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NUMERICAL MODELING OF TRANSIENT FLOWS INVOLVING EROSION AND DEPOSITION OF SEDIMENTS

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Abstract. *We present a robust numerical methods for solving transient flows involving erosion and deposition of sediments. The governing equations consist of three components, namely a hydrodynamical part described by the shallow water equations, a morphologycal part described by the Exner equation and a sediment transport part modelled by an advection equation accounting for erosion and deposition effects. The coupled equations form a hyperbolic system of conservation laws with source terms. Approximating numerical solution to this system is not trivial due to the coupling between the hydrodynamics and morphodynamics, presence of the source terms and the disparity of time scales for the water waves and sediment loads. In the current study we propose a finite volume method formulated for the one-dimensional problems. The numerical fluxes are reconstructed using a modified Roe's scheme that incorporates, in its reconstruction, the sign of the Jacobian matrix in the morphodynamic system. A well-balanced discretization is used for the treatment of source terms. The method is well-balanced, non-oscillatory and suitable for both slow and rapid interactions between hydrodynamics and morphodynamics. Numerical results are presented for a test example of dam-break over a movable bed. The obtained results for this test example are considered to be representative, and might be helpful for a fair rating of finite volume solution schemes, particularly in transient flow regimes.*

1 INTRODUCTION

The main concern of the sediment transport (or morphodynamics) is to determine the evolution of bed levels for hydrodynamics systems such as rivers, estuaries, bays and other nearshore regions where water flows interact with the bed geometry. Example of applications include among others, beach profile changes due to severe wave climates, seabed response to dredging procedures or imposed structures, and harbour siltation. The ability to design numerical methods able to predict the morphodynamics evolution of the coastal seabed has a clear mathematical and engineering relevances. In practice, morphodynamics involve coupling between a hydrodynamics model, which provides a description of the flow field leading to a specification of local sediment transport rates, and an equation for bed level change which expresses the conservative balance of sediment volume and its continual redistribution with time. Here, the hydrodynamic model is described by the shallow water equations, the bed-load is modelled by the Exner equation, and the suspended sediment transport is modelled by an advection equation accounting for erosion and deposition effects. The coupled models form a hyperbolic system of conservation laws with a source term.

Nowadays, much effort has been devoted to develop numerical schemes for morphodynamic models able to resolve all hydrodynamic and morphodynamic scales. Special attention has been given to the treatment of source term and the bed-load flux. It is well known that shallow water equations on nonflat topography have steady-state solutions in which the flux gradients are nonzero but exactly balanced by the source terms. This well-balanced concept is also known by conservation property (C-property), compare [9, 15] among others. The well-established Roe's scheme [13] has been modified in [6] for the sediment transport problems. However, for practical applications, this method may become computationally demanding due to its treatment of the source terms. Numerical methods based on Euler-WENO techniques have also been applied to sediment transport equations in [12]. Authors in [14] extended the ENO and WENO schemes to sediment transport equations, whereas the CWENO method has been applied to sediment transport problems in [10]. Unfortunately, most ENO, WENO and CWENO methods that solves real morphodynamic models correctly are still very computationally expensive. On the other hand, numerical methods using the relaxation approximation have also been applied to sediment transport equations in [11]. It is well known that TVD schemes have their order of accuracy reduced to first order in the presence of shocks due to the effects of limiters.

In the current study, a class of finite volume methods is proposed for numerical simulation of transient flows involving erosion and deposition of sediments. The method consists of a predictor stage where the numerical fluxes are constructed and a corrector stage to recover the conservation equations. The sign matrix of the Jacobian matrix is used in the reconstruction of the numerical fluxes. Most of these techniques have been recently investigated in [3, 2] for solving sediment transport models without accounting for erosion and deposition effects. The current study presents an extension of this method to transient flows involving erosion and deposition of sediments. A detailed formulation of the sign matrix and the numerical fluxes is presented. The proposed method also satisfies the property of well-balancing flux-gradient and source-term in the system. Numerical results are shown for a dam-break problem over a Mobil bed.

2 GOVERNING EQUATIONS FOR SEDIMENT TRANSPORT

In the present work, we consider the following one-dimensional sediment transport model [16]

$$\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} &= \frac{E-D}{1-p}, && \text{(1a)}\\
\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) &= -gh\left(\frac{\partial Z}{\partial x} - S_f\right) - \frac{(\rho_s-\rho_w)g}{2\rho}h^2\frac{\partial c}{\partial x} - \\
&\quad \frac{(\rho_0-\rho)(E-D)u}{\rho(1-p)}, && \text{(1b)}\\
\frac{\partial (hc)}{\partial t} + \frac{\partial (huc)}{\partial x} &= E-D, && \text{(1c)}\\
\frac{\partial Z}{\partial t} + A\xi\frac{\partial u^3}{\partial x} &= -\frac{E-D}{1-p}, && \text{(1d)}
\end{aligned}$$

where t is the time, x is the streamwise coordinate, h is the water depth, u is the depth-averaged streamwise velocity, Z is the bed elevation, c is the flux-averaged volumetric sediment concentration, g is the gravitational acceleration, S_f is the friction slope, p is the bed sediment porosity, E and D are sediment entrainment and deposition fluxes across the bottom boundary of flow, representing the sediment exchange between the water column and the bed. The density of the water-sediment mixture is defined by

$$\rho = \rho_w(1-c) + \rho_s c, \tag{2}$$

where ρ_w and ρ_s are the densities of water and suspended sediment, respectively. The density of the saturated bed is given by

$$\rho_0 = \rho_w p + \rho_s(1-p), \tag{3}$$

with p denotes the bed sediment porosity. In (1),

$$\xi = \frac{1}{1-p},$$

and A is the Grass constant for the sediment transport flux. To close the governing equations, a conventional empirical relation is used to determine the friction slope involving the Manning roughness n_b,

$$S_f = \frac{n_b^2 u^2}{h^{4/3}}. \tag{4}$$

Since the sediment exchange between the water and the bed is a vital process for morphological evolution, a large number of empirical relations have been proposed to determine the entrainment and deposition fluxes. For deposition of non-cohesive sediment, this study uses the relation :

$$D = w(1 - C_a)^m C_a, \tag{5}$$

where w is the settling velocity of a single particle in tranquil water

$$\omega = \frac{\sqrt{(36\nu/d)^2 + 7.5\rho_s g d - 36\nu/d}}{2.8}, \tag{6}$$

with ν is the kinematic viscosity of water, d is the averaged diameter sediment particles, m is the exponent indicating the effects of hindered settling due to high sediment concentrations, C_a is the near-bed volumetric sediment concentration, $C_a = \alpha_c c$ where c is the depth averaged volumetric sediment concentration and α_c is a coefficient larger than unity. In order so that the near-bed concentration does not exceed $(1 - p)$, the coefficient α_c is computed by $\alpha_c = min(2, \frac{1-p}{c})$. For the entrainment of cohesive material the following relation is used

$$E = \begin{cases} \varphi(\theta - \theta_c) u h^{-1} d^{-0.2}, & \text{if } \theta \geq \theta_c \\ 0, & \text{else} \end{cases} \tag{7}$$

where φ is a coefficient to control the erosion forces, θ_c is the critical value of Shield parameter for the initiation of sediment motion and the Shields parameter

$$\theta = \frac{u_*^2}{sgd},$$

with

$$u_*^2 = \sqrt{\frac{f}{8}}\,|u|,$$

is the friction velocity, with f is the Darcy-Weisbach friction factor and s is the submerged specific gravity of sediment defined as

$$f = \frac{8gn^2}{h^{\frac{1}{3}}}, \qquad s = \frac{\rho_s}{\rho_w} - 1.$$

For simplicity in presentation we rewrite the system (1) can be rearranged in the following vector form

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} = \mathbf{S}(\mathbf{W}). \tag{8}$$

where

$$\mathbf{W} = \begin{pmatrix} h \\ hu \\ hc \\ Z \end{pmatrix}, \qquad \mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huc \\ A\xi u^3 \end{pmatrix},$$

$$\mathbf{S} = \begin{pmatrix} \dfrac{E-D}{1-p} \\ -gh(\dfrac{\partial Z}{\partial x} + S_f) - \dfrac{(\rho_s - \rho_w)g}{2\rho}h^2\dfrac{\partial c}{\partial x} - \dfrac{(\rho_0 - \rho)(E-D)u}{\rho(1-p)} \\ E-D \\ -\dfrac{E-D}{1-p} \end{pmatrix}.$$

Note that most of existing formulations for sediment transport models are empirical to differing extents and have been derived from experiments and measured data. It should be stressed that the method described in this paper can be applied to other forms of sediment transport fluxes without major conceptual modifications. For instance, the bed-load sediment transport functions proposed in [7, 8] can also be handled by the proposed finite volume method.

3 SOLUTION PROCEDURE

As pointed out in [2], the bed-load involves different physical mechanisms occurring within different time scales according to their time response to the hydrodynamics. In practice, the sediment transport of the bed occurs on a transport time scale much longer than the flow time scale, compare for example [6, 2, 3]. It is therefore desirable to construct numerical schemes that preserve stability for all time scales and for all forms of sediment discharges. In the present work, to numerically solve the equations (1) we apply a method early developed in [3, 2] for solving sediment transport equations without accounting for erosion-deposition effects. The method was also investigated by the authors in [4] for shallow water flows on fixed beds. Our focus in the current study is to check the performance of the method for solving morphodynamic models involving erosion and deposition of sediments. Therefore, we briefly describe the numerical method and we refer the reader to [4, 3, 2] for more details.

The system (1) can be reformulated in an advective form as

$$\frac{\partial \mathcal{W}}{\partial t} + \mathcal{B}(\mathcal{W})\frac{\partial \mathcal{W}}{\partial x} = \mathcal{G}(\mathcal{W}), \tag{9}$$

where

$$\mathcal{B}(\mathcal{W}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ gh - u^2 - \dfrac{(\rho_s - \rho_w)}{2\rho} ghc & 2u & \dfrac{(\rho_s - \rho_w)}{2\rho} gh & gh \\ -uc & c & u & 0 \\ -3A\xi \dfrac{u^3}{h} & 3A\xi \dfrac{u^2}{h} & 0 & 0 \end{pmatrix},$$

$$\mathcal{G}(\mathcal{W}) = \begin{pmatrix} \dfrac{E-D}{1-p} \\ -ghS_f - \dfrac{\rho_0(E-D)}{\rho(1-p)h} u \\ E - D \\ -\dfrac{E-D}{1-p} \end{pmatrix}.$$

Let us discretize the spatial domain into control volumes $[x_{i-1/2}, x_{i+1/2}]$ with uniform size $\Delta x = x_{i+1/2} - x_{i-1/2}$ and divide the temporal domain into subintervals $[t_n, t_{n+1}]$ with uniform size Δt. The proposed finite volume method consists of a predictor and corrector satges. In this approach, the physical variables $\mathcal{W}^n_{i+1/2}$ are used to compute the averaged states in the predictor stage, while the conservative variables $\mathbf{W}^{n+1}_i$ are updated in the corrector stage. In the preditor stage an operator splitting is used to compute the intermediate states as

$$\begin{aligned} W^*_i &= W^n_i + \Delta t\, \mathcal{G}(W^n_i), \\ W^n_{i+\frac{1}{2}} &= \frac{W^*_i + W^*_{i+1}}{2} - \frac{1}{2}\mathrm{sgn}\left[\mathcal{B}\Big(\nu(W^*_i, W^*_{i+1})\Big)\right](W^*_{i+1} - W^*_i), \end{aligned} \tag{10}$$

where

$$\mathcal{B}\Big(\nu(V^*_i, V^*_{i+1}))\Big) = \mathcal{R}^*_{i+\frac{1}{2}} \Lambda^*_{i+\frac{1}{2}} \mathcal{R}^{*-1}_{i+\frac{1}{2}},$$

with $\mathcal{R}$ is the matrix of right eigenvectors of $\mathcal{A}$, Λ is the diagonal matrix with eigenvalues of $\mathcal{B}$ as its elements. In (10) $\mathrm{sgn}\,[\mathcal{B}]$ denotes the sign matrix of $\mathcal{B}$ defined by

$$sign\Big(\mathcal{B}(\nu(V^*_i, V^*_{i+1}))\Big) = \mathcal{R}^*_{i+\frac{1}{2}} |\Lambda^*_{i+\frac{1}{2}}|^{-1} \Lambda^*_{i+\frac{1}{2}} \mathcal{R}^{*-1}_{i+\frac{1}{2}},$$

and $\nu(V_i^*, V_{i+1}^*)$ is approximated by the Roe's average state

$$\nu(V_i^*, V_{i+1}^*) = \begin{pmatrix} h^*_{i+\frac{1}{2}} \\ h^*_{i+\frac{1}{2}} u^*_{i+\frac{1}{2}} \\ h^*_{i+\frac{1}{2}} c^*_{i+\frac{1}{2}} \\ Z^*_{i+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \dfrac{h_i + h_{i+1}}{2} \\ \dfrac{h_i + h_{i+1}}{2} \dfrac{\sqrt{h_i}\, u_i + \sqrt{h_{i+1}}\, u_{i+1}}{\sqrt{h_i} + \sqrt{h_{i+1}}} \\ \dfrac{h_i + h_{i+1}}{2} \dfrac{\sqrt{h_i}\, c_i + \sqrt{h_{i+1}}\, c_{i+1}}{\sqrt{h_i} + \sqrt{h_{i+1}}} \\ \dfrac{Z_i + Z_{i+1}}{2} \end{pmatrix}.$$

In the corrector stage, the solution is updated as

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\Big(F(W^n_{i+\frac{1}{2}}) - F(W^n_{i-\frac{1}{2}})\Big) + \Delta t S_i^n \tag{11}$$

A detailed formulation for the sign matrix in (10) and for the approximation of the source term $\mathbf{S}_i^n$ are given in [1] and will not be repeated here. It should be stressed that the discretization of the source term in [4, 3, 2] was reconstructed such that the well-known C-property is satisfied.

4 A NUMERICAL EXAMPLE

To verify the finite volume method we consider the test example of dam-break flow over movable beds studied in [5]. Here, the experiment is carried out in a rectangular channel 1.2 m long, 0.2 m wide, and 0.7 m deep. The dam is represented by a sluice gate located in the middle of the test channel. The upstream initial water depth is set to 0.1 m and the dam-break wave was released by rapidly lifting the sluice gate, see [5] for a detailed description of the experiment. The sediment porosity is $p = 0.28$, the constant $A = 0.00215$, the mesh spacing $\Delta x = 0.005\ m$ and results are displayed at time $t = 0.404\ s$. The initial conditions are displayed in Figure 1. In Figure 2 we present the simulated results obtained for the bed-load and the water surface. The erosion magnitude and wavefront location are well predicted by the numerical model. As expected, a hydraulic jump is formed near the initial dam place and propagates upstream. However, the location of the hydraulic jump is accurately predicted by the numerical model.

5 CONCLUDING REMARKS

A coupled model of shallow water flow and suspended sediment equations was used to simulate erosion and deposition of sediments. To solve the model we have implemented a finite volume method using a predictor and corrector stages. The method is simple, robust and can be implemented for large system of species transport in flow field driven by water flows. The method captures the correct bed-load dynamics and shows it ability

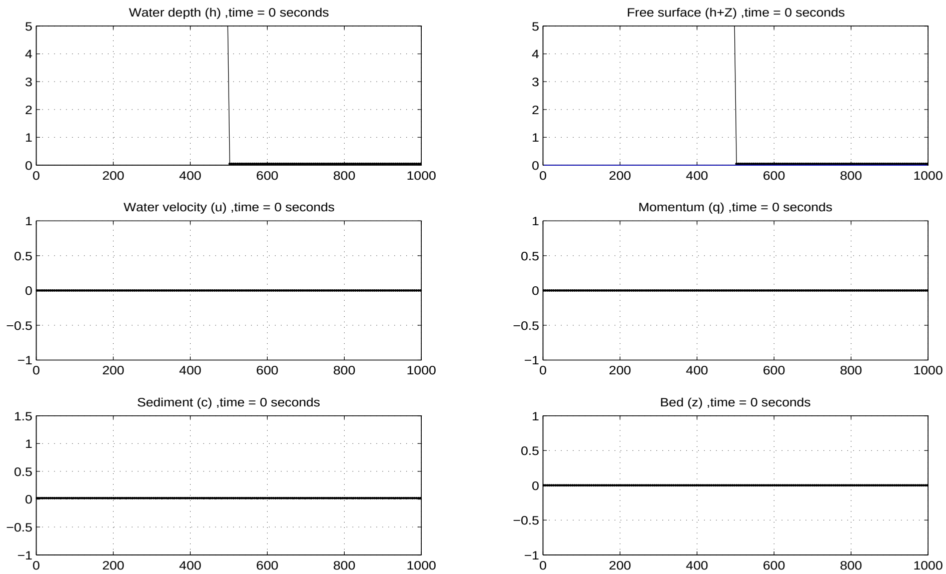

Figure 1: Initial conditions used in the computations.

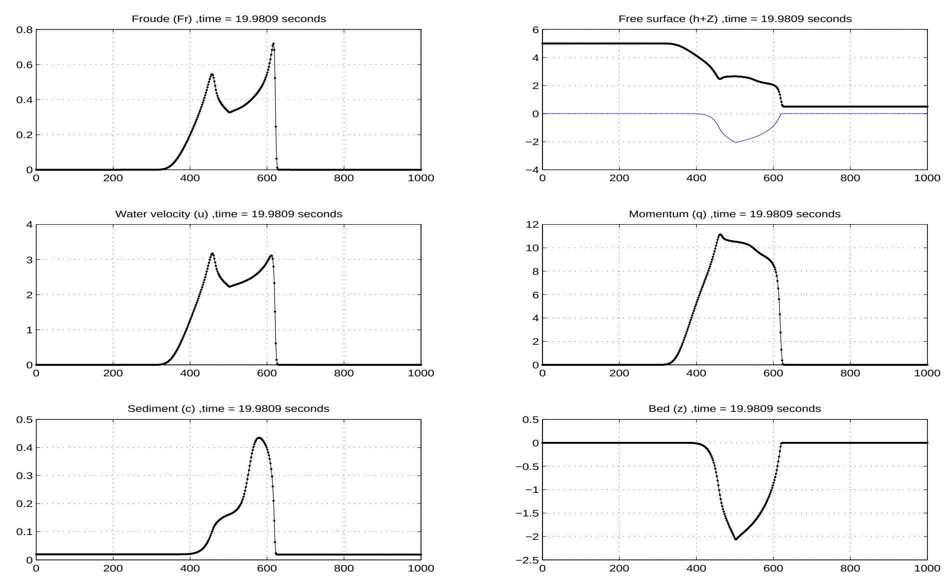

Figure 2: Computed results at different simulation times.

to resolve sediment transport in the shallow water flows under extreme hydrodynamical and morphological regimes. Further studies, using two-dimensional models should be performed.

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