# CHARACTERISTIC BOUNDARY CONDITIONS FOR THE NUMERICAL SOLUTION OF EULER EQUATIONS BY THE DISCONTINUOUS GALERKIN METHODS.

## Ioannis Toulopoulos\*<br/>and John A. Ekaterinaris $^{\dagger}$

\*Institute of Mathematics University of Freiburg and Institute of Applied and computational Mathematics/FORTH, Crete Greece, Universitat of Freiburg, Hermann-Herder-Str. 10, 79104 Freiburg, Germany e-mail: ioannis.toulopoulos@math.uni-freiburg.de

<sup>†</sup>Department of Mechanical and Aerospace Engineering University of Patras, Greece and Institute of Applied and computational Mathematics/FORTH, Crete Greece IACM/FORTH, Nikolaou Plastira 100, Vassilika Vouton, GR 700 13 Heraklion, Crete GREECE e-mail:ekaterin@iacm.forth.gr

**Key words:** Euler equations, discontinuous Galerkin finite element methods, artificial boundary conditions, characteristic boundary conditions, characteristic waves.

Abstract. We present artificial boundary conditions for the numerical simulation of nonlinear Euler equations with the discontinuous Galerkin (DG) finite element method. The construction of the proposed boundary conditions is based on characteristic analysis which follows the Euler equations and are applied for boundaries with arbitrary shape and orientation. Numerical experiments demonstrate that the proposed boundary treatment enables to convect out of the computational domain complex flow features with little distortion. In addition, it is shown that small-amplitude acoustic disturbances could be convected out of the computational domain, with no significant deterioration of the overall accuracy of the method.

### **1** INTRODUCTION

In this paper we present in briefly the method and the numerical computational examples of our work [1]. In the numerical simulation of realistic flow problems the computational domain is truncated in order to enclose to region of interest and to minimize the computational effort. Therefore the boundary of the computational domain includes artificial boundary parts. Usually, on these parts no analytical (physical) boundary data are available and we turn of the use of artificial boundary conditions, hereafter denoted as ABCs. In general the construction of ABCs that will not have any affection to the accuracy of the whole scheme is not an easy task. This task is relatively easier when the flow near the boundary can be linearized and analytical solutions in the truncated part of the domain may be used to derive ABCs, [2], [3].

In the literature, the most frequently effective way to construct ABCs is to use the characteristic analysis of Euler equations. Specificaly, applying the characteristic analysis on the artificial boundaries, the original system of Euler equations is expressed in relation to incoming and outgoing characteristic waves. By this expression the values of the state variables (conservative variables) are related with the values of the characteristic waves. This methodology has been extensively studied and successfully applied by Thompson [4],[5], Poinsot and Lele, [6], Sele et al., [7] and Colonious, [8],[9],[3], for finite difference numerical methods. The construction of this type of ABCs requires the estimation of the waves. The outgoing waves are described by the solution coming from the interior of the domain. The incoming waves are depended on the solution outside of the domain. The exterior solution is not known and an implicit way should be introduced, in order to obtain an estimation of the incoming waves (for example in [6], using physical conditions for the exterior solution, the incoming waves are expressed in relation to outgoing waves).

The main objective of our work is to generalize this methodology for arbitrary shape of boundaries, to present characteristic type ABCs compatible with the high-order discontinuous Galerkin finite element method. In order to build up our methodology and incorporate the proposed boundary treatment in the DGFEM framework, we use the *mirror (ghost) element* technique. The new here is that, the proposed method does not require any physical boundary condition for the exterior solution in order to estimate the incoming characteristic waves which crossing the outflow boundary. Applying the characteristic analysis in the mirror element, we derive a system of ordinary differential equations (ODE system). This system relates the time variations of the state variables on the artificial element to the characteristic waves, (we call it as ABCs-ODE). ABCs-ODE system is advanced in time using the same method as the interior original problem. By the solution of ABCs-ODE, we obtain the values of the state variables at the current time step (Dirichlet type boundary conditions). For the current time step, the characteristic waves are computed (and so the right hand side of ABCs-ODE) using the boundary solution data on the artificial element of the previous time step.

Numerical experiments demonstrate that use of these artificial boundary conditions

with DG discretizations makes convection of vortical disturbances away from the computational domain with little distortion possible. The proposed boundary treatment was further applied for aeroacoustic problems. For this case, the full nonlinear-Euler equations are used and aeroacoustic disturbances are specified as small perturbations in pressure.

The rest of this paper is organized as follows. In section 2, we present the formulation of the discontinuous Galerkin method for the Euler equations. Section 3 contains the description of the proposed ABC's for an arbitrary shape boundary, including the characteristic analysis. The approximation of the characteristic waves on the local polynomial space of the mirror elements, the expression of ABC-ODE system and the estimation of the values of the state vector in the mirror elements. Finally, in section 4, the accuracy of the DGFEM utilizing the proposed ABC's is evaluated with numerical examples. In all examples high order of accuracy at the artificial boundary is achieved.

## 2 GOVERNING EQUATIONS AND SPACE DISCRETIZATION

The equations of the two dimensional Euler equations are

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) + \partial_y \mathbf{g}(\mathbf{u}) = 0, \ in \ \Omega \times (0, T)$$
(1)

where T > 0 is the length of the time interval and  $\Omega$  is a bounded domain of  $\mathbf{R}^2$ . In (1) the conservative variable vector and the Cartesian flux vectors are given by

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix}, \ \mathbf{f}(\mathbf{u}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (\rho e + p)u \end{bmatrix}, \ \mathbf{g}(\mathbf{u}) = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (\rho e + p)v \end{bmatrix}.$$
(2)

Here, p is the pressure,  $\rho e$  is the total energy, and  $\mathbf{v} = (u, v)$  is the velocity. The system is completed by the equation of state for a perfect gas,

$$p = (\gamma - 1)[\rho e - \rho \frac{1}{2}(u^2 + v^2)], \qquad (3)$$

where  $\gamma = 1.4$  is the ratio of the specific heats. The Euler equations are a hyperbolic system; initial and appropriate boundary conditions supplementing (1) should be specified.

## 2.1 The discontinuous Galerkin space discretization

Let  $T_h$  be a triangulation of the domain  $\Omega$  consisting of a collection of non-overlapping triangular elements  $\bigcup_{el=1}^{Nel} E_{el}$ , thus  $T_h = \bigcup_{el=1}^{Nel} E_{el} = \Omega$ . For every element  $E \in T_h$ we define a local polynomial space  $V_h(E)$  of dimension k whose basis functions  $P_j$  are polynomials of degree at most m, i.e.  $V_h(E) = \mathbf{P}^m(E)$  and  $P_j$ ,  $j = 1, ..., k \in \mathbf{P}^m(E)$ . The dimension k of the space  $V_h(E)$  is given by the formula  $k = \frac{(2+m)!}{2!m!}$ . In our numerical examples third order polynomials are used and so m = 3, k = 10. The approximate solution  $\mathbf{u}_h$  is sought in the *discontinuous* finite element space  $\mathbf{V}_h$ 

$$\mathbf{V}_h = \{ v \in L^2(\Omega) : v |_E \in V_h(E) \ \forall E \in T_h \}.$$

Therefore, for each element  $E \in T_h$ , the components  $u_h^i$ , i = 1, 2, 3, 4 of the approximate conservative variable vector  $\mathbf{u}_h$  have the representation

$$u_h^i = \sum_{j=1}^k c(t)_j^i P_j(x, y), \ i = 1, 2, 3, 4,$$
(4)

where  $c(t)_j^i$  are the degrees of freedom to be advanced in time. For every  $E \in T_h$ , we write the degrees of freedom of the components of  $\mathbf{u}_h$ , as a vector denoted by  $\mathbf{U}$ . Note that, the finite element space does not require any continuity across inter-element boundaries.

#### 2.2 The space-discrete weak formulation

For an element  $E \in T_h$ , let  $\mathbf{u}_h \in V_h(E)$  be an approximation of  $\mathbf{u}$ . In order to determine the approximate solution of (1) in  $V_h(E)$  we consider the weak form of (1)

$$\int_{E} \partial_t \mathbf{u}v - \int_{E} (\mathbf{f}(\mathbf{u}), \mathbf{g}(\mathbf{u})) \cdot \nabla v dx dy + \int_{\partial E} (\mathbf{f}(\mathbf{u}), \mathbf{g}(\mathbf{u})) \cdot \mathbf{n}v ds = 0$$
(5)

for any smooth function v. Replace  $\mathbf{u}$  by  $\mathbf{u}_h$ , v by a test function  $\varphi$  that belongs to the finite element space  $V_h$ , and obtain

$$\int_{E} \partial_t \mathbf{u}_h \varphi - \int_{E} (\mathbf{f}(\mathbf{u}_h), \mathbf{g}(\mathbf{u}_h)) \cdot \nabla \varphi dx dy + \int_{\partial E} (\mathbf{f}(\mathbf{u}_h), \mathbf{g}(\mathbf{u}_h)) \cdot \mathbf{n} \varphi ds = 0, \ \forall \ \varphi \in V_h(E).$$
(6)

Here **n** denotes the outward unit normal to the boundary  $\partial E$  of E. The line integral in (6) is not well defined since the finite element space  $V_h(E)$  does not require continuity of the approximate solution at the interfaces. Therefore, following the finite volume approach, we replace the flux  $(\mathbf{f}, \mathbf{g}) \cdot \mathbf{n}$  by a numerical flux function h, which depends on the two values of  $\mathbf{u}_h$  at the interface of the element. One is the value obtained from the interior of the element E, denoted as  $\mathbf{u}_h|_E$ , and the other is the value obtained from the adjacent element  $E_{adj}$  sharing common edge with E and is denoted as  $\mathbf{u}_h|_{E_{adj}}$ ,

$$h = h(\mathbf{u}_h|_E, \mathbf{u}_h|_{E_{adi}}, \mathbf{n})$$

For simplicity, in hereafter  $\mathbf{u}_{\mathbf{h}}|_{E}$  is denoted as  $\mathbf{u}_{\mathbf{h}}{}^{in}$  and  $\mathbf{u}_{\mathbf{h}}|_{E_{adj}}$  as  $\mathbf{u}_{\mathbf{h}}{}^{out}$ . Any consistent approximate Riemann solver can be used to evaluate the numerical flux, [10]. We use the Lax-Friedrich's (LF) numerical flux in our numerical implementation.

The line integral in (6) with the LF flux takes the form,

$$\int_{\partial E} h(\mathbf{u}_{h}^{in}, \mathbf{u}_{h}^{out}, \mathbf{n})\varphi ds =$$

$$\int_{\partial E} \left( (\mathbf{f}^{in} + \mathbf{f}^{out}, \mathbf{g}^{in} + \mathbf{g}^{out}) \cdot \mathbf{n} - \lambda(\mathbf{u}_{h}^{out} - \mathbf{u}_{h}^{in}) \right) \varphi ds, \qquad (7)$$

$$\lambda = \max_{(x,y) \in \partial E} \{ |\lambda_{i}(x, y)| \}, \qquad i = 1, 2, 3, 4$$

where  $\lambda_i$  are the eigenvalues of the flux Jacobian  $\mathbf{J} = \frac{\partial(\mathbf{f}, \mathbf{g}) \cdot \mathbf{n}}{\partial \mathbf{u}}$ . With these notations, the discrete problem can be expressed as: for every  $E \in T_h$  find  $\mathbf{u}_h \in V_h(E)$  such that

$$\int_{E} \partial_{t} \mathbf{u}_{h} \varphi dx dy - \int_{E} (\mathbf{f}(\mathbf{u}_{h}), \mathbf{g}(\mathbf{u}_{h})) \cdot \nabla \varphi dx dy + \int_{\partial E} h(\mathbf{u}_{h}^{in}, \mathbf{u}_{h}^{out}, \mathbf{n}) \varphi ds = 0, \ \forall \varphi \in V_{h}(E)$$
with appropriate boundary conditions on  $\partial \Omega$ 

#### 2.3 Temporal Discretization

Replacing  $\varphi$  in Eq. (8) by the polynomial basis functions  $P_j$ , a system of Ordinary Differential Equation (ODE) is obtained for the degrees of freedom **U**,

$$M\frac{d\mathbf{U}}{dt} = L(\mathbf{U}) \tag{9}$$

(8)

where M is the mass matrix. This ODE system may be advanced in time by an explicit scheme. We use the explicit third order TVD Runge-Kutta method introduced in [11]:

Let  $\{t^n\}_{n=1}^{N-1}$  be a uniform partition of the time interval [0, T]. Set  $\Delta t = t^{n+1} - t^n$ , n = 1, 2, ..., N - 1 where  $N\Delta t = T$ . The three intermediate stages of one Runge-Kutta cycle, from  $t^n$  to  $t^{n+1}$  are

$$\mathbf{U}^{n,1} = \mathbf{U}^{n} + \Delta t M^{-1} L(\mathbf{U}^{n}), 
\mathbf{U}^{n,2} = \frac{3}{4} \mathbf{U}^{n} + \frac{1}{4} \mathbf{U}^{n,1} + \frac{1}{4} \Delta t M^{-1} L(\mathbf{U}^{n,1}), 
\mathbf{U}^{n,3} = \frac{1}{3} \mathbf{U}^{n} + \frac{2}{3} \mathbf{U}^{n,2} + \frac{2}{3} \Delta t M^{-1} L(\mathbf{U}^{n,2}), 
\mathbf{U}^{n+1} = \mathbf{U}^{n,3},$$
(10)

where  $\mathbf{U}^{n,l}$ , l = 1, 2, 3 are the solutions at the intermediate stages of the Runge-Kutta method.

#### 2.4 Boundary treatment

Consider an interior element with an edge on the boundary of the computational domain,  $\partial\Omega$ . Denote this element by  $E_b$  and the corresponding edge on the domain boundary as  $e_b \in \partial E_b \cap \partial\Omega$ . Consider the mirror element  $E_m$  of  $E_b$  be reflection about  $e_b$  outside of the computational domain, such that  $\partial E_m \cap \partial E_b = e_b$ , (see Fig. 1).

On the boundary  $\partial\Omega$  physical and artificial boundary conditions must be imposed. The boundary conditions determine the amount of the normal flux through the boundary edge  $e_b$ . This flux is computed through the numerical flux function

$$h(\mathbf{u}_h^{in}, \mathbf{u}_h^{out}),\tag{11}$$

where  $\mathbf{u}_{h}^{in} = \mathbf{u}_{h}|_{E_{b}}$  is the interior solution coming from the boundary element and  $\mathbf{u}_{h}^{out} = \mathbf{u}_{h}|_{E_{m}}$  is the boundary data information that should come from the mirror element. If there

are physical boundary conditions which specify the behavior of all depended variables (for example on a solid wall), then the  $\mathbf{u}_h|_{E_m}$  can be directly computed. In general, analytic boundary conditions do not exist at the inflow and outflow boundaries and thus in the next section, we describe how the values  $\mathbf{u}_h|_{E_m}$ , can be determined by constructing characteristic type artificial boundary conditions.



Figure 1: Boundary and mirror element.



Figure 2: The direction of the characteristic waves for an outflow boundary.

## 3 CHARACTERISTIC BOUNDARY CONDITIONS

The construction of ABCs for time depended problems is not an easy task, since ABCs must mimic the *exact* boundary data and to prevent the genneration of numerical reflected waves which can affect the stability of the numerical scheme. A systematic way to construct artificial BC's is by utilizing the characteristic theory. In the last few years

characteristic type ABCs, hereafter denoted as CHBC, have been proposed by many investigators, [3], [12], [13], In [4], [5], [6], for finite difference methods in case of simple boundaries. We generalize this technique for the DG framework and boundaries of arbitrary shape.

#### 3.1 The characteristic system

The quasi-linear form of the system (1) for the element  $E_m$  is,

$$\partial_t \mathbf{u} + A \partial_x \mathbf{u} + B \partial_y \mathbf{u} = 0 \tag{12}$$

where A, B are the Jacobian matrices of the fluxes  $\mathbf{f}, \mathbf{g}$  defined as

$$A = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}, \ B = \frac{\partial \mathbf{g}}{\partial \mathbf{u}}.$$
 (13)

Let  $\mathbf{n} = (n_x, n_y)$  be the unit outward normal vector to the boundary edge  $e_b$ , see Fig. 2. We construct the matrix  $R = R(\mathbf{n})$  [14] which diagonalizes the matrix  $An_x + Bn_y$ . Hence,

$$\Lambda = R^{-1}(An_x + Bn_y)R \tag{14}$$

is the diagonal eigenvalue matrix whose diagonal entries are

$$\lambda_{1} = \mathbf{v} \cdot \mathbf{n},$$
  

$$\lambda_{2} = \mathbf{v} \cdot \mathbf{n},$$
  

$$\lambda_{3} = \mathbf{v} \cdot \mathbf{n} + c,$$
  

$$\lambda_{4} = \mathbf{v} \cdot \mathbf{n} - c$$
(15)

where  $c = \sqrt{\gamma p/\rho}$  is the local speed of sound. The matrices R,  $R^{-1}$  relate variations of the conservative variables to variations of the characteristic variables **w** through the relations

$$\delta(\mathbf{w}) = R^{-1}\delta(\mathbf{u}), \ \delta(\mathbf{u}) = R\delta(\mathbf{w}).$$
(16)

The characteristic waves which cross the boundary edge  $e_b$ , have the form [14]:

$$\delta(w_1) = \delta \rho - \frac{1}{c^2} \delta p,$$
  

$$\delta(w_2) = n_y \delta u - n_x \delta v,$$
  

$$\delta(w_3) = n_x \delta u + n_y \delta v + \frac{1}{\rho c} \delta p,$$
  

$$\delta(w_4) = -n_x \delta u - n_y \delta v + \frac{1}{\rho c} \delta p.$$
(17)

The matrices R,  $R^{-1}$  (see Eq. (14)) are associated with the normal to the boundary edge direction **n**. As a result, the characteristic waves  $w_i$ , i = 1, 2, 3, 4 and the corresponding wave speeds  $\lambda_i$  are functions of **n**.

We multiply (12) by the matrix  $R^{-1}$  and after some algebra, we obtain the following hyperbolic type system for the characteristic waves ,

$$(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla)w_{1} = 0,$$

$$(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla)w_{2} = \frac{c}{2}(n_{x}\partial_{y} - n_{y}\partial_{x})(w_{3} + w_{4}),$$

$$(\frac{\partial}{\partial t} + (\mathbf{v} + c\mathbf{n}) \cdot \nabla)w_{3} = c(n_{x}\partial_{y} - n_{y}\partial_{x})w_{2},$$

$$(\frac{\partial}{\partial t} + (\mathbf{v} - c\mathbf{n}) \cdot \nabla)w_{4} = c(n_{x}\partial_{y} - n_{y}\partial_{x})w_{2}.$$
(18)

If  $\mathbf{F}_w$  and  $\mathbf{G}_w$  are the fluxes of the left and the right side of (18) respectively, then (18) can be written as

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{F}_w = \mathbf{G}_w. \tag{19}$$

The right hand side of (19) expresses the variation of the waves along the tangential to the boundary direction  $\mathbf{s} = (n_y, -n_x)$ . Following the approach applied for the interior Riemann solver, where the normal to the interface flux is evaluated, the right-hand side tangential flux  $\mathbf{G}_w$  of the system (19) is neglected and the following evolution-type equation for the time evaluation of the characteristic waves on  $E_m$  is obtained:

$$\partial_t \mathbf{w} + \mathbf{F}_w = 0. \tag{20}$$

The system (20) is referred to as the *characteristic system*, CHS.

This system is solved numerically for all mirror elements of the artificial boundary, according to an upwind discretization in order to obtain a discrete approximation of the time variation of the waves  $\partial_t(\mathbf{w})|_{E_m}$ . Our goal is to relate this discrete time variation of the waves  $\partial_t(\mathbf{w}_h)$  which cross the artificial boundary through edge  $e_b$  as closely as possible with the corresponding discrete time variation of the conservative variable  $\partial_t(\mathbf{u}_h|_{E_m})$ .

## 3.1.1 Discrete formulation of the CHS

For the element  $E_m$ , the solution  $\mathbf{w}$  of (20) is approximated, with the discontinuous Galerkin method by  $\mathbf{w}_h \in V_h(E_m)$  such that,

$$\int_{E_m} \frac{\partial \mathbf{w}_h}{\partial t} \varphi dx dy = -\int_{E_m} \mathbf{F}_w(\mathbf{w}_h) \varphi dx dy + \int_{e_b} \Lambda(\mathbf{u}_h|_{E_b}, \mathbf{u}_h|_{E_m}) R^{-1}(\mathbf{u}_h|_{E_b}, \mathbf{u}_h|_{E_m}) (\mathbf{u}_h|_{E_b} - \mathbf{u}_h|_{E_m}) \varphi ds, \ \forall \varphi \in V_h(E_m)$$
(21)

where  $\Lambda$ ,  $R^{-1}$  are evaluated at the Roe's average state on the midpoint of  $e_b$ . Replacing in (21) the  $\varphi$  by the polynomial elements of the base of  $V_h(E_m)$ , we obtain a system of ODE's for the degrees of freedom **W** of  $\mathbf{w}_h$ 

$$\frac{d\mathbf{W}}{dt} = \mathcal{L}_{E_m}(\mathbf{w}_h) \tag{22}$$

where the right side  $\mathcal{L}_{E_m}$  results from the right hand side of (21) multiplied by the inverse mass matrix  $M^{-1}$ .

From (22) and (16), the following system for the time variation of the conservative variables in  $E_m$  is obtained

$$\frac{d\mathbf{U}}{dt} = R\frac{d\mathbf{W}}{dt} = R\mathcal{L}_{E_m}(\mathbf{w}_h) = \mathcal{L}_{bc}(\mathbf{u}_h)|_{E_m}$$
(23)

where R is computed at the Roe' average state on midpoint of  $e_b$  and U is the vector of the degrees of freedom of  $\mathbf{u}_h|_{E_m}$ , to be advanced in time.

The system (23) defines the ABCs (referred to as the ABC-ODE system) and is advanced in time using the Runge-Kutta method (10). We note that for the numerical solution of ABC-ODE by Runge-Kutta method, the characteristic waves (the right hand side) at the intermediate stage l, l = 1, 2, 3, are computed using the solution data from the previous stage  $\mathbf{U}^{n,l-1}|_{E_m}$ , see (10). The numerical solution obtained from (23) yields values of the conservative state variable vector  $\mathbf{u}_h$  on the mirror element  $E_m$  of the artificial boundary. Using these values the numerical flux  $h(\mathbf{u}_h|_{E_b}, \mathbf{u}_h|_{E_m})$ , (see (11)), on  $e_b$ can be computed.

## 4 NUMERICAL EXAMPLES

The first problem is the propagation of a vortex in a uniform stream. Convection of a vortex through an artificial outflow computational boundary is a difficult time-depended test problem. For this problem, boundary treatment procedures based on one-dimensional Riemann invariant extrapolation, usually fails. For this case, the flow near the boundary cannot be represented as a small amplitude disturbance to a uniform state and any artificial boundary treatment is not fully nonreflective. The convection of the isentropic vortex is governed by the Euler equations. For this problem it is found that the incoming numerical waves do not strongly affect the overall accuracy of the numerical solution. Furthermore, the numerical perturbations generated by the incoming wave are not able to destroy the stability of the method.

We used the same boundary conditions for the numerical solution of aeroacoustic problems. The first problem of this category concerns the scattering of an acoustic pulse by a circular cylinder, [15], [16]. The second problem of this category has a time periodic solution and is the propagation of acoustic pulses generated by a time harmonic source, [16].

In the numerical examples we denote the numerical solutions computed by the application of the proposed boundary treatment as *CHBC*. For certain problems where the exact solution is known, we also obtained numerical solutions using boundary values specified by the exact solution. In these cases, the analytical specification of boundary data is used for comparison. We denote the corresponding numerical solutions as *ExBc*. The local polynomial space for all the problems is  $V_h(E) = \mathbf{P}^3(E), \forall E \in T_h$ .

### 4.0.2 Isentropic Vortex Convection

The DG finite element method, with the proposed characteristic boundary conditions, is used for the convection of a vortex through the computational boundary. The free stream flow velocity, pressure, and density are prescribed as:

$$[\rho_{\infty}, u_{\infty}, v_{\infty}, p_{\infty}] = [1, 1, 0, 1].$$

The initial condition for the isentropic vortex is the addition of perturbations in the velocity components (u, v) and temperature T (no perturbation in entropy) to the free stream. These perturbations are given by:

$$(\delta u, \delta v) = \left(\frac{\beta}{2\pi} e^{\frac{1-r^2}{2}} (-y', x')\right)$$
(24)  
$$\delta T = -\frac{(\gamma - 1)\beta^2}{8\gamma\pi^2} e^{1-r^2}$$

where  $\beta$  is the vortex strength,  $\gamma = 1.4$ , and  $(x_{vo}, y_{vo})$  are the initial coordinates of the vortex center,  $(x', y') = (x - x_v, y - y_v)$ , and  $r^2 = x'^2 + y'^2$ . In the absence of viscous terms, the entire flow field is required to be isentropic,  $\delta S = 0$ , and the initial condition for the conservative state variables vector  $\mathbf{u}(x, y, 0) = \mathbf{u}_0$  is

$$\mathbf{u}_{0} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix} = \begin{bmatrix} (1+\delta T)^{\frac{1}{\gamma-1}} \\ \rho(u_{\infty}+\delta u) \\ \rho(v_{\infty}+\delta v) \\ \frac{p}{\gamma-1} + \frac{\rho(u^{2}+v^{2})}{2} \end{bmatrix}.$$
 (25)

The exact solution remains self similar for all times and represents a passive convection of the vortex with the freestream velocity (u, v). In the numerical examples,  $\beta = 2$ . and  $(x_{vo}, y_{vo}) = (8., 0)$ .

Fig. 3(a) shows the initial density field for discretization of the domain with  $\Delta x = 0.5$ . The entire flow-field is mixed subsonic and supersonic

$$\left\{\begin{array}{ll} \frac{u}{c} < 1, & for \ y > 0\\ \frac{u}{c} > 1, & for \ y < 0, \end{array}\right.$$

as it is shown in Fig.3(b). The numerical solution is computed on regular triangular meshes using grid spacing  $\Delta x = 0.5, 0.25, 0.125$ . Time iterations start and the vortex is convected downstream. At computational time t = 2 the center of the vortex is located at the outflow boundary point (10,0), and half vortex has passed through the artificial boundary out of the computational domain. Fig. 4 shows the crossing of the vortex moving with  $u_{\infty} = 1$  through the outflow boundary for the grid size  $\Delta x = 0.125$  at t = 2. In Fig. 4(a) CHBC have been used and in Fig 4(b) the boundary values have been specified by the analytic solution, ExBC. It is seen, Fig. 4(a), that the shape of the vortex of the supersonic part y < 0 has not changed, while the shape of the vortex corresponding to the subsonic part y > 0 has been deformed.

The vortex continues to escape through the outflow boundary and for a later time t = 4 the vortex has moved out of the computational domain. However, Fig. 5 shows that perturbations generated during the passage of the vortex through the outflow boundary remain in the computational domain.

At time t = 4 the  $L_2$  error of the density variable

$$||\rho_h - \rho_{exact}||_2 = \frac{\sum_{E \in T_h} \sqrt{\int_E (\rho_h - \rho_{exact})^2}}{N_{el}}$$

is computed for the three different grids. The convergence rate of the  $L_2$  error is shown in Fig. 6. The dashed line shows the reduction of the  $L_2$  error using characteristic BC versus  $\Delta x$ . The dotted line shows the error reduction using exact (analytical) boundary data specified by the analytical solution, (ExBC). It is seen that the slope of the characteristic BC line is one order less than the corresponding slope of the ExBC line.

At later times the vortex has disappeared. However due to imperfections of the characteristic boundary conditions, spurious numerical waves which have been generated during the passage from the outflow boundary remain in the computational domain. These numerical waves propagate in the computational domain towards the inflow. The contours of these numerical disturbances at time t = 50 are shown in Fig. 7 and have small magnitude, about 2% of the free stream density. The variations of the global  $L_2$  error versus time are shown in Fig. 8. It can be seen that the  $L_2$  error decreases with the mesh refinement, however even after long time integration the numerical computation remains stable.



(a) Initial position of the vortex, density contours

(b) Contours of u/c

Figure 3: Vortex propagation through a subsonic outflow boundary,  $\Delta x = 0.5$ , t = 0.



(a) Subsonic outflow boundary, CHBC

(b) Subsonic outflow boundary, ExBC

Figure 4: Vortex crossing through the computational boundary: density contours computed on mesh  $\Delta x = 0.125$  at t = 2. (a) Half vortex has crossed the subsonic outflow boundary using CHBC, (b) Half vortex has crossed the subsonic outflow boundary using ExBC.



Figure 5: Remaining perturbations of density field on mesh  $\Delta x = 0.125$  at t = 4. The vortex has left the computational domain and spurious numerical waves generated by the crossing on the vertical boundary.



Figure 6: Vortex problem. The convergence rate of the  $L_2$  error of the density, dashed line CHBC, dotted line ExBC.



Figure 7: Vortex problem. The magnitude of the numerical disturbances in the density field for the mesh  $\Delta x = 0.125$  at time t=50.



Figure 8: Vortex problem. The variation of the density  $L_2$  error for long time integration.

### 4.1 Performance of CHBC for aeroacoustic problems

## 4.1.1 Scattering of an acoustic pulse by a cylinder surface

Scattering of an aeroacoustic pulse by the surface of a circular cylinder is considered next. The center of the cylinder is (x, y) = 0.0 and its radius r = 0.5. The outflow boundary part of  $\Omega$  is the hemicycle  $\Gamma_{outflow \ bd} = \{(x, y) : \sqrt{x^2 + y^2} = 10, y \ge 0\}$ . The rigid boundary comprises the up half of the surface of the cylinder  $\Gamma_{cyl} = \{(x, y) : \sqrt{x^2 + y^2} = 0.5, y \ge 0\}$  and the points  $\{(x, y), -10 \le x \le 0.5, y = 0\} \cup \{(x, y), 0.5 \le x \le 10, y = 0\}$ . The pulse is introduced by the initial pressure disturbance centered at (x, y) = (4, 0). The initial conditions are

$$\mathbf{u}_{0} = \begin{pmatrix} \rho_{0} = 1 \\ u_{0} = 0 \\ v_{0} = 0 \\ p_{0} = \varepsilon \exp(-\frac{\ln(2)}{w}((x-4)^{2} + y^{2})) \end{pmatrix}$$

where in our numerical experiments we set the Gaussian half-width w = 1 and  $\varepsilon = 0.01$ . Further details on the set up of this problem can be found in [15]. The computational domain is divided in consequence of three triangular meshes using in every mesh finer discretization for both the  $\Gamma_{outflow \ bd}$  and for the  $\Gamma_{cyl}$ . For the coarse mesh the  $\Gamma_{outflow \ bd}$ ,  $\Gamma_{cyl}$  are discretized by  $N_{outflow \ bd} = 6$  elements. For the medium mesh the surface are discretized using  $N_{outflow \ bd} = 12$  and for the fine mesh using  $N_{outflow \ bd} = 24$ elements. We can approximately consider that the size of the  $\Gamma_{outflow \ bd}$  discretization is  $\Delta\Gamma_{outflow \ bd} = \frac{\Gamma_{outflow \ bd}}{N_{outflow \ bd}}$ . The problem for the three meshes is solved at final time t = 8. The pressure field computed on the finer mesh  $(N_{outflow \ bd} = 24)$  is shown in Fig. 9, where the incident and scattered acoustic pulse leave the computational domain through the  $\Gamma_{outflow \ bd}$  without the appearance of incoming numerical waves. For each of the three meshes, we computed the  $L_2^{E_b}$  error on the boundary elements of the outflow boundary at t = 8. The grid converge of the  $L_2^{E_b}$  error is presented in Fig. 10. The order of the convergence is  $\mathbf{r} = 2.97$ , which is close to the optimal order for the outflow surfaces is  $\mathbf{r} = m + \frac{1}{2}$ , [17], ( for our polynomial space  $\mathbf{r} = 3 + \frac{1}{2}$ ).

## 4.1.2 The time harmonic source problem

The last aeroacoustic problem examined is the calculation of the perturbation pressure field generated by a time dependent source. This problem provides a stringent case for the test of the artificial BC because acoustic waves continuously cross the outflow boundary. The time depended acoustic source term has the form

$$S(x, y, t) = \varepsilon \exp(-\frac{\ln(2)}{w}((x-4)^2 + y^2))\sin(\omega t)$$

and is added to the right-hand side of (1) for the energy equation. In the numerical experiments we chose  $\varepsilon = 0.01$ , w = 1,  $\omega = 0.5\pi$ .



Figure 9: Acoustic pulse scattering problem. The final pressure field at T=8 computed on the fine mesh using the characteristic boundary conditions on the outflow boundary.



Figure 10: Acoustic pulse scattering problem. The converge rate of the  $L_2^{E_b}$  error at t = 8.

The outflow boundary computational domain is

$$\Gamma_{outflow \ bd} = \{(x, y) : x = rcos(\theta), y = rsin(\theta), \ 0 \le \theta \le 2\pi, r = 10\}$$

The computational domain is discretized in two triangular meshes, coarse and fine. For the first mesh the grid size on the outflow boundary is

$$\Delta \Gamma_{outflow \ bd} = \frac{\Gamma_{outflow \ bd}}{40}$$

and for the second mesh

$$\Delta \Gamma_{outflow \ bd} = \frac{\Gamma_{outflow \ bd}}{80}.$$

The problem has numerically solved at final time t = 350. The computed pressure fields on the two meshes are shown in Fig. 11. For both meshes the pressure values are recorded at the point (x, y) = (10, 0) of the boundary. In Fig. 12(a) the numerical point values corresponding to the coarse mesh are compared with the exact pressure values. In Fig. 12(b) we perform the same comparison for the numerical point values corresponding to the fine mesh. The  $L_2^{E_b}$  error is computed on the elements of the boundary for both meshes. In Fig 13, the variations of the  $L_2^{E_b}$  error versus time are presented. For both meshes the values of the  $L_2^{E_b}$  error show periodic variation but remain bounded. Therefore it appears that the proposed characteristic BC approach is accurate and does not deteriorate the stability of the overall numerical scheme.



Figure 11: Acoustic time harmonic source problem. (a) The numerical pressure field computed at t = 350 using CHBC on the coarse mesh. (b) The numerical pressure field computed at t = 350 using CHBC on the fine mesh.



Figure 12: Acoustic time harmonic source problem. (a) Coarse mesh. The exact and the numerical boundary point values of the pressure versus time. (b) Fine mesh. The exact and the numerical boundary point values of the pressure versus time.



Figure 13: Acoustic time harmonic source problem. The time variation of the  $L_2^{E_b}$  error for the two meshes.

### 5 CONCLUSIONS

Artificial boundary conditions based on the characteristic analysis of the Euler's equations were developed and applied in the DG framework. These boundary conditions are applicable to artificial boundaries with arbitrary shape. An auxiliary system was constructed through application of characteristic analysis. The numerical solution of this system is used for the evaluation of the time variation of the characteristic waves which cross the artificial boundary. The time variation of these waves was related to the conservative variables time variation resulting in an ODE system. The numerical solution of this system mimics adequately the analytical boundary data. Computed results with the proposed boundary conditions demonstrated that complex flow features can be convected outside the computational domain with little distortion. In addition, propagation of small amplitude acoustic-type perturbations with the proposed boundary conditions demonstrated equally good performance with exact boundary data.

## REFERENCES

- [1] Toulopoulos I. and Ekaterinaris J. Artificial boundary conditions for the numerical solution of the euler equations by the discontinuous galerkin method. *Under review*, 2009.
- [2] D. Givoli. Non-reflecting boundary conditions. Journal of Computational Physics, 91:1–29, 1991.
- [3] T Colonious. Modeling artificial boundary conditions for compressible flow. Annual Review of Fluid Mechanics, 36:315–345, 2004.
- [4] K. W. Thompson. Time dependent boundary conditions for hyperbolic systems. Journal of Computational Physics, 68:1–24, 1987.
- [5] K. W. Thompson. Time-dependent boundary conditions for hyperbolic systems, ii. Journal of Computational Physics, 89:439–461, 1990.
- [6] T.J. Poinsot and S.K. Lele. Boundary conditions for direct simulations of compressible viscous flows. *Journal of Computational Physics*, 101:104–129, 1992.
- [7] L. Selle, F. Nicoud, and T. Poinsot. Actual impedance of nonreflective boundary conditions: Implications for computation of resonators. *AIAA Journal*, 42(5):958– 964, 2004.
- [8] T. Colonious, S. K. Lele, and P. Moin. Boundary conditions for direct computation of aerodynamic sound generation. AIAA Journal, 31(9):1574–1582, 1993.
- [9] T Colonious. Numerical nonreflecting boundary and interface conditions for compressible flow and aeroacoustic computations. *AIAA Journal*, 35(7):1126–113, 1997.

- [10] Jianxian Qiu, Boo Cheong Khoo, and Chi Wang Shu. A numerical study for the performance of the runge-kutta discontinuous Galerkin method based on different numerical fluxes. *Journal of Computational Physics*, 212(2):540–565, 2006.
- [11] B Cockburn and C.W. Shu. The runge-kutta discontinuous galerkin method for conservation laws, multidimensional systems. *Journal of Computational Physics*, 141:199–224, 1998.
- [12] M Giles. Nonreflecting boundary conditions for euler calculations. AIAA Journal, 28:2050–2058, 1990.
- [13] G. W. Hedstrom. Nonreflective boundary conditions for the nonlinear hyperbolic systems. Journal of Computational Physics, 30:222–237, 1979.
- [14] C. Hirsch. Numerical Computation of Internal and External Flows, volume 2, Computational methods for inviscid and viscous flows of Wiley series in numerical methods in engineering. Wiley, 1988.
- [15] I. Toulopoulos and J. A. Ekaterinaris. High-order discontinuous-galerkin discretizations for for computational aeroacoustics in complex domains. AIAA Journal, 44(3):502–511, 2006.
- [16] J. C. Hardin, J. R. Ristorcelli, and C. K. Tam. Icase/larc workshop on benchmark problems in computational aeroacoustics (caa), nasa cp 3300. 1995.
- [17] P. Monk and G. R Richter. A Discontinuous Galerkin Method for Linear Symetric Hyperbolic Systems in Inhomogeneous Media. *Journal of Scientific Computing*, 22-23:443–477, 2005.