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WHEN DOES EDDY VISCOSITY RESTRICT THE DYNAMICS TO LARGE EDDIES?

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Abstract. *Large-eddy simulation (LES) seeks to predict the dynamics of spacially filtered turbulent flows. The very essence is that the LES-solution contains only scales of size* $\geq \Delta$*, where* Δ *denotes the width of the spatial filter. This property enables us to perform a LES when it is not feasible to compute the full, turbulent solution of the Navier-Stokes equations. A large-eddy simulation based on an eddy-viscosity model differs from a Navier-Stokes simulation only in the use of a modified viscosity; hence, the desired effect thereof is a restriction of the dynamics. In the paper, we focus on the question: "when does eddy viscosity restrict the dynamics to scales of size* $\geq \Delta$*?". From this it is deduced that the eddy viscosity* ν_e *has to depend on the invariants* q *and* r *of the filtered strain tensor. The simplest model is then given by* $\nu_e = c^2\,\Delta^2 r^+/q$.

1 Problem setting

The Navier-Stokes equations provide an appropriate model for turbulent flow. In the absence of compressibility ($\nabla \cdot u = 0$), the equations are

$$\partial_t u + (u \cdot \nabla)u + \nabla p - 2\nu \nabla \cdot S(u) = 0, \tag{1}$$

where u is the fluid velocity field, p stands for the pressure, ν denotes the viscosity, and $S(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ is the symmetric part of the velocity gradient. Turbulent flow is generally visualized as a cascade of kinetic energy from large to small scales of motion (eddies). The energy introduced at the large-scale components of the flow is transferred to smaller and smaller eddies until eddies become sufficiently small to dissipate energy efficiently. The entire spectrum - ranging from the wavenumbers where energy is injected to the dissipation range - is to be resolved when turbulence is computed directly from Eq. (1). This is, however, not feasible in many cases (see e.g. Ref. [1]). Large-eddy simulation (LES) can then be a practicable solution. LES seeks to predict the dynamics of spacially filtered turbulent flows. Formally, a spatial operator $u \mapsto \overline{u}$ is introduced that maps the turbulent velocity field u to the velocity $\overline{u}$ of the large scales of motion. Applying this operator to Eq. (1) yields

$$\partial_t \overline{u} + (\overline{u} \cdot \nabla)\overline{u} + \nabla \overline{p} - 2\nu \nabla \cdot S(\overline{u}) = \nabla \cdot (\overline{u}\,\overline{u}^T - \overline{uu^T}) \tag{2}$$

provided $u \mapsto \overline{u}$ commutes with differentiation. The right-hand side depends on both u and $\overline{u}$, due to the nonlinearity. Modeling the right-hand side in terms of $\overline{u}$ yields a simplified representation of the large eddies. The most commonly used model in LES is given by

$$\partial_t v + (v \cdot \nabla)v + \nabla \tilde{p} - 2\nu \nabla \cdot S(v) = 2\nabla \cdot (\nu_e S(v)), \tag{3}$$

where ν_e denotes the eddy viscosity. Here, the variable name is changed from $\overline{u}$ to v to stress that the solution of Eq. (3) differs from that of Eq. (2), because the closure model is not exact.

The classical Smagorinsky model reads

$$\nu_e = C_S^2 \Delta^2 |S(v)| \tag{4}$$

where C_S is the Smagorinsky constant, Δ is the characteristic length scale set by the operator $u \mapsto \overline{u}$ (i.e., the width of the filter) and $|S(v)| = \sqrt{2\,\mathrm{tr}\,(S(v)^2)}$. It may be noticed that the precise definition of $\overline{u}$ is not of much significance in case the Smagorinsky model is used, since Eq. (4) depends only on the characteristic length scale Δ set by the map $u \mapsto \overline{u}$, and not on the details of the mapping. Various value for the Smagorinsky constant have been proposed, mainly ranging from $C_S = 0.1$ to $C_S = 0.17$, see [2], e.g. Instead of adhering to a constant value, C_S is also treated as a model coefficient which is determined during the simulation, that is $C_S = C_S(v)$. In the well-known dynamical procedure the coefficient C_S is computed with the help of the Jacobi identity (in least-square sense) [3].

The solution v of (3) is composed of eddies of different size. The very essence of large-eddy simulation is that v contains only eddies of size $\geq \Delta$, where Δ is the smallest characteristic length scale set by the filter $u \mapsto \overline{u}$. This property enables us to solve (3) numerically when it is not feasible to compute the solution of (1). A simulation based on Eq. (3) differs from a Navier-Stokes simulation only in the use of a modified viscosity; hence, the desired effect thereof is a restriction of the dynamics. Strictly speaking, the eddy viscosity is to be determined such that the corresponding solution v of Eq. (3) forms the 'best' approximation of $\overline{u}$. To that end, however, we have to rely on phenomenological arguments, because we do not know how to derive the 'best' eddy viscosity from the Navier-Stokes equations. In the present approach we try not to make any specific assumptions (about the spectrum, e.g.). Rather, we focus on the question: "when does eddy viscosity damp subfilter scales sufficiently?" Therefore we view the eddy viscosity as a function of v that is to be determined such that any length scales smaller than Δ are damped sufficiently fast. Although the principle is quit apparent it turns out to be nontrivial to obtain a sharp estimate of the minimum amount of eddy viscosity. The basic problem is that the evolution of the subfilter scales is needed for that. To circumvent this problem, we will make use of the Poincaré inequality.

To start, we consider an arbitrary part Ω_Δ with diameter Δ of the flow domain and define the filtering operator $u \mapsto \overline{u}$ by

$$\overline{u} = \frac{1}{|\Omega_\Delta|} \int_{\Omega_\Delta} u(x,t)\, dx$$

In other words, the filtered value of u is equal to the average value of u over Ω_Δ. This filter is known as a box or top-hat filter. Furthermore, we suppose that Ω_Δ is a periodic box. The underlying reason for this assumption is that boundary terms resulting from integration by parts (in the computations to come) vanish. In fact, we can generalize our results to any set of boundary conditions for which the boundary terms vanish. Poincaré's inequality states that there exists a constant C_Δ, depending only on Ω_Δ, such that for every function v in the Sobolev space $W^{1,2}(\Omega_\Delta)$,

$$\int_{\Omega_\Delta} ||v - \overline{v}||^2\, dx \ \leq \ C_\Delta \int_{\Omega_\Delta} ||\nabla v||^2\, dx \tag{5}$$

The optimal constant C_Δ - the Poincaré constant for the domain Ω_Δ - is the inverse of the smallest (non-zero) eigenvalue of the dissipative operator $-\nabla^2$ on Ω_Δ [4]. In Ref. [5] it is shown that the Poincaré constant is given by $C_\Delta = (\Delta/\pi)^2$ for convex domains Ω_Δ.

The residual field $v' = v - \overline{v}$ contains eddies of size smaller than Δ. The eddy viscosity must keep them from becoming dynamically significant. Poincaré's inequality (5) shows that the $L^2(\Omega_\Delta)$ norm of the residual field v' is bounded by a constant (independent of v) times the $L^2(\Omega_\Delta)$ norm of ∇v. Consequently, we can confine the dynamically significant part of the motion to scales $\geq \Delta$ by damping the velocity gradient with the help of an eddy viscosity. This will be worked out in Section 2. As a result the eddy viscosity has to depend on the invariants q and r of the filtered strain tensor $S(v)$. The simplest model is then given by $\nu_e = c^2\,\Delta^2 r^+/q$ (Section 3).

2 When does eddy viscosity counteract the production of sub-filter scales?

We view the eddy viscosity as a function of the velocity v that is to be determined such that the dynamics stays confined to eddies of size $\geq \Delta$; in particularly, the smallest scales of motion that dissipate the energy are to be of size Δ, or larger. To start, we consider both the incompressible Navier-Stokes equations (1) and the LES-model given by Eq. (3) on an arbitrary part Ω_ℓ of the flow domain, where Ω_ℓ has diameter $\ell \geq \Delta$ and is supplied with periodic boundary conditions. Initially, say at time $t = 0$, the Navier-Stokes solution u is given by $u(x, 0) = u_0(x)$, for all $x \in \Omega_\ell$. The initial condition for the LES-model reads $v(x, 0) = \overline{u}_0(x)$. The initial conditions supply energy to the flow. This energy cannot escape from Ω_ℓ, since we have applied periodic conditions. Hence, the energy is to be dissipated within Ω_ℓ. According to the Navier-Stokes equations the evolution of the energy $E(t) = \int_{\Omega_\ell} \frac{1}{2}||u||^2 dx$ is given by $dE/dt = -\epsilon$ with $\epsilon = \nu \int_{\Omega_\ell} ||\nabla u||^2 dx$. The dissipation rate of the LES-model becomes $\int_{\Omega_\ell} (\nu + \nu_e)||\nabla v||^2 dx$. In the absence of eddy viscosity, i.e., $\nu_e = 0$, this integral is much smaller than ϵ if $v \approx \overline{u}$. Indeed, the mapping $u \mapsto \overline{u}$ reduces the velocity gradient. Now suppose that the amount of eddy-viscosity is taken too little. Then, $||\nabla v||^2$ will have (a tendency) to increase, because the energy that is supplied to the flow has to be dissipated anyway. Since the norm of the velocity gradient $||\nabla v||$ provides a consistent characterization of the reciprocal of the time scale, an increase of $||\nabla v||$ implies that smaller time-scales are produced. Then, the eddies of scale ℓ in the velocity field v are unstable and break up, transferring their energy to smaller eddies. These smaller eddies undergo a similar break-up process, and transfer their energy to smaller eddies, and so on till the energy can be dissipated effectively. So, in conclusion, an increase of $\int_{\Omega_\ell} ||\nabla v||^2 dx$ indicates that scales with a length smaller than ℓ are produced. In a LES this causes no problem if $\ell > \Delta$. But, in order to confine the dynamics to scales $\geq \Delta$ this process has to stop at the scale set by the filter. Therefore, we determine the eddy viscosity from the requirement that

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \tfrac{1}{2}||\nabla v||^2 \, dx = -\nu \int_{\Omega_\Delta} ||\nabla^2 v||^2 \, dx \tag{6}$$

This condition can also be derived in a more formal way by utilizing Poincaré's inequality (5). Poincaré's inequality states that the $L^2(\Omega_\Delta)$ norm of the residual field $v' = v - \overline{v}$ is bounded by a constant C_Δ (independent of v) times the $L^2(\Omega_\Delta)$ norm of ∇v. Consequently, the dynamically significant part of the motion can be confined to scales $\geq \Delta$ by damping the velocity gradient with the help of an eddy viscosity. To see how the evolution of the $L^2(\Omega_\Delta)$ norm of ∇v is to be damped, we consider the residual field v' first:

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \tfrac{1}{2}||v'||^2 \, dx = \int_{\Omega_\Delta} T\left(\overline{v}, v'\right) \, dx - \nu_e \int_{\Omega_\Delta} ||\nabla v'||^2 \, dx - \nu \int_{\Omega_\Delta} ||\nabla v'||^2 \, dx$$

Here, $\int_{\Omega_\Delta} T(\overline{v}, v')\, dx$ represents the energy transfer from $\overline{v}$ to v'. It goes without saying that the energy of v' has to decrease quickly, since Eq. (3) should not produce subfilter scales. Ideally, the eddy viscosity is taken such that the first two terms in the right-hand side above cancel each other out, that is $\int_{\Omega_\Delta} T(\overline{v}, v')\, dx = \nu_e \int_{\Omega_\Delta} ||\nabla v'||^2\, dx$. Then we have

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \tfrac{1}{2}||v'||^2\, dx = -\nu \int_{\Omega_\Delta} ||\nabla v'||^2\, dx \tag{7}$$

This equation shows that the evolution of the energy of v' is not depending on $\overline{v}$. Stated otherwise, the energy of subfilter scales dissipates at a natural rate, without any forcing mechanism involving scales larger than Δ. In this way, the scales $< \Delta$ are separated from scales $\geq \Delta$. With the help of the Poincaré inequality (5), we obtain from Eq. (7) that

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \tfrac{1}{2}||v'||^2(x,t)\, dx \;\leq\; -2\nu/C_\Delta \int_{\Omega_\Delta} \tfrac{1}{2}||v'||^2(x,t)\, dx$$

The Gronwall lemma leads then to

$$\int_{\Omega_\Delta} \tfrac{1}{2}||v'||^2(x,t)\, dx \;\leq\; \exp\left(-2\nu t/C_\Delta\right) \int_{\Omega_\Delta} \tfrac{1}{2}||v'||^2(x,0)\, dx$$

In other words, the energy of the subfilter scales decays at least as fast as $\exp(-2\nu t/C_\Delta)$, for *any* filter length Δ. Applying Poincaré's inequality and Gronwalls lemma to Eq. (6) results into the same rate of decay:

$$\begin{aligned} \int_{\Omega_\Delta} \tfrac{1}{2}||v'||^2(x,t)\, dx &\overset{(5)}{\leq} C_\Delta \int_{\Omega_\Delta} \tfrac{1}{2}||\nabla v||^2(x,t)\, dx \\ &\overset{(6)}{\leq} C_\Delta \exp\left(-2\nu t/C_\Delta\right) \int_{\Omega_\Delta} \tfrac{1}{2}||\nabla v||^2(x,0)\, dx \end{aligned}$$

The decay rate of individual subfilter scales of motion can be determined from the dissipative operator $-\nabla^2$ on Ω_Δ in the space-periodic case with vanishing space average. This positive, self-adjoint operator possesses a sequence of positive eigenvalues μ_n. We can order them such that $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots$ The first eigenvalue, μ_1, equals $1/C_\Delta$. The eigenfunctions e_n form an orthogonal basis; hence we can write, $v'(x,t) = \sum_n v'_n(t) e_n(x)$, and associate a scale of motion with a term of the sum. Since the dissipative condition (7) holds for any residual field v', we obtain that the behavior of the n-th scale is given by

$$v'_n(t) \sim \exp\left(-\nu \mu_n t\right) \tag{8}$$

where $n = 1, 2, \ldots$ This holds for any n if and only if Eq. (7) holds for any v'. In Ω_Δ the velocity can be written as $v = \overline{v} + v'$, where $\overline{v}$ is constant (the average of v over Ω_Δ) and v' is periodic with vanishing average. Consequently, $\nabla v = \nabla(\overline{v} + v') = \nabla v'$. Therefore, Eq. (8) holds for every n if and only if Eq. (6) holds for any v with $\nabla \overline{v} = 0$ in Ω_Δ. So, in conclusion, we can impose Condition (6) on the decay rate of the $L^2(\Omega_\Delta)$ norm of ∇v, and thus keep the subfilter component v' under control. The (minimum) amount of eddy viscosity needed to satisfy the dissipative condition (6) can be derived by taking the L^2 innerproduct of Eq. (3) with $\nabla^2 v$. Integration by parts yields

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \tfrac{1}{2} ||\nabla v||^2 \, dx = \int_{\Omega_\Delta} \Big(\, (v \cdot \nabla) v \cdot \nabla^2 v \; - \; (\nu + \nu_e) \, ||\nabla^2 v||^2 \, \Big) \, dx \tag{9}$$

where ν_e is assumed to be constant in Ω_Δ. As remarked before, the boundary terms that result from the integration by parts vanish because Ω_Δ is a periodic box. By comparing Eq. (6) with Eq. (9) we get

$$\nu_e \int_{\Omega_\Delta} ||\nabla^2 v||^2 \; dx = \int_{\Omega_\Delta} (v \cdot \nabla) v \cdot \nabla^2 v \; dx \tag{10}$$

In Ref. [8] (page 791-792) it is shown that, for a periodic box, the convective term in the right-hand side of (10) is equal to $4 \int_{\Omega_\Delta} r(v) \, dx$, where

$$r(v) = -\tfrac{1}{3} \mathrm{tr}(S^3(v))$$

is an invariant of the rate of strain tensor $S(v)$. It may be remarked here that the calculations in Ref. [8] are done for the 3D Euler equations; yet one can add the viscous term to each step of the calculations in Ref. [8]. The other invariant of $S(v)$,

$$q(v) = \tfrac{1}{2} \mathrm{tr}(S^2(v)),$$

has the property that $4 \int_{\Omega_\Delta} q(v) \, dx = \int_{\Omega_\Delta} ||\nabla \times v||^2 \, dx$. See again Ref. [8] for instance. Since $\nabla^2 v = -\nabla \times \omega$ with $\omega = \nabla \times v$, it follows that Eq. (10) is equivalent to

$$\nu_e \int_{\Omega_\Delta} q(\omega) \, dx \;\; = \;\; \int_{\Omega_\Delta} r(v) \, dx \tag{11}$$

So, in conclusion, the eddy-viscous damping in Eq. (9) counteracts the nonlinear production in Eq. (9) if the eddy viscosity is taken according to Eq. (11). A noticeable difference between this result and the standard Smagorinsky model (4) with C_S constant is that the standard model depends only on the invariant q (read: not on r). Notice that $|S(v)| = \sqrt{2\mathrm{tr}(S^2)} = \sqrt{4q(v)}$. The role of the invariant $r(v)$ can be explained with the help of the vorticity ω. By taking the curl of Eq. (3) we find the vorticity equation and from that we obtain that the enstrophy is governed by

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \tfrac{1}{2} ||\omega||^2 \, dx = \int_{\Omega_\Delta} \omega \cdot S\omega \, dx - (\nu + \nu_e) \int_{\Omega_\Delta} ||\nabla \omega||^2 \, dx$$

In the right-hand side we recognize the vortex stretching term that can produce smaller scales of motion and the dissipative term that should counteract the production of smaller scales at the scale Δ. Now, Eq (11) leads to

$$\nu_e \int_{\Omega_\Delta} ||\nabla\omega||^2 \, dx = \nu_e \int_{\Omega_\Delta} 4\, q(\omega)\, dx \stackrel{(11)}{=} \int_{\Omega_\Delta} 4\, r(v)\, dx = \int_{\Omega_\Delta} \omega \cdot S\omega \, dx$$

Notice that the latter equality shows that $r(v)$ is a measure for the vortex stretching [8]. Thus, Eq. (11) can also be interpreted as follows: the eddy viscosity is taken such that the corresponding damping of the enstrophy, $\nu_e \int_{\Omega_\Delta} ||\nabla\omega||^2 \, dx$, equals the production by means of the vortex stretching mechanism, $\int_{\Omega_\Delta} \omega \cdot S\omega \, dx$. In other words, the eddy viscosity prevents the intensification of vorticity at the scale Δ set by the map $u \mapsto \overline{u}$.

3 An eddy-viscosity model

The dissipative term in Eq. (11) can be bounded from below with the help of Poincaré's inequality:

$$\int_{\Omega_\Delta} q(\omega)\, dx \;=\; \int_{\Omega_\Delta} \tfrac{1}{4} ||\nabla\omega||^2 \, dx \;\geq\; \frac{1}{C_\Delta} \int_{\Omega_\Delta} \tfrac{1}{4} ||\omega||^2 \, dx \;=\; \frac{1}{C_\Delta} \int_{\Omega_\Delta} q(v)\, dx$$

where the equality-sign holds if ω is fully aligned with the first eigenfunction, e_1, of the dissipative operator $-\nabla^2$ on Ω_Δ. So, in conclusion, the eddy-viscous term in Eq. (11) dominates the nonlinear, convective term if

$$\nu_e \int_{\Omega_\Delta} q(\omega)\, dx \;\geq\; \boxed{\frac{\nu_e}{C_\Delta} \int_{\Omega_\Delta} q(v)\, dx \;\geq\; \int_{\Omega_\Delta} r(v)\, dx} \tag{12}$$

The latter condition ensures that subfilter scales are dynamically insignificant, meaning that their energy is bounded by (5) where the upper bound evolves according to

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \tfrac{1}{2} ||\nabla v||^2 \, dx \leq -\nu \int_{\Omega_\Delta} ||\nabla^2 v||^2 \, dx$$

Since this condition is stronger than (6), we can conclude that the energy of the scales of size $\leq \Delta$ decays at least as fast as $\exp(-2\nu t/C_\Delta)$, for any filter length Δ, if the eddy-viscosity is taken such that (12) holds. Now, the question is: does the minimal amount of eddy viscosity satisfying (12), i.e.,

$$\nu_e = C_\Delta \frac{\int_{\Omega_\Delta} r(v)\, dx}{\int_{\Omega_\Delta} q(v)\, dx}, \tag{13}$$

adequately model the subfilter contributions to the evolution of the filtered velocity?

Here, it may be noted that Eq. (13) relates the eddy viscosity to $q(v)$, $r(v)$ and C_Δ:

$$\nu_e = \nu_e(C_\Delta, q, r)$$

where $C_\Delta = (\Delta^2/\pi)^2$. A dimensional analysis shows that the general form of an eddy viscosity model depending on Δ, r and q is

$$\nu_e = \Delta^2 \sqrt{q} \;\; \sum_n c_n \left(r/\sqrt{q^3}\right)^n$$

where c_n denotes a dimensionless constant. Thus, in terms of a non-constant coefficient, we have

$$C_S^2 = C_S^2 \left(r/\sqrt{q^3}\right)$$

The latter relation can be characterized further by imposing (exact) properties of the filtered Navier-Stokes equations. The reversibility property, for example, limits C_S^2 to odd functions of $r/\sqrt{q^3}$. Consequently, we have $C_S^2(0) = 0$.

3.1 Modeling consistency

Applying a box filter to a solution u of the Navier-Stokes equations (1) yields the field $\overline{u}$. This filtered field is, by construction, a solution to the filtered Navier-Stokes equation (2). By substituting $\overline{u}$ into Eq. (3) we can get an idea about the consistency of the eddy-viscosity model. Notice that all data comes from the Navier-Stokes equations: we take $v = \overline{u}$ in Eq. (3) and compare the result with Eq. (2). This is also called *a priori* testing. In this way, we see that the approximation

$$(\overline{u}\,\overline{u}^T - \overline{uu^T})_{tr} \;\approx\; 2\,\nu_e\, S(\overline{u})$$

causes the difference between Eq. (2) and Eq. (3) with $v = \overline{u}$. Here, we use the notation $(A)_{tr} = A - \frac{1}{3}\mathrm{tr}(A)I$. Notice that we can focus on the traceless part because the trace of $\overline{u}\,\overline{u}^T - \overline{uu^T}$ can be incorporated into the pressure; hence need not be modeled. The series expansion (up to terms of the order Δ^4) of the left-hand side in the approximation above reads

$$(\overline{u}\,\overline{u}^T - \overline{uu^T})_{tr} = -\frac{\Delta^2}{12}(\nabla\overline{u}\,\nabla\overline{u}^T)_{tr} + \mathcal{O}(\Delta^4)$$

The leading term is known as the gradient model [9]. Unfortunately, the gradient model cannot be used as a stand-alone LES model, since it produces a finite time blow-up of the kinetic energy [10]-[11]. In other words, the gradient model can produce length-scales smaller than Δ. However, we can project both the eddy-viscosity model given by Eq. (13) and gradient model onto $S(v)$, and study the differences between the models under this projection:

$$C_\Delta \; \frac{\overline{r}}{\overline{q}} \int_{\Omega_\Delta} 2S(v) : S(v)\, dx \stackrel{?}{=} -\frac{\Delta^2}{12} \int_{\Omega_\Delta} \nabla v \nabla v^T : S(v)\, dx \qquad (14)$$

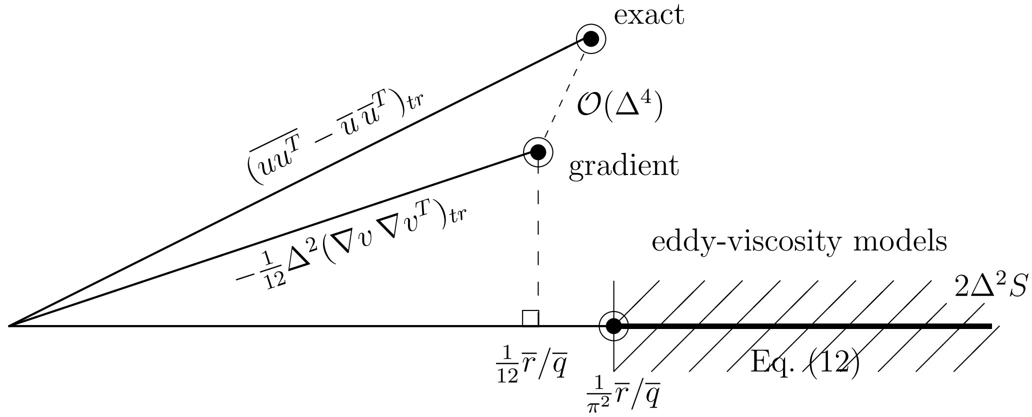

Figure 1: Some LES-models in the space of trace-less 3x3 tensors.

The integral in the right-hand side equals $-4\int_{\Omega_\Delta} r(v)\,dx$; details of the computation of the integral can be found in Ref. [8] on page 791-792. This shows that $r(v)$ provides a measure of the alignment of the gradient model and $S(v)$. By definition we have $S(v) : S(v) = 2q(v)$. Consequently, we obtain from Eq. (14) that $C_\Delta \approx \Delta^2/12$. This value is in fair agreement with the Poincaré constant, $C_\Delta = \Delta^2/\pi^2$; yet, it is a slightly lower. The overall situation is sketched in Fig. 3.1. The horizontal axis in this figure represents all possible eddy-viscosity models; the axis is parameterized by the eddy viscosity. The shaded part of the horizontal axis in Fig. 3.1 depicts the subset of eddy viscosities that satisfy Eq. (12). The projection of the gradient model onto the horizontal axis falls outside the shaded area; hence it cannot be guaranteed that this projection damps subfilter scales adequately. This seems to reflect that the gradient model produces subfilter scales in some cases. The model given by Eq. (13) forms the best approximation of the projection of the gradient model provided the eddy viscosity model is restricted by Eq. (12). This justifies to take the minimal value that satisfies (12), i.e., to take the eddy viscosity as in Eq. (13).

3.2 Properties

Since vortex stretching is absent in laminar flows, we have $r = 0$ in any laminar flow. Hence, Eq. (13) yields $\nu_e = 0$ in any laminar (part of the) flow. Here it may be remarked that the Smagorinsky model (4) predicts a nonvanishing eddy viscosity in regions where the flow is laminar. Furthermore, we have $r = 0$ in any two-dimensional flow. Therefore, Eq. (13) leads to $\nu_e = 0$ for any two-dimensional flow. In this manner, Eq. (13) recognizes the salient differences of 2D and 3D turbulence. Another shortcoming of the Smagorinsky model (4) is that the eddy viscosity does not vanish at no-slip walls. The near-wall behavior of the invariants $r(v)$ and $q(v)$ is given by $r \propto y^3$ and $q \propto y^0$, respectively, where y denotes the distance to the wall. Thus (13) results into an eddy viscosity that vanishes at the wall (as it should): $\nu_e \propto y^3$. In homogeneous, isotropic turbulence, the invariants

$r(v)$ and $q(v)$ scale with the Reynolds number Re like $r \propto Re^{3/2}$ and $q \propto Re^1$, respectively. Hence, the quotient of r and q scales like $r/q \propto Re^{3/2}/Re = Re^{1/2}$. Therefore $\nu_e/\nu \propto Re^{3/2}$ for fixed Δ. Additionally, we obtain that $\nu_e + \nu \rightarrow \nu$ if $\nu \propto Re^{-1} \propto \Delta^2 r/q \propto \Delta^2 Re^{1/2}$, that is if $\Delta \propto Re^{-3/4}$. This shows that the eddy viscosity given by Eq. (13) vanishes as Δ is of the order of $Re^{-3/4}$, i.e., if Δ approaches the smallest (Kolmogorov) scale in homogeneous, isotropic turbulence.

3.3 A qr-model

To compute the eddy viscosity ν_e according to Eq. (13), we need know how q and r vary within Ω_Δ. Here, we cannot simply take $\overline{q(v)} = q(\overline{v})$, because the relation between q and v is nonlinear. On the other hand, however, we do not want to compute v' explicitly. In a finite volume approximation, for instance, we resolve only $\overline{v}$ if the grid size is taken equal to Δ, cf. [12]. To express the eddy viscosity in terms of $\overline{v}$, we have to apply an approximate deconvolution method that recovers some of the information lost in the filtering process. To recover an approximation for v' we consider the series expansion of v around $\overline{v}$. Ignoring terms that are of the order Δ^4, we get the approximation

$$v' \approx -\tfrac{1}{24}\,\Delta^2\ \nabla^2\overline{v} \tag{15}$$

In homogeneous, isotropic turbulence we have $r \propto Re^{3/2}$ and $q \propto Re^{1/2}$; hence the ratio of r and $q^{3/2}$ scales like Re^0. This scaling law suggests to take $\overline{r(v)}/\overline{q(v)}^{3/2} \approx r(\overline{v})/q(\overline{v})^{3/2}$. Thus Eq. (13) leads to

$$\nu_e \;\approx\; C_\Delta\ \frac{r(\overline{v})}{q(\overline{v})^{3/2}}\ \left(\frac{\overline{q(v)}}{q(\overline{v})}\right)^{1/2}\ \sqrt{q(\overline{v})}$$

Furthermore with the help of Eq. (15) it can be shown that

$$\begin{aligned}\overline{q(v)} \;&=\; \tfrac{1}{4}\overline{||\nabla v||^2} \;\overset{(15)}{\approx}\; \tfrac{1}{4}\overline{||\nabla(\overline{v} - \tfrac{1}{24}\,\Delta^2\,\nabla^2\overline{v})||^2}\\ &\leq\; \tfrac{1}{4}(1+\tfrac{1}{24}\Delta^2/C_\Delta)^2\overline{||\nabla\overline{v}||^2} \;=\; (1+\tfrac{1}{24}\pi^2)^2\ \overline{q(\overline{v})}\end{aligned}$$

So by taking $\overline{q(\overline{v})} \approx q(\overline{v})$ we obtain $\nu_e = c^2\Delta^2\, r(\overline{v})/q(\overline{v})$ where the constant c^2 is given by $\frac{1}{\pi^2} + \frac{1}{24} \approx 0.143$. Since the eddy viscosity is determined such that the solution v of Eq. (3) is restricted to scales of size $\geq \Delta$, we can avoid the explicit application of the filter here by taking $r(\overline{v})/q(\overline{v}) \approx r(v)/q(v)$. Actually, this approximation forms the basis of finite-volume methods. Then the eddy-viscosity model becomes

$$\nu_e = c^2\Delta^2\,\frac{r(v)}{q(v)} \tag{16}$$

This expression is invariant under rotation of coordinate axis, since it depends on the invariants of $S(v)$. The qr-model (16) can be put into the standard notation (4) by introducing the relation

$$C_S^2 = \frac{c^2\, r}{2\sqrt{q^3}} \tag{17}$$

where again $c^2 = \frac{1}{\pi^2} + \frac{1}{24}$.

In homogeneous, isotropic turbulence we have $C_S^2 \propto r/\sqrt{q^3} \propto Re^0$, i.e., the Smagorinsky coefficient is (in lowest order) independent of the Reynolds number Re. So, if we average Eq. (17) over the homogeneous directions we obtain an approximately constant coefficient C_S^2 that is valid for a wide range of Reynolds numbers (in case of homogeneous, isotropic turbulence). This partially agrees with Smagorinsky's reasoning, in which C_S^2 is taken constant (once again: provided that $r \propto Re^{3/2}$ and $q \propto Re^{1/2}$).

The three roots of the characteristic polynomial of $S(v)$ must be real-valued because $S(v)$ is symmetric. This requirement leads to the constraint $27r^2 \leq 4q^3$. Hence, Eq. (16) yields an eddy viscosity in the range

$$-\frac{c^2\Delta^2}{3\sqrt{3}}\,|S| \;\leq\; \nu_e \;\leq\; \frac{c^2\Delta^2}{3\sqrt{3}}\,|S|$$

Consequently, the largest value of the Smagorinsky coefficient C_S is equal to $\frac{c}{27^{1/4}} \approx 0.17$. Remarkably this maximum value is identical to Lilly's value, $C_S = 0.17$ [7], which implies that the standard Smargorinsky model (4) with $C_S = 0.17$ satisfies (12). Stated differently, the standard model with $C_S = 0.17$ stops the production of scales $< \Delta$. Interestingly, the value $C_S = 0.17$ has been found too large in many numerical experiments. In turbulent shear flow, for instance, the value of the coefficient C_S is often reduced to the relatively low value $C_S = 0.1$ to give the standard model a fair change for success.

3.4 Backscatter

The qr-model given by Eq. (16) predicts a negative eddy viscosity in case $r < 0$. In that case, energy is transferred from the smallest scale Δ to larger scales (backscatter). This does not contradict with the energy cascade concept: at certain times and positions backscatter occurs, indeed, but the average transfer is from large scales to smaller ones. Eddy-viscosity models are intrinsically not very suited to represent backscatter, since backscatter is often anisotropic and negative values of the eddy viscosity are to be limited in order to keep the problem well posed. That is, we have to impose the condition $\nu + \nu_e \geq \alpha\nu$, with $\alpha > 0$, to ensure that (3) is well posed.

Taking $\alpha = 1$ leads to a purely dissipative model:

$$C_S^2 = \frac{c^2\, r^+}{2\sqrt{q^3}} \tag{18}$$

where $r^+ = \max\{0, r\}$. In other words, we set the eddy viscosity ν_e equal to zero if $r < 0$. Notice that Condition (12) is trivially satisfied by taking $\nu_e = 0$ in case $r < 0$.

In this way, we get a solution v depending on the molecular viscosity ν, i.e., on the Reynolds number Re. Furthermore, it may be remarked that $r < 0$ implies that $\lambda_2 < 0$. An infinitesimal fluid volume is then stretched in one direction and compressed in the other two directions (linear stretching), whereas $r > 0$ corresponds to stretching in two directions and compression in one (planar stretching). In other words, the dissipative model given by Eq. (18) damps only planar stretching.

Of course, we may include backscatter. The amount of backscatter can be controlled with the help of the parameter α. Accordingly, the eddy viscosity is to be limited such that $\nu_e \geq (\alpha - 1)\nu$ with $0 < \alpha \leq 1$. However, care needs to be taken, because the clipping procedure (in particularly, the choice of α) is just an ad hoc approach to include backscatter in an originally dissipative model.

3.5 First results

In summary, the qr-model given by Eq. (16) has the following properties: (a) $v_e = 0$ in any laminar flow; (b) $v_e = 0$ in any 2D flow; (c) $v_e \propto y^3$ near a wall $y = 0$; (d) $v_e \to 0$ if $\Delta \propto Re^{-3/4}$; (e) $C_S \leq 0.17$. It goes without saying that the performance of the eddy-viscosity model (16) has to be investigated for many cases. As a first step it was tested (without backscatter, that is with C_S given by Eq. (18)) for turbulent channel flow by means of a comparison with direct numerical simulations. This flow forms a prototype for near-wall turbulence: virtually every LES has been tested for it. The results are compared to the DNS data of Moser *et al.* [13] at $\mathrm{Re}_\tau = 590$. In fact, we should compare the LES-solution v to the filtered DNS-solution $\overline{u}$. Yet, since the filtered DNS-solution is not presented in Ref. [13] we will compare v directly to u. The dimensions of the channel are taken identical to those of the DNS of Moser *et al.* The computational grid used for the large-eddy simulation consists of 64^3 points. The DNS was performed on a 384x257x384 grid, i.e., the DNS uses about 144 times more grid points than the present LES. The LES-results were obtained with an incompressible code that uses a fourth-order, symmetry-preserving, finite-volume discretization. Details about the numerics can be found in Ref. [14]. The width of the filter is the only LES-parameter that need be specified. On a uniform grid with spacing dx, dy and dz we can simply take $\Delta = dx = dy = dz$. On a nonuniform grid we use the Poincaré constant C_Δ to specify Δ. Analytically, the Poincaré constant is equal to the inverse of the smallest (non-zero) eigenvalue of the dissipative operator $-\nabla^2$ on Ω_Δ. Since this eigenvalue cannot be represented on the grid, we replace it by the largest, representable eigenvalue, which is given by $4/dx^2 + 4/dy^2 + 4/dz^2$. In this way, we get

$$\Delta^2 = \frac{3}{1/dx^2 + 1/dy^2 + 1/dz^2}$$

where the constant in the numerator is chosen such that $\Delta = dx = dy = dz$ on a uniform grid. It may be stressed that this definition differs from the usual expression $\Delta = (dx\,dy\,dz)^{1/3}$, if the grid is (strongly) nonuniform. Further it may be emphasized

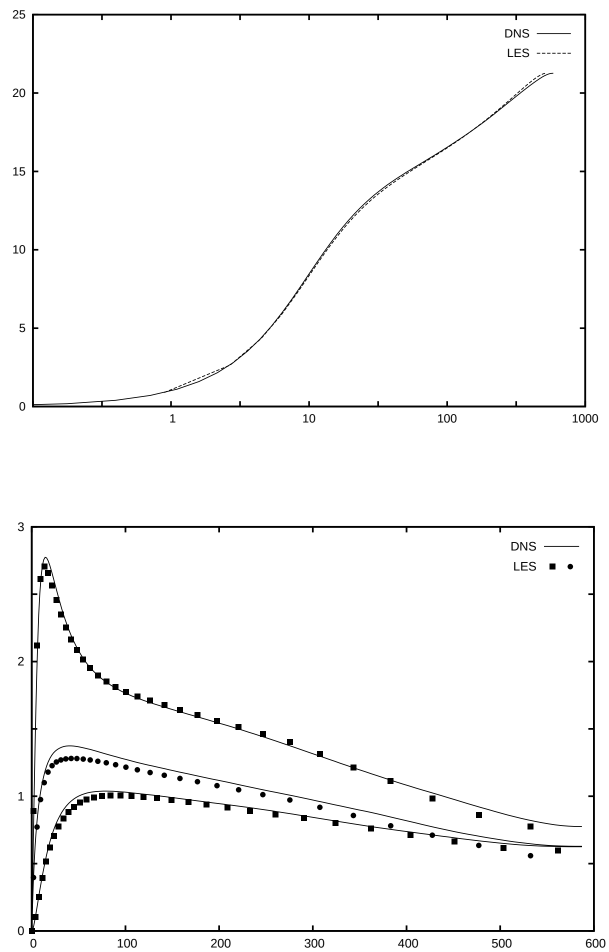

Figure 2: The upper figure shows the mean velocity (in wall coordinates) obtained with the help of the 64^3 LES and the DNS by Moser et al. The lower figure displays the root-mean-square of the fluctuating velocities. The boxes and circles represent LES data; every symbol corresponds to data in a grid point.

that the eddy-viscosity model (16) is essentially not more complicated to implement in a LES-code than the standard Smagorinsky model (with C_S constant). Indeed, the qr-model is expressed in terms of the invariants of the rate-of-strain tensor and does not involve explicit filtering. The invariant $q = \frac{1}{4}|S|^2$ is to be computed in any case; the computation of r is just as difficult. Unlike the standard Smagorinsky model (even with the relatively low value $C_S = 0.1$), the present model showed an appropriate behavior. As can be seen in Figure 2 both the mean velocity and the root-mean-square of the fluctuating velocity are in good agreement with the DNS.

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