \tag{20}$$

where $\omega(\boldsymbol{u})=\partial_1 u_2-\partial_2 u_1$, $\boldsymbol{u}^\perp=(-u_2,u_1)$ and $q \;:=\; p+\frac{|\boldsymbol{u}|^2}{2}$. $\omega(\boldsymbol{u})$ denotes the vorticity of the flow and q is the Bernoulli pressure. The incompressibility condition is described by the continuity equation

$$\mathrm{div}\;\boldsymbol{u} \;=\; 0\;\;. \tag{21}$$

4.2 Boundary conditions

Let us suppose that the boundary condition on the boundaries Γ_i, Γ_w, Γ_- and Γ_+ are identical to the conditions used in previous problem.

The Bernoulli pressure q is naturally supposed to be τ–periodic in the x_2–direction too, i.e.

$$q(x_1,x_2+\tau) \;=\; q(x_1,x_2) \qquad\qquad \text{for } (x_1,x_2)\in\Gamma_-\;\;. \tag{22}$$

The boundary condition used on the outflow Γ_o is

$$q = h_1, \qquad -\nu\, \omega(\boldsymbol{u}) = h_2 \tag{23}$$

where $\boldsymbol{h} = (h_1, h_2)$ is a given function on Γ_o. This condition will naturally arise (as a boundary condition of the "do nothing" type) from an appropriate weak formulation.

Remark Considering the nonlinear term in the Navier–Stokes equation in the form $\omega(\boldsymbol{u})\, \boldsymbol{u}^{\perp}$ has the advantage that if we formally multiply (20) by $\boldsymbol{u}$, the product $\omega(\boldsymbol{u})\, \boldsymbol{u}^{\perp} \cdot \boldsymbol{u}$ equals zero point–wise a.e. in Ω and it is therefore not necessary to integrate by parts in order to remove or transform this term. Thus, we avoid problems on the part Γ_o of the boundary, caused by possible backward flows. On the other hand, this approach implies that we must deal with the Bernoulli pressure $q = p + \frac{1}{2}|\boldsymbol{u}|^2$ instead of the physical pressure p on the right hand side of (20) and consequently, also in the first of the boundary conditions (23). The pressure in the form q is sometimes called the "total" pressure in the difference to the "static" pressure p.

4.3 Weak formulation

In order to arrive formally at the weak formulation of the problem (20)–(23), we multiply (20) by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$, integrate over Ω, apply Green's theorem and use the condition of incompressibility (21), and the boundary conditions and conditions of periodicity (8), (9), (10). We obtain the equation

$$\begin{aligned} \nu \int_{\Omega} \omega(\boldsymbol{u}) \cdot \omega(\boldsymbol{v})\, \mathrm{d}\boldsymbol{x} + \int_{\Omega} \omega(\boldsymbol{u})\, \boldsymbol{u}^{\perp} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \,-\, \nu \int_{\Gamma_o} \omega(\boldsymbol{u})\, (v_2 n_1 - v_1 n_2)\, \mathrm{d}S \\ + \int_{\Gamma_o} q\, \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{x} \;=\; \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}\ . \end{aligned}$$

Using the identities $n_1 = 1$ and $n_2 = 0$ on Γ_o and substituting here for the terms in the integrand on Γ_o from (23), we obtain

$$\nu \int_{\Omega} \omega(\boldsymbol{u}) \cdot \omega(\boldsymbol{v})\, \mathrm{d}\boldsymbol{x} + \int_{\Omega} \omega(\boldsymbol{u})\, \boldsymbol{u}^{\perp} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_o} [h_2\, v_2 + h_1\, v_1]\, \mathrm{d}S = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}.$$

This integral equation can be written in the form

$$a(\boldsymbol{u}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v}) + b(\boldsymbol{h}, \boldsymbol{v}) \tag{24}$$

where $\boldsymbol{h}$ is given by (23) and the forms a and b are defined below:

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) \;&=\; a_1(\boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})\ , \quad a_1(\boldsymbol{u}, \boldsymbol{v}) \;=\; \nu \Big(\omega(\boldsymbol{u}), \omega(\boldsymbol{v}) \Big)_{L^2(\Omega)}\ , \\ a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \;&=\; \int_{\Omega} \omega(\boldsymbol{u})\, \boldsymbol{v}^{\perp} \cdot \boldsymbol{w}\, \mathrm{d}\boldsymbol{x}\ , \quad b(\boldsymbol{h}, \boldsymbol{v}) \;=\; -\int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v}\, \mathrm{d}S\ . \end{aligned}$$

The weak problem now reads as follows:

Let function $\boldsymbol{g} \in H^s(\Gamma_i)^2$ (for some $s \in (\frac{1}{2}, 1])$ satisfy the condition $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$ (where A_0 and A_1 are the end points of Γ_i). Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. The weak solution of the problem (20)–(23) is a vector function $\boldsymbol{u} \in H^1(\Omega)^2$ which satisfies the condition of incompressibility (21) a.e. in Ω, the identity (24) for all test functions $\boldsymbol{v} \in V$, the boundary conditions (6), (7) in the sense of traces on Γ_i and Γ_w and the condition of periodicity (8) in the sense of traces on Γ_- and Γ_+.

4.4 Existence and Uniqueness of a weak solution

Now we shall seek for the weak solution $\boldsymbol{u}$ in the form $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z} \in V$ is a new unknown function. This guarantees that $\boldsymbol{u}$ satisfies all the boundary and periodicity conditions ((6)–(8),(22),(23)). Substituting the sum $\boldsymbol{g}^* + \boldsymbol{z}$ for $\boldsymbol{u}$ into the (24), we arrive at the following problem: *Find a function $\boldsymbol{z} \in V$ such that it satisfies the equation*

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v}) \;+\; b(\boldsymbol{h}, \boldsymbol{v}) \tag{25}$$

for all $\boldsymbol{v} \in V$.

Let us formulate the problem of existence of the weak solution as the following theorem:

Theorem There exists $\varepsilon > 0$ such that if $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} < \varepsilon$ then there exists a solution $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ of the problem. Moreover $\boldsymbol{z}$ satisfies the estimate

$$|||\boldsymbol{z}||| \;\leq\; R_2 \;\; .$$

Consequently, the weak problem (25) has a solution $\boldsymbol{u}$ $(= \boldsymbol{z} + \boldsymbol{g}^*)$ that satisfies

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)^2} \;\leq\; R_1 + \|\nabla \boldsymbol{g}^*\|_{L^2(\Omega)^2} \;\leq\; R_1 + c\,\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \;:=\; R_3 \;\; . \tag{26}$$

Here R_2 and c are constants based on the construction of the weak solution in the proof of the theorem. The proof is carried out by using the method of Galerkin approximations. We need to prove the coercivity of the form a and construct the weak solution. The value of the constant ε comes from the proof to ensure coercivity. The complete proof can be found in [3] and in [4].

Remark The special Navier–Stokes formulation (20) naturally leads to the linear "do-nothing" type boundary condition (23) which, on the contrary to the basic linear "do-nothing" boundary condition, enables us to prove existence of the weak solution. The proof is less complicated than in the case of nonlinear boundary condition because of the perpendicular property of the term a_2 and because there is no term a_3.

Let us suppose that $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the weak problem (25).

Theorem There exists $R > 0$ such that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the problem (25) such that $\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \leq R$ then $\boldsymbol{u}_1 \;=\; \boldsymbol{u}_2$.

Remark The proof of the uniqueness for the liner type of do nothing boundary condition is in agreement to the literature and to the theory of the partial differential equations more simple than the proof in the nonlinear case. Moreover we are able to prove uniqueness within the weaker assumption than it the case of nonlinear outlet boundary condition in section three.

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