THE DISCONTINUOUS GALERKIN METHOD WITH DIVERGENCE-FREE ELEMENTS FOR INCOMPRESSIBLE FLOWS

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ABSTRACT

An important advantage of discontinuous Galerkin finite-element methods over classical continuous finite elements, is the fact that the dG basis functions can be endowed with certain special properties that are not available for continuous functions. For instance, one can straightforwardly construct vectorial basis functions that are element-wise divergence-free. However, if one imposes as an additional constraint that these functions are to be continuous, then the construction becomes prohibitively complicated.

Discontinuous Galerkin methods with divergence-free elements, so called solenoidal dG formulations, have remarkable stability properties for incompressible flow problems: due to the upwinding in the convective term and the exact solenoidality within each element, the convective part adds non-negative terms to the variational formulation and, consequently, the dG formulation inherits its stability directly from the underlying Stokes problem. Conventional mixed finite-element methods such as Taylor-Hood and Crouzeix-Raviart elements typically fail to reproduce this behavior. If the pressure space in the finite-element approximation is ”too large”, then the inf-sup condition is violated, and the formulation is unstable. On the other hand, if the pressure space is ”too small”, then the velocity field will not be exactly divergence free, and the contribution of the convective term can become negative. In that case, stability of the Stokes problem does not imply stability of the Oseen problem.

A complication of the solenoidal dG method pertains to the fact that the method is of mixed type. However, instead of the volumetric pressure terms that appear in the conventional mixed formulated, the solenoidal dG formulation contains so-called edge-pressure terms. The edge-pressure terms act as Lagrange multipliers to enforce edge-wise continuity of the normal-velocity component. Hence, to ensure stability of the overall formulation, one must show that these edge-pressure terms satisfy an inf-sup condition, in the appropriate norms for the velocity and the pressure. A stability proof in 2D has been presented by Carrero, Cockburn and Schotzau in 2006. However, the pressure norm that is used in this proof is unsuitable from a continuum viewpoint. In addition, the proof is based on a vorticity formulation, which renders its extension to 3D nontrivial.

In this presentation, we provide an overview of the solenoidal dG method, with particular emphasis on the stability properties of the method. In addition, we present numerical experiments which illustrate the stability and convergence of the solenoidal dG formulation.

REFERENCES

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