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RECENT PROGRESS IN HYBRID TEMPORAL-LES/RANS MODELING

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Abstract. *The majority of flows of practical relevance are inhomogeneous, such that seamless hybrid RANS-LES models suffer from a conceptual weakness: LES is based on spatial filtering and RANS on statistical averaging, which are only compatible in homogeneous flows. On the contrary, in the limit of statistically steady (stationary) flows,* Temporal Large-Eddy Simulation *(TLES), with a filter width continuously going to infinity, is compatible with the RANS approach. An explicit filter can be defined that ensures the Galilean invariance, and, coupled with the correct limiting behavior associated with the temporal filter, forms the basis for consistently defined variables that can represent both filtered or averaged field variables.*

This formalism can be readily coupled to an adaptation of the Partially Integrated Transport Model (PITM) to a temporal-PITM (TPITM) model. The result is a hybrid method that incorporates transport equations used to govern the subfilter scales (SFS). Additionally, the resulting equation system within this TPITM framework can be amended in order to account for nonlocal wall-blockage by employing the elliptic blending methodology.

The formalism adopted for the TPITM theory can be used to interpret widely used hybrid methods, such as DES, as hybrid TLES/RANS approaches. Indeed, it can be shown that the cutoff frequency does not explicitly appears in the model equations. The relevant parameter is k_m/k, the ratio of modeled energy *to* total energy. *In particular, it can be analytically shown that DES is* equivalent *to TPITM in that it provides the same partition of energy and, thus, very similar statistics of the resolved field if the coefficient C_{DES} is made a function of the energy ratio r. This result enables DES to be interpreted as a hybrid Temporal-LES/RANS approach, i.e., as a model to close the temporally filtered Navier-Stokes equations, and compatible with the limit of infinite temporal filter width (RANS).*

1 INTRODUCTION

Many hybrid RANS-LES models can be described as seamless, in the sense that the computation progressively transitions from a RANS model to an LES in other region. However, the majority of flows of practical relevance are inhomogeneous, and in that case such models suffer from an important conceptual weakness: LES gives spatially filtered fields; whereas, RANS gives long-time averaged fields.

As will be shown in section 2, this issue can be addressed in the frame of temporal filtering, in the context of statistically steady (stationary) flows, with a filter width Δ_T continuously going to infinity, which is the limit corresponding to the RANS approach. Models derived within this formalism are then hybrid RANS/Temporal Large-Eddy Simulation (TLES) models. All the flows for which the boundary conditions are not varying in time are stationary, i.e., are statistically independent on a shift in time, including massively separated wakes with vortex shedding, such that the formalism can be rigorously applied to many flows of practical importance. In practice, for flows with time-dependent boundary conditions, the present approach can also be applied although the theoretical link between RANS and TLES is then lost, similar to what is usually done for approaches developed in homogeneous turbulence.

In section 3, it will be shown that this formalism enables the adaptation of the Partially Integrated Transport Model (PITM) [1] to a temporal-PITM (TPITM) model. The resulting hybrid method is based on transport equations for the subfilter scales, and, in order to account for nonlocal wall-blockage, the elliptic blending methodology [2] is applied.

Finally, section 4 is dedicated to the investigation of the equivalence between DES and TPITM. Indeed, in TPITM, the temporal filter width Δ_T is not explicitly involved in the model, and the fundamental parameter is r, the energy ratio of modeled energy to total energy. Therefore, many approaches, can be regarded as temporally filtered approaches in which this energy ratio is empirically related to the local grid step (e.g., DES) or the integral length scale (e.g., URANS) [3]. Moreover, the equivalence between DES and TPITM can be explicitly found in some particular cases, by analyzing the equilibrium states of the two approaches.

2 TEMPORAL FILTERING: A CONSISTENT FORMALISM FOR HYBRID MODELS

2.1 Definition of the filter

A Galilean invariant, time filtering operator can be defined, in order to decompose the instantaneous velocity $\mathbf{u}^*$ into filtered (resolved) $\widetilde{\mathbf{U}} = <\mathbf{u}^*>$ and residual $\mathbf{u}''$ parts, where the resolved part is

$$<\mathbf{u}^*>(\mathbf{x},t) = \iint \mathcal{G}(\mathbf{x}' - \boldsymbol{\xi}(\mathbf{x},t'-t), t'-t)\ \mathbf{u}^*(\mathbf{x}',t')\ \mathrm{d}\mathbf{x}'\mathrm{d}t', \quad (1)$$

with a filter of the form

$$\mathcal{G}(\mathbf{x}' - \boldsymbol{\xi}(\mathbf{x}, t' - t), t' - t) = \delta(\mathbf{x}' - \boldsymbol{\xi}(\mathbf{x}, t' - t))\ G_{\Delta_T}(t' - t) \tag{2}$$

with Δ_T the temporal filter width. The introduction of $\boldsymbol{\xi}(\mathbf{x}, t'-t) = \mathbf{x} + \mathbf{V}_{\mathrm{ref}}\,(t'-t)$, with a $\mathbf{V}_{\mathrm{ref}}$ reference velocity related to the flow configuration, e.g., the boundary conditions, aims at ensuring that the filtering operation preserves the Galilean invariance [3].

In the case of a stationary flow, Reynolds averaged quantities are obtained as the limit of filtered quantities when the temporal width of the filter goes to infinity. For example, if the filter kernel G_{Δ_T} is a top-hat filter, the filtering operation is equivalent to a running average from $t - \Delta_T$ to t, denoted by $[\,.\,]_{\{\Delta_T;\mathbf{V}_{\mathrm{ref}}\}}$,

$$<\mathbf{u}^*> (\mathbf{x}, t) = \frac{1}{\Delta_T} \int_{t-\Delta_T}^{t} \mathbf{u}^*(\boldsymbol{\xi}, t')\,\mathrm{d}t' = [\mathbf{u}^*]_{\{\Delta_T;\mathbf{V}_{\mathrm{ref}}\}}(\mathbf{x}, t)\ , \tag{3}$$

and the generalized long-time averaging, defined by

$$[\mathbf{u}^*]_{\{\infty;\mathbf{V}_{\mathrm{ref}}\}}(\mathbf{x}, t) = \lim_{\Delta_T \to \infty} <\mathbf{u}^*> (\mathbf{x}, t)\ , \tag{4}$$

inherits the Galilean invariance from the filter. This property ensures that, in any reference frame, the generalized long-time average (4) is equivalent to the Reynolds average. These definitions thus provide the consistent formalism for seamless hybrid TLES–RANS for flows that are stationary in a particular reference frame.

2.2 Filtered equations

The filtered velocity is denoted by $\widetilde{\mathbf{U}} = <\mathbf{u}^*>$ and the residual velocity is defined by

$$\mathbf{u}'' = \mathbf{u}^* - \widetilde{\mathbf{U}}. \tag{5}$$

The Reynolds-averaged, or long-time-averaged, velocity is denoted by $\mathbf{U}$, and the fluctuating part of the filtered velocity by

$$\mathbf{u}' = \widetilde{\mathbf{U}} - \mathbf{U}, \tag{6}$$

such that the total fluctuation is

$$\mathbf{u} = \mathbf{u}^* - \mathbf{U} = \mathbf{u}' + \mathbf{u}''. \tag{7}$$

For a commutative filter, the filtered momentum equation reads

$$\frac{\partial \widetilde{U}_i}{\partial t} + \widetilde{U}_k \frac{\partial \widetilde{U}_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial \widetilde{P}}{\partial x_i} + \nu \frac{\partial^2 \widetilde{U}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij_{\mathrm{SFS}}}}{\partial x_j}. \tag{8}$$

The subfilter-scale (SFS) tensor $\tau_{ij_{\text{SFS}}}$ is defined as the generalized central second moment $\tau_{ij_{\text{SFS}}} = \tau(u_i^*, u_j^*)$, where $\tau(a,b) =< ab > - < a >< b >$. The transport equation for the subfilter stress reads

$$\frac{\partial \tau_{ij_{\text{SFS}}}}{\partial t} + \widetilde{U}_k \frac{\partial \tau_{ij_{\text{SFS}}}}{\partial x_k} = \underbrace{-\frac{\partial \tau(u_i^*, u_j^*, u_k^*)}{\partial x_k}}_{D^T_{ij_{\text{SFS}}}} + \underbrace{\nu \frac{\partial^2 \tau_{ij_{\text{SFS}}}}{\partial x_k \partial x_k}}_{D^\nu_{ij_{\text{SFS}}}} \underbrace{- 2\nu\tau\left(\frac{\partial u_i^*}{\partial x_k}, \frac{\partial u_j^*}{\partial x_k}\right)}_{\varepsilon_{ij_{\text{SFS}}}}$$
$$\underbrace{-\frac{1}{\rho}\tau\left(u_i^*, \frac{\partial p^*}{\partial x_j}\right) - \frac{1}{\rho}\tau\left(u_j^*, \frac{\partial p^*}{\partial x_i}\right)}_{\phi_{ij_{\text{SFS}}}} \underbrace{- \tau_{ik_{\text{SFS}}} \frac{\partial \widetilde{U}_j}{\partial x_k} - \tau_{jk_{\text{SFS}}} \frac{\partial \widetilde{U}_i}{\partial x_k}}_{P_{ij_{\text{SFS}}}}, \tag{9}$$

where

$$\tau(a,b,c) =<abc> - <a> \tau(b,c) - <b> \tau(a,c) - <c> \tau(a,b) - <a><b><c> .$$

It is worth emphasizing that this equation is formally identical to the Reynolds stress ($\overline{u_i u_j}$) transport equation: this feature forms the basis for the adaptation of a RANS second moment closure to the hybrid TLES–RANS context presented in section 3.3.

For stationary flows, the temporal filter satisfies

$$\overline{<\mathbf{u}^*>} = \overline{\mathbf{u}^*}, \tag{10}$$

such that the total Reynolds stress $\overline{u_i u_j}$ and turbulent energy are exactly decomposed as

$$\overline{u_i u_j} = \left(\overline{\widetilde{U}_i \widetilde{U}_j} - U_i U_j\right) + \overline{\tau_{ij_{\text{SFS}}}}. \tag{11}$$

and

$$k = \frac{1}{2}\overline{u_i u_i} = k_m + k_r, \tag{12}$$

where the subfilter scale (modeled) and resolved parts are

$$k_m = \overline{k_{\text{SFS}}} = \frac{1}{2}\overline{\tau_{ii\text{SFS}}} \qquad \text{and} \qquad k_r = \frac{1}{2}\left(\overline{\widetilde{U}_i \widetilde{U}_i} - U_i U_i\right) = \frac{1}{2}\overline{u_i' u_i'}. \tag{13}$$

The formalism introduced in this section provides an appropriate framework for hybrid methodologies. The definition of the filtering operator, based on a temporal kernel, ensures that the variables $\widetilde{U}_i$, $\widetilde{P}$ and $\tau_{ij_{\text{SFS}}}$ consistently and continuously tend to their corresponding RANS counterparts, U_i, P and $\overline{u_i u_j}$, when the temporal filter width goes to infinity [4]. The main issue is thus the modeling of the subfilter stress, which must be compatible with a characteristic cutoff frequency ω_c ranging from 0 (RANS) to the inertial range (TLES). The form-invariance of TLES and RANS equations suggests that the form of the models used in TLES and RANS regions can be identical, with an appropriate modification of the coefficients or scales to ensure the transition.

3 TEMPORAL PITM (TPITM)

In order to model the subfilter stresses, Fadai-Ghotbi *et al.* [3] proposed the Temporal Partially Integrated Transport Model (TPITM), an adaptation to time filtering of the PITM model [1]. Only a summary of the analytical derivation is provided here. For details, the reader is referred to the original article [3].

3.1 Energy partition in frequency space

The Eulerian frequency spectrum is defined by the relation

$$k(\mathbf{x}) = \int_0^\infty E_T(\mathbf{x}, \omega)\mathrm{d}\omega, \tag{14}$$

and is the temporal Fourier transform of the two-time correlation tensor

$$Q_{i,j}(\mathbf{x}, \tau) = \overline{u_i(\mathbf{x}, t)u_j(\mathbf{x}, t + \tau)} \tag{15}$$

The resolved part of the turbulent energy is given by

$$k_r(\mathbf{x}) = \int_0^\infty \widehat{G}_{\Delta_T}(\omega)\, \widehat{G}^*_{\Delta_T}(\omega) E_T(\mathbf{x}, \omega)\mathrm{d}\omega \ , \tag{16}$$

and the modeled energy, $k_m = k - k_r$,

$$k_m(\mathbf{x}) = \int_0^\infty \left[1 - \widehat{G}_{\Delta_T}(\omega)\, \widehat{G}^*_{\Delta_T}(\omega)\right] E_T(\mathbf{x}, \omega)\mathrm{d}\omega \ , \tag{17}$$

where $\widehat{G}_{\Delta_T}(\omega)$ is the Fourier transform of the temporal filter kernel $G_{\Delta_T}(\tau)$, as shown in Fig. 1.

From the Fourier transform of the transport equation for $Q_{i,i}$, the Eulerian temporal energy spectrum equation can be written as [3]

$$\frac{DE_T}{Dt} \ = \ \hat{\mathbb{P}} + \hat{\mathbb{D}} - \hat{\mathbb{E}} + \hat{\mathbb{T}} \ , \tag{18}$$

in which $\hat{\mathbb{P}}$ is the production by the mean velocity, $\hat{\mathbb{D}}$ the diffusion term, sum of the turbulent $\hat{\mathbb{D}}^T$, molecular $\hat{\mathbb{D}}^\nu$ and pressure $\hat{\mathbb{D}}^P$ diffusions, and $\hat{\mathbb{E}}$ the dissipation rate. The spectral flux $\hat{\mathbb{T}}$ originates from the non-linear interactions.

Using Eq. (17), the transport equation for the subfilter energy is

$$\frac{Dk_m}{Dt} = P_m + D_m - \varepsilon_m - T_G \tag{19}$$

where

$$D_m = \int_0^\infty (1 - \widehat{G}_{\Delta_T}\, \widehat{G}^*_{\Delta_T})\hat{\mathbb{D}}\, \mathrm{d}\omega; \quad P_m = \int_0^\infty (1 - \widehat{G}_{\Delta_T}\, \widehat{G}^*_{\Delta_T})(\hat{\mathbb{P}} + \hat{\mathbb{T}})\, \mathrm{d}\omega; \tag{20}$$

$$\varepsilon_m = \int_0^\infty (1 - \widehat{G}_{\Delta_T}\, \widehat{G}^*_{\Delta_T})\hat{\mathbb{E}}\, \mathrm{d}\omega; \quad T_G = \int_0^\infty E_T\, \frac{D}{Dt}(\widehat{G}_{\Delta_T}\widehat{G}^*_{\Delta_T})\, \mathrm{d}\omega. \tag{21}$$

T_G is a transfer term arising from the variations of filter width. The other terms are the subfilter parts of the corresponding terms in the total turbulent energy k, which is recovered term by term when $\Delta_T \to \infty$.

3.2 Dissipation rate equation

In order to ensure the transition from RANS to LES, the model must be made dependent on the filter width, aiming at controlling the amount of resolved energy. In order to know how to modify the equations to make the solution dependent on the characteristic frequency of the filter $\omega_c = \pi/\Delta_T$, a second filter G'_{Δ_T} is introduced, at the frequency ω_d, such that the turbulent spectrum is divided into three parts, the resolved range $[0;\omega_c]$, the subfilter energetic range $[\omega_c;\omega_d]$ and the subfilter dissipative range $[\omega_d;\infty]$, as illustrated in Fig. 1. Similar to the case of spatially filtered PITM [1], ω_d is defined as

$$\omega_d = \omega_c + \chi_m \frac{\varepsilon_m}{k_m}\ , \tag{22}$$

where χ_m is a constant chosen in such a way that the energy in the range $[\omega_d;\infty]$ is negligible compared to the energy in the range $[\omega_c;\omega_d]$.

Taking the material derivative of Eq. (22) and using Eq. (19), a transport equation for the subfilter dissipation rate can be written as

$$\frac{D\varepsilon_m}{Dt} = \frac{\varepsilon_m}{k_m} P_m - \frac{\varepsilon_m^2}{k_m}\left[1 - \left(\frac{k_m}{\varepsilon_m}\right)\frac{\mathrm{d}\omega_d/\mathrm{d}t - \mathrm{d}\omega_c/\mathrm{d}t}{\omega_d - \omega_c} + \frac{T_G}{\varepsilon_m}\right] + \frac{\varepsilon_m}{k_m} D_m\ . \tag{23}$$

Fadai-Ghotbi *et al.* [3] have shown that, after some algebraic manipulations, the equation can be recast under the form

$$\frac{D\varepsilon_m}{Dt} = C_{\varepsilon_1}\frac{\varepsilon_m}{k_m} P_m - \underbrace{\Big[C_{\varepsilon_1} + r\,(C_{\varepsilon_2} - C_{\varepsilon_1})\Big]}_{C^*_{\varepsilon_2}} \frac{\varepsilon_m^2}{k_m} + D_{\varepsilon_m}\ , \tag{24}$$

which is similar to the equation found in the spatial PITM approach [1], with the additional diffusion term D_{ε_m}. The fundamental parameter in this formulation is the energy ratio $r = k_m/(k_m + k_r)$ that controls the transition from RANS to TLES via the variable coefficient $C^*_{\varepsilon_2}$. In principle, k_r can be computed from the resolved field, such that r will be adapted to the amount of energy contained in the resolved turbulent structures.

3.3 Transport model for the subfilter stresses

The cutoff frequency can be located in the large-scale region of the turbulent spectrum in hybrid TLES–RANS, such that complex production and redistribution mechanisms can play a major role. Therefore, the model is based on the subfilter-stress transport equations. Because of the formal similarity between these equations and the transport equations for the Reynolds-stress tensor, using a model formally identical to a RANS second-moment closure is legitimate, provided that length and time scales are modified to account for the variable cutoff frequency.

Fadai-Ghotbi *et al.* [5] proposed, in the context of spatial PITM, an adaptation of the Elliptic Blending Reynolds-Stress Model (EB-RSM), usually applied in a RANS context, which is an extension of the SSG model [6] to the near-wall region. The velocity–pressure-gradient correlation and dissipation terms are modeled using a blending of near-wall and homogeneous models

$$\phi_{ij_{\text{SFS}}} = (1-\alpha^3)\phi_{ij}^w + \alpha^3\phi_{ij}^h, \tag{25}$$

$$\varepsilon_{ij_{\text{SFS}}} = (1-\alpha^3)\frac{\tau_{ij_{\text{SFS}}}}{k_{\text{SFS}}}\varepsilon_{\text{SFS}} + \alpha^3\frac{2}{3}\varepsilon_{\text{SFS}}\delta_{ij} \ , \tag{26}$$

where, the blending parameter α goes from zero at the wall, to unity far from the wall. The elliptic relaxation equation

$$\alpha - L_{\text{SFS}}^2\boldsymbol{\nabla}^2\alpha = 1 \tag{27}$$

is solved to preserve the nonlocal character of the pressure. The near-wall form of the model

$$\phi_{ij}^w = -5\frac{\varepsilon_{\text{SFS}}}{k_{\text{SFS}}}\left[\tau_{ik_{\text{SFS}}}n_jn_k + \tau_{jk_{\text{SFS}}}n_in_k - \frac{1}{2}\tau_{kl_{\text{SFS}}}n_kn_l\left(n_in_j+\delta_{ij}\right)\right] \tag{28}$$

ensures the correct asymptotic behavior of all the variables at the wall. As shown by Fadai-Ghotbi *et al.* [5], the length scale of the elliptic equation (27) must be made a function of the parameter r

$$L_{\text{SFS}} = C_L \max\left(\frac{k_{\text{SFS}}^{3/2}}{\varepsilon_{\text{SFS}}}, r^{3/2}C_\eta\frac{\nu^{3/4}}{\varepsilon^{1/4}}\right). \tag{29}$$

in order to account for the fact that a part of the turbulent scales are explicitly resolved.

3.4 Control of the energy partition

The choice of the temporal filter width Δ_T is not as obvious as in spatial LES: it would seem straightforward to link Δ_T to the time step used in the computation, Δt, for instance by $\Delta_T = 2\Delta t$, but, in order to allow local variations of the filter width, the possibility of linking the temporal filter width to the local mesh refinement is investigated.

The dispersion relation $\omega = f(\kappa)$, necessary to evaluate the frequency ω_c corresponding to a given cutoff wavenumber κ_c, is not known in general. However, the Eulerian frequency spectrum $E_T(\omega)$ and the wavenumber spectrum $E(\kappa)$ are related by

$$\mathrm{d}k = E(\kappa)\,\mathrm{d}\kappa = E_T(\omega)\,\mathrm{d}\omega \ . \tag{30}$$

Therefore, a simple analytical expression of the energy ratio r can be obtained in the case of a cutoff filter

$$r = \frac{1}{k}\int_{\omega_c}^{\infty} E_T(\omega)\mathrm{d}\omega = \frac{1}{k}\int_{\kappa_c}^{\infty} E_T(\omega)\frac{E(\kappa)}{E_T(\omega)}\mathrm{d}\kappa = \frac{1}{k}\int_{\kappa_c}^{\infty} E(\kappa)\,\mathrm{d}\kappa \ . \tag{31}$$

Since the model does not directly involve ω_c but only r, Eq. (31) shows that the dispersion relation is actually not required. Using a standard Kolmogorov spectrum in wavenumber space yields

$$r = \frac{3}{2}C_\kappa\left(\kappa_c\frac{k^{3/2}}{\varepsilon}\right)^{-\frac{2}{3}} \ . \tag{32}$$

The use of this relation does not imply that the temporal filtering is replaced by spatial filtering: it is only a convenient way, justified by Eq. (31), of making the equations of the model sensitive to the local grid step. As shown by Fadai-Ghotbi *et al.* [5], a better control of the parameter r in the near-wall region is obtained with the empirical formulation

$$r = (1-\alpha^3) + \alpha^3\beta_0^{-1}\left(\kappa_c\frac{k^{3/2}}{\varepsilon}\right)^{-\frac{2}{3}} , \tag{33}$$

used to enforce the RANS mode in the near-wall region.

This relation gives the energy ratio r the user *targets*, which drives the coefficient $C^*_{\varepsilon 2}$ in the dissipation rate equation. However, one of the difficulties is that this value is in general not exactly *observed* in the solution. In particular in a channel flow, the model has a tendency to underestimate the resolved energy, and to eventually tend to a steady solution.

Therefore, a dynamical correction of the coefficient $C^*_{\varepsilon_2}$

$$C^*_{\varepsilon_2} = C_{\varepsilon_1} + r\,(C_{\varepsilon_2} - C_{\varepsilon_1}),$$

is used. The energy ratio $k_m/(k_m+k_r)$, called the *observed* ratio r^o is monitored during the calculation and compared with the ratio given by Eq. (33), which is called the *target* ratio r^t. The coefficient $C^*_{\varepsilon_2}$ in the dissipation equation is replaced by $C^*_{\varepsilon_2}+\delta C^*_{\varepsilon_2}$, in order to drive the observed ratio toward the target ratio.

As will be shown in section 4.3 (see, for instance, Eq. 58), the desired variation of the level of modeled energy δk_m and the variation $\delta C^*_{\varepsilon 2}$ of the coefficient are linked by a relation of the form

$$\delta C^*_{\varepsilon 2} = \mathcal{C}\,\frac{\delta k_m}{k_m} \ . \tag{34}$$

Eq. (58) shows that the parameter $\mathcal{C}$ is equal to $3\,(C^*_{\varepsilon 2} - C_{\varepsilon 2}) = 3\,r\,(C_{\varepsilon_2} - C_{\varepsilon_1})$ and, thus, is dependent on the energy ratio r. In practice, this parameter is taken constant. In order to reach the target ratio r^t, the desired variation δk_m is

$$\delta k_m = k_m\frac{\delta r^o}{r^o} = k_m\left(\frac{r^t}{r^o}-1\right) \ , \tag{35}$$

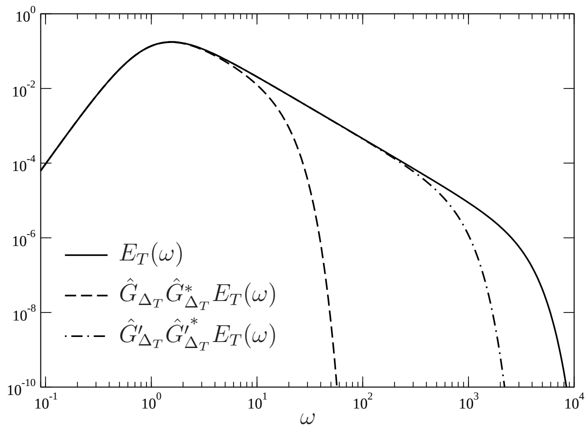

Figure 1: Example of application of the two filters of the TPITM approach to a generic spectrum.

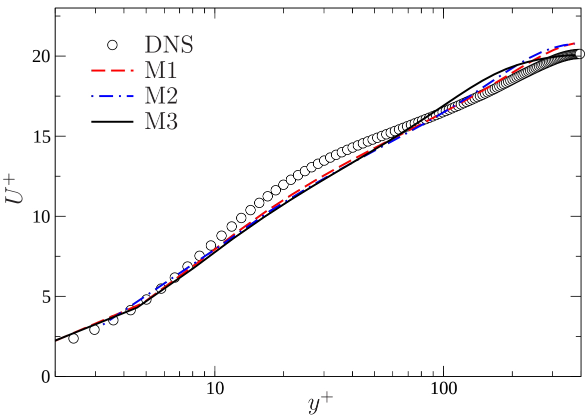

Figure 2: Channel flow at $Re_\tau = 395$. Mean velocity profiles.

such that the following relation is obtained

$$\delta C^*_{\varepsilon_2} = \mathcal{C}\left(\frac{r^t}{r^o} - 1\right) \ , \tag{36}$$

which provides an estimate of the dynamic correction to be applied.

3.5 Validation in channel flow

A channel flow at $Re_\tau = 395$ is computed using the open-source software Code_Saturne, a parallel, finite volume solver on unstructured grids, developed at EDF [7], distributed under Gnu GPL license[1]. The numerical method is based on a SIMPLEC algorithm, with a Rhie & Chow interpolation, and is second-order accurate in space and time.

Computations are performed in a domain spanning $8h \times 2h \times 4h$, corresponding to $3160 \times 790 \times 1580$ in wall units. The reference mesh, called M2, contains $64 \times 54 \times 64$ cells. In order to assess the ability of the model to tend to a RANS mode on a coarser mesh and to a TLES mode on a finer mesh, the reference mesh is respectively coarsened by a factor of 2 (mesh M1) or refined by a factor 1.5 (mesh M3) in streamwise and spanwise directions.

Fig. 3 shows the profiles of the modeled, resolved and total shear stress, in comparison with the DNS data [8]. It can be seen that the method indeed provides the appropriate control of the respective contributions of the resolved (filtered) and modeled (subfilter) fields. Relation (33) enforces a RANS solution in the near-wall region, independently of the grid. The computation continuously transitions to TLES toward the outer part of the flow. Moreover, Fig. 4 shows a remarkable feature of the model: when the mesh is refined, the partition of energy is radically modified, but the total energy remains almost constant.

[1]http://www.code-saturne.org

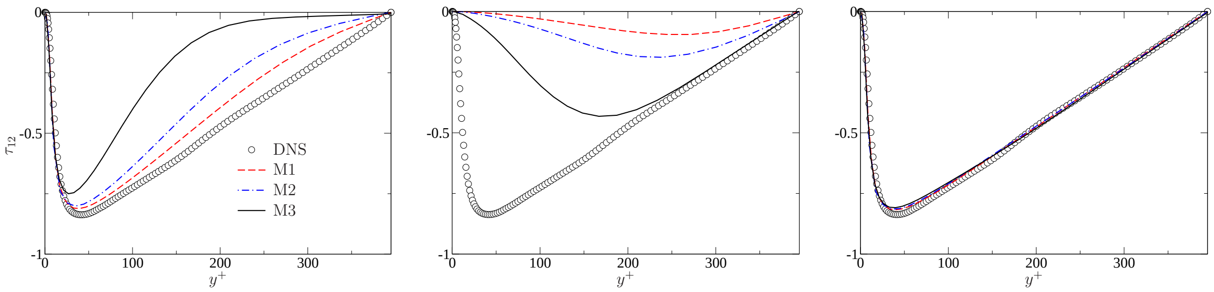

Figure 3: Channel flow at $Re_\tau = 395$. Shear stress profiles. Left: Contribution of the subfilter scales; Middle: Contribution of the resolved scales; Right: Total.

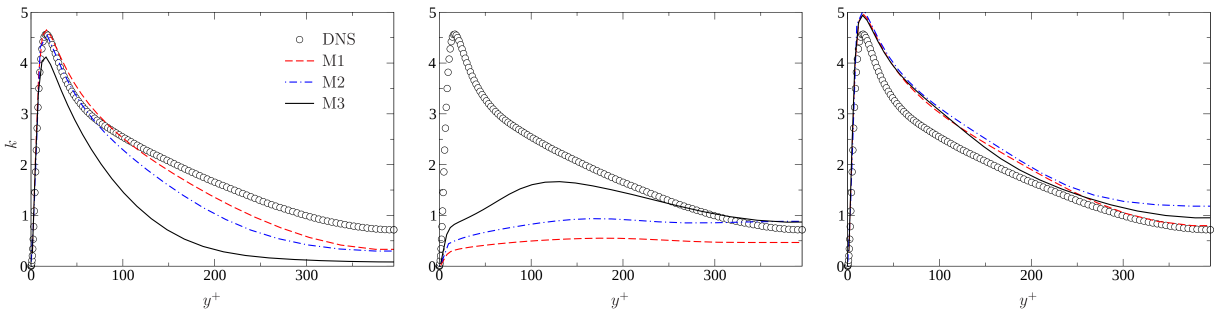

Figure 4: Channel flow at $Re_\tau = 395$. Turbulent energy profiles. Left: Contribution of the subfilter scales; Middle: Contribution of the resolved scales; Right: Total.

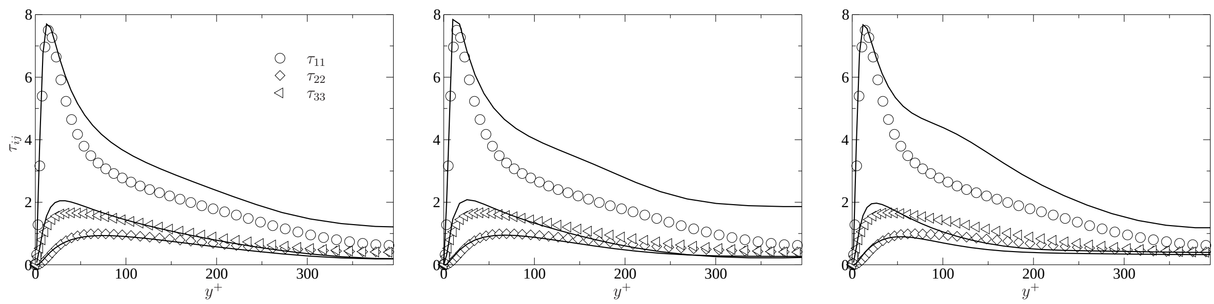

Figure 5: Channel flow at $Re_\tau = 395$. Normal stresses. Left: Mesh M1; Middle: Mesh M2; Right: Mesh M3.

Fig. 5 shows a satisfactory reproduction of the normal components of the Reynolds stress. In the near-wall region, these results are to be credited to the RANS model. The case of the reference mesh M2 is the most difficult configuration for the model, since the resolution is not sufficient to compute a true TLES. It can be seen that the streamwise component is overestimated, but, as shown in Figs. 2 and 4, the predictions of the turbulent energy and of the mean velocity profile remain very satisfactory.

Fig. 2 shows an important feature of this approach: the mean velocity profile is very acceptable, even using meshes much coarser than a LES-type mesh. When the mesh is refined, although the relative weights of the resolved and modeled contributions to the flow field are drastically modified, the prediction of the mean velocity profile shows a very moderate sensitivity to the mesh, the variation of the flow rate being less than 0.5%.

4 EQUIVALENT DES

It can be argued that many, if not all, seamless methods should be regarded as RANS–TLES hybrid methods [3]. This remark also applies to the basic URANS in stationary flows. Indeed, although the RANS equations are recovered by making the temporal filter width go to infinity, the time scale entering the RANS models is the integral time scale; therefore, the equations of the model are formally identical to the equations of a subfilter stress model with a filter width equal to the integral time scale.

In the following of this section, the case of DES [9, 10] is investigated in details. DES, as with TPITM, consists of a modification of RANS model equations. DES uses an increase of the dissipation term in the turbulent energy equation, and TPITM a reduction of the destruction term in the dissipation equation, in order to control the *energy partition* among resolved and modeled scales. Therefore, the question arises whether some equivalence, following the definition given in section 4.2, can be found between the two approaches.

4.1 Definition of self-consistency

A hybrid model is called henceforth *self-consistent* if it satisfies the behavior expected from a filtered approach: firstly, the transition from RANS to LES corresponds to a variation of location of the cutoff frequency, modifying the energy partition while leaving the total turbulent energy unchanged; secondly, since the cutoff frequency is located below the dissipative range of the spectrum, the dissipation rate is not affected by this transition.

Self-consistency is desirable in flow regions close to equilibrium. In regions far from equilibrium, such as massively separated wakes, hybrid methods are on the contrary expected to improve the flow prediction compared to the RANS model.

4.2 Definition of equivalence

Two *self-consistent hybrid models* based on the same RANS model are distinguished by the method used to control the energy partition. In this case, if their control parameters are adjusted such a way that the level of modeled energy is the same for the two models, their respective resolved fields will be very similar. In order to be able to proceed in the analysis, we will admit the following postulate:

> Postulate: *Two hybrid approaches based on the same underlying RANS model, but using a different method of control of the energy partition, yield similar resolved velocity fields provided that they give the same level of subfilter energy.*

Similar simply means here that the statistics will not differ significantly. In this case, the two approaches will be considered *equivalent*.

4.3 Equivalence between DES and TPITM

In DES based on two-equation RANS model, the transition from RANS to LES is obtained by introducing the factor

$$\psi = \max\left(1; \frac{k^{3/2}/\varepsilon}{C_{\text{DES}}\Delta}\right) \tag{37}$$

in the term ε in the turbulent energy equation, where Δ is the grid step.

The present section addresses the issue of establishing a relation between the functions $C^*_{\varepsilon 2}$ of TPITM and ψ of DES in particular cases.

Equilibrium layers

For flows such as homogeneous shear or in the logarithmic region of a boundary layer, the turbulence evolves to a state where equilibrium values are reached for quantities such as the production-to-dissipation ratio, the ratio of turbulent to mean shear time scales, and the turbulent stress anisotropies.

In these cases, the system of equations for the TPITM model reduces to

$$\begin{aligned} \frac{\mathrm{d}k_m}{\mathrm{d}t} &= P_m - \varepsilon \\ \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} &= C_{\varepsilon 1}\frac{\varepsilon}{k}P_m - C^*_{\varepsilon 2}\frac{\varepsilon^2}{k} \end{aligned} \tag{38}$$

For eddy-viscosity models, the average subfilter production is

$$P_m = \overline{P_{\text{SFS}}} = -\overline{\tau_{ij\text{SFS}}\frac{\partial \tilde{U}_i}{\partial x_j}} = C_\mu \gamma \frac{k_m^2}{\varepsilon} S^2 \ , \tag{39}$$

where the definitions $S = \sqrt{2\overline{\tilde{S}^2}}$ and

$$\gamma = \frac{\varepsilon}{k_m^2 S^2}\overline{\frac{k_{\text{SFS}}^2}{\varepsilon_{\text{SFS}}}\tilde{S}^2} \tag{40}$$

have been used. The equilibrium solution is given by

$$\tau^2 = \frac{C^*_{\varepsilon 2} - 1}{C_\mu \left(C_{\varepsilon 1} - 1\right) \gamma S^2} \ . \tag{41}$$

The partition of energy, characterized by the ratio $r = k_m/k$, cannot be explicitly determined, since the resolved energy k_r can only be given by a full computation.

The question that arises now is how to modify the coefficient $C^*_{\varepsilon 2}$ to obtain an uniform variation of the energy partition, such that $\delta k_m/k_m$ is a constant over the domain. Introducing the infinitesimal variation $\delta C^*_{\varepsilon 2}$ of the coefficient $C^*_{\varepsilon 2}$ in the system of equations, a new the equilibrium solution can be evaluated, showing that the relative variation $\delta\tau/\tau$ of the equilibrium time scale is

$$\frac{\delta\tau}{\tau} = \frac{1}{2}\frac{\delta C^*_{\varepsilon 2}}{C^*_{\varepsilon 2} - 1} - \frac{1}{2}\frac{\delta\gamma}{\gamma} - \frac{\delta S}{S} \tag{42}$$

The same result can be obtained by logarithmic differentiation of Eq. (41).

For a *self-consistent* hybrid model, as defined in section 4.2, the dissipation rate is not affected by the modification of the energy partition, such that

$$\frac{\delta\tau}{\tau} = \frac{\delta k_m}{k_m} \ , \tag{43}$$

and the variation $\delta C^*_{\varepsilon 2}$ to be applied is

$$\delta C^*_{\varepsilon 2} = \left(C^*_{\varepsilon 2} - 1\right)\left(2\frac{\delta k_m}{k_m} + \frac{\delta\gamma}{\gamma} + 2\frac{\delta S}{S}\right) \ . \tag{44}$$

Now, for the DES system

$$\begin{aligned} \frac{\mathrm{d}k_m}{\mathrm{d}t} &= P_m - \psi\varepsilon \\ \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} &= C_{\varepsilon 1}\frac{\varepsilon}{k}P_m - C_{\varepsilon 2}\frac{\varepsilon^2}{k} \ , \end{aligned} \tag{45}$$

it can easily be shown that the variation $\delta\psi$ to apply in order to modify the modeled energy by the factor $\delta k_m/k_m$ is

$$\delta\psi = -\left(C_{\varepsilon 2} - \psi\right)\left(2\frac{\delta k_m}{k_m} + \frac{\delta\gamma}{\gamma} + 2\frac{\delta S}{S}\right) \ . \tag{46}$$

Applying the postulate presented in section 4.2, the same modification δk_m of the subfilter energy for the two models, TPITM and DES, corresponds to the same modification of the statistics. Assuming that the two models are equivalent with the initial values of $C^*_{\varepsilon 2}$ and ψ, they remain equivalent with modified coefficients provided that the relation

$$\frac{\delta C^*_{\varepsilon 2}}{C^*_{\varepsilon 2} - 1} = -\frac{\delta\psi}{C_{\varepsilon 2} - \psi} \tag{47}$$

is satisfied. Therefore, the two models remain equivalent if, starting from a common state, the partition of energy is progressively modified by applying successive infinitesimal variations of the coefficients $C^*_{\varepsilon 2}$ and ψ related by Eq. (47). Such a common state exists: the RANS limit. Indeed, when $C^*_{\varepsilon 2} = C_{\varepsilon 2}$ for TPITM and $\psi = 1$ for DES, the equations of the two models are identical.

Integrating between the RANS state and any arbitrary state

$$\int_{C_{\varepsilon 2}}^{C^*_{\varepsilon 2}} \frac{1}{x-1} \mathrm{d}x = -\int_{1}^{\psi} \frac{1}{C_{\varepsilon 2} - y} \mathrm{d}y \,,$$

and using the definition of $C^*_{\varepsilon 2}$ given in Eq. (24), shows that, if

$$\psi = 1 + (C_{\varepsilon 2} - C_{\varepsilon 1})\,(1-r)\ , \tag{48}$$

DES is *equivalent* to TPITM.

In the regions where DES is in LES mode, the ψ coefficient is

$$\psi = \frac{k_m^{3/2}}{\varepsilon L}\ , \tag{49}$$

where the length scale is $L = C_{\text{DES}}\Delta$. Eq. (48) shows that the length scale to be used in DES to ensure the equivalence with TPITM is

$$L = \frac{r^{3/2}}{1 + (C_{\varepsilon 2} - C_{\varepsilon 1})(1-r)} \frac{k^{3/2}}{\varepsilon}\ . \tag{50}$$

Similar to TPITM, in the so-called *equivalent DES*, defined by Eq. (50), the transition from RANS to LES is driven by variations of the energy ratio $r = k_m/(k_m + k_r)$, which can be computed during the simulation, such that the model does not explicitly involve the grid step Δ.

Influence of diffusion

The influence of diffusion effects, neglected in the previous section, is now examined. In order to keep a tractable system of equations, k_m and ε themselves, not only the time-scale $\tau = k_m/\varepsilon$, are assumed to be in equilibrium along streamlines, which is for instance the case for fully developed flows in straight ducts.

The TPITM system of equations now reads

$$P_m - \varepsilon - D_m = 0 \tag{51}$$

$$C_{\varepsilon 1}\frac{\varepsilon}{k} P_m - C^*_{\varepsilon 2}\frac{\varepsilon^2}{k} - D_\varepsilon = 0\ , \tag{52}$$

where D_m and D_ε stand for the turbulent diffusion of k_m and ε, respectively,

$$D_m = \frac{\partial}{\partial x_k}\left(C_\mu \frac{k_m^2}{\varepsilon}\frac{\partial k_m}{\partial x_k}\right) \qquad \text{and} \qquad D_\varepsilon = \frac{\partial}{\partial x_k}\left(C_\mu \frac{k_m^2}{\varepsilon}\frac{\partial \varepsilon}{\partial x_k}\right) . \tag{53}$$

The same analysis as for equilibrium layers can be followed. Introducing infinitesimal perturbations of the different terms of the equations (δk_m, δP_m, δD_m, δD_ε), making use of Eqs. (51) and (52) and keeping only terms linear in the infinitesimal perturbations yields

$$\delta P_m - \delta D_m = 0 \tag{54}$$

$$C_{\varepsilon 1}\frac{\varepsilon}{k_m}P_m\left(\frac{\delta P_m}{P_m} - \frac{\delta k_m}{k_m}\right) - C^*_{\varepsilon 2}\frac{\varepsilon^2}{k_m}\left(\frac{\delta C^*_{\varepsilon 2}}{C^*_{\varepsilon 2}} - \frac{\delta k_m}{k_m}\right) + \delta D_\varepsilon = 0. \tag{55}$$

The variation δP_m satisfies

$$\frac{\delta P_m}{P_m} = 2\frac{\delta k_m}{k_m} + \frac{\delta\gamma}{\gamma} + 2\frac{\delta S}{S} \tag{56}$$

and, since $\delta k_m/k_m$ is constant, the variations of the diffusion terms are

$$\frac{\delta D_m}{D_m} = 3\frac{\delta k_m}{k_m} \qquad \text{and} \qquad \frac{\delta D_\varepsilon}{D_\varepsilon} = 2\frac{\delta k_m}{k_m} \tag{57}$$

The 7 unknowns, $\delta C^*_{\varepsilon 2}$, δP_m, δD_m, δD_ε, δS, D_m and D_ε, are thus solution of a system of 7 equations, which gives

$$\delta C^*_{\varepsilon 2} = 3\left(C^*_{\varepsilon 2} - C_{\varepsilon 1}\right)\frac{\delta k_m}{k_m} . \tag{58}$$

The same procedure can be followed for the DES system, leading to

$$\delta\psi = -3\,\frac{C_{\varepsilon 2} - C_{\varepsilon 1}\,\psi}{C_{\varepsilon 1}}\frac{\delta k_m}{k_m} . \tag{59}$$

Similar to the equilibrium layer case, TPITM and DES remain equivalent if

$$\psi = 1 + \left(\frac{C_{\varepsilon 2}}{C_{\varepsilon 1}} - 1\right)(1-r) . \tag{60}$$

It is observed that the only difference with Eq. (48) is the slope of the linear function. This same type of influence of the diffusion term was also found by Rumsey *et al.* [11] in their nullcline analysis of two-equation models.

In the analysis leading to Eq. (60), the self-consistency of the hybrid approaches has been used (ε remains constant), but not the postulate of section 4.2. If, on the contrary,

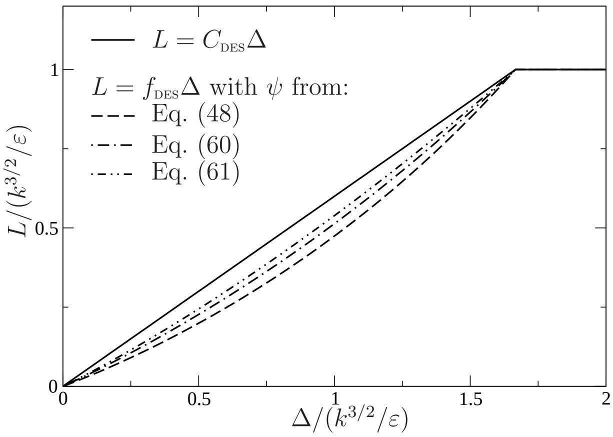

Figure 6: Comparison of the length scales.

the same analysis is followed by allowing variations of ε, for the two approaches to remain equivalent, the variations of the coefficients of TPITM and DES must satisfy

$$\psi = 1 + \left(\frac{C_{\varepsilon 2}}{C_{\varepsilon 1}} - 1\right)\left(1 - r^{C_{\varepsilon 1}/C_{\varepsilon 2}}\right) . \tag{61}$$

which is very close to Eq. (60), since the power $C_{\varepsilon 1}/C_{\varepsilon 2}$ is about 0.75, and r only varies between 0 and 1. In order to obtain the relation between ψ and $C^*_{\varepsilon 2}$, invoking the postulate is necessary, simply because allowing variations of ε removes a constraint on the factor γS^2.

4.4 Relation between r and Δ

The results obtained above show that DES is *equivalent* to TPITM if the usual DES length scale $L = C_{\text{DES}}\Delta$ is replaced by some function of the energy ratio $r = k_m/k$, such that DES can be interpreted as a hybrid RANS/temporal LES, which does not involve the grid step Δ.

However, introducing the analytical expression (32) for r, with $\kappa_c = \pi/\Delta$, into the length scales $L = k_m^{3/2}/(\varepsilon\,\psi)$ obtained for equilibrium layers or duct flows, shows that, for the DES model to be equivalent to TPITM, the constant coefficient C_{DES} must be replaced by the function

$$f_{\text{DES}} = \frac{1}{\beta_0^{3/2}\pi\psi(r)} . \tag{62}$$

If the coefficients C_{DES} and β_0 are chosen in such a way that the length scale L of the model reaches the integral scale for the same value of Δ, Fig. 6 shows that the standard DES is a simple linear approximation of formulation (62), called *equivalent DES*.

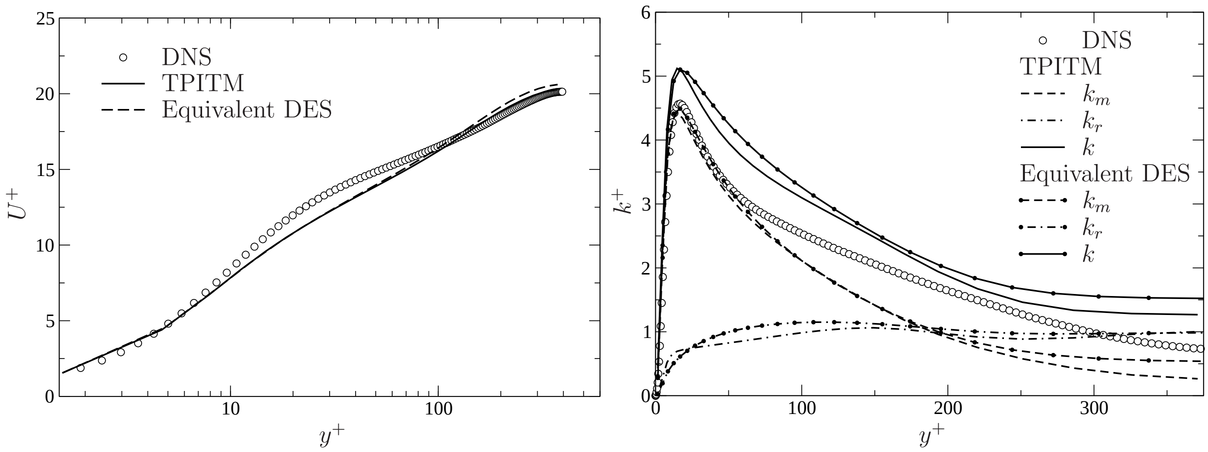

Figure 7: Channel flow at $Re_\tau = 395$: mean velocity (left) and turbulent kinetic energy profiles.

4.5 Computations

Comparisons between TPITM and the equivalent DES are performed for the case described in section 3.5, using the reference mesh M2.

The results obtained using the equivalent DES, with $\psi(r)$ given by Eq. (61), shown in Fig. 7, are very close to those of the TPITM. In particular, the same transition from RANS to TLES when moving away from the wall is obtained, with approximately the same energy partition.

Moreover, it is worth paying some attention to the value of the coefficient β_0. Indeed, in TPITM, the key parameter is the energy ratio r, which, as shown by Eq. (32), can be evaluated as a function of the grid step, and the theoretical value of the coefficient is $\beta_0 = 2/(3C_\kappa) \simeq 0.44$, where C_κ is the Kolmogorov constant. Therefore, contrary to standard DES, for which the coefficient C_{DES} is calibrated against decaying isotropic turbulence, the function f_{DES} of the equivalent DES is entirely determined by the analysis, such that the ability of the f_{DES} function to provide the correct level of dissipation needs to be validated.

The validation is performed in the case of incompressible decaying isotropic turbulence. The initial velocity field at $Re_\lambda = 104.5$ and the reference DNS data are from [12]. The triply-periodic domain is a $(2\pi)^3$ box. The underlying RANS model is the standard k–ε model. Two grids, consisting of 32^3 and 64^3 cells, are used. The initial velocity field is obtained by filtering the DNS field using a cutoff filter adapted to the grid step. The initial fields of k and ε are obtained by running a preliminary computation with frozen velocities (initial velocity field).

The decay of the resolved turbulent energy is shown in Fig. 8. It can be seen that, with the theoretical value $\beta_0 = 0.44$, the two versions of the equivalent DES compared in Fig. 8 provide a correct amount of dissipation, very close to the standard DES with the optimized coefficient $C_{\text{DES}} = 0.6$.

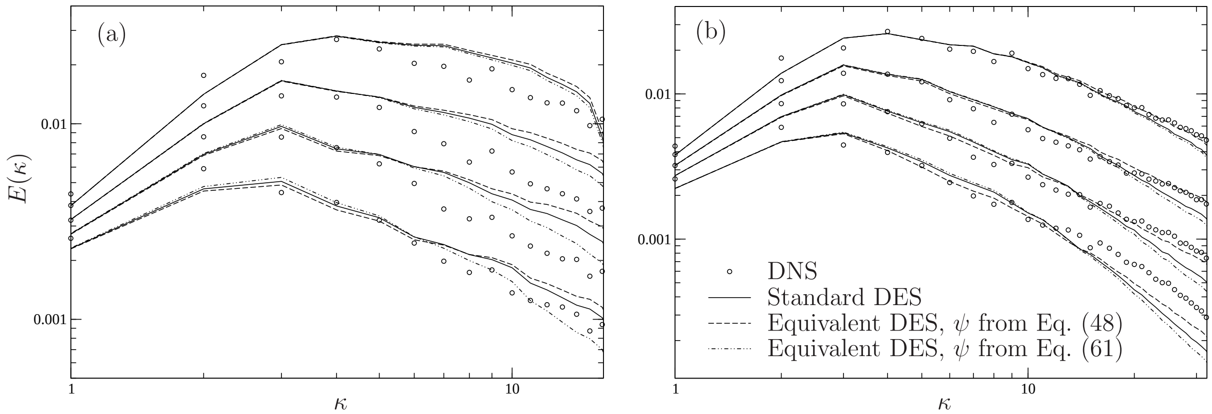

Figure 8: Decaying isotropic turbulence: time evolutions of the energy spectra. (a) 32^3 (b) 64^3.

5 CONCLUSION

In inhomogeneous and stationary turbulent flows, temporal filtering consistently bridges the two approaches, RANS, based on Reynolds averaging, and TLES, based on temporal filtering. The form-invariance between the RANS and TLES equations provides the basis for the use of similar models for the subfilter stresses.

In order to control the energy partition among resolved and modeled scales, and thus seamlessly transition from RANS to TLES, the TPITM, the temporal version of the PITM, has been developed, in which the key parameter is the ratio r of modeled energy to total energy.

An important consequence of this analysis is the fact that a temporally filtered approach does not necessarily need to use explicitly the temporal filter width. In particular, by analyzing the equilibrium solutions of the system of equations for DES and TPITM, a version of DES equivalent to TPITM has been proposed, in which the length scale is parametrized by the energy ratio r, and does not explicitly depend on the grid step. However, if r is evaluated from a Kolmogorov spectrum assumption, it can be shown that the standard DES length scale $L = C_{\text{DES}}\Delta$ is recovered, however with a variable C_{DES} coefficient.

This analysis and the results shown in the present article lead to the conclusion that, although DES is originally an empirical blend of RANS and LES, it is *equivalent* to a particular example of hybrid TLES–RANS model, the TPITM. Therefore, DES can be interpreted as a model for the subfilter stress appearing in the temporally filtered Navier-Stokes equations, compatible with the limit of infinite temporal filter width (RANS) in stationary flows.

These results support the hypothesis originally put forth by Fadai-Ghotbi *et al.*[3]: the existing seamless hybrid methodologies, such as the widely used DES, or the so-called *second-generation URANS models* [13], such as SAS, and even the basic URANS, can be regarded as temporally filtered approaches. This observation then reconciles an apparent inconsistency in the formalisms adopted in many previously proposed hybrid methods.

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