V European Conference on Computational Fluid Dynamics
ECCOMAS CFD 2010
J. C. F. Pereira and A. Sequeira (Eds)
Lisbon, Portugal,14-17 June 2010

STABILIZED DISCONTINUOUS GALERKIN APPROXIMATIONS FOR FOURTH-ORDER STOKES-LIKE PROBLEMS

A.G. Barbosa da Cruz*, **E.G. Dutra do Carmo**† **and F.P. Duda**††

*,††Mechanical Engineering Departament/COPPE/ Federal University of Rio de Janeiro,
Ilha do Fundão, 21945-970, P.B. 68503, Rio de Janeiro, RJ, Brasil
e-mail: aguicruz@mecanica.ufrj.br
†Nuclear Engineering Departament/COPPE/ Federal University of Rio de Janeiro,
Ilha do Fundão, 21945-970, P.B. 68509, Rio de Janeiro, RJ, Brasil
e-mail: aguicruz@mecanica.ufrj.br

Key words: Second grade incompressible fluids, Fourth-order problems, GLS stabilization, Discontinuous Galerkin, Finite element methods

Abstract. *In this work we present a stabilized discontinuous finite element method for a second-gradient theory for treating incompressible fluid recently proposed in the literature. This theory produces a fourth-order flow equation since it accounts for constitutive dependencies on gradient of the velocity field and delivers boundary conditions involving an additional length scale that characterizes the eddies found near walls. The developed finite element approach combines concepts from both stabilized and discontinuous Galerkin methods, avoiding the need for mixed finite element approach or interpolation functions with a high degree of continuity mainly employed to treat similar equations involving fourth-order spatial derivatives. Here, we adopt an alternative approach in which the pressure may also be discontinuous. Furthermore, we employ a GLS formulation with an appropriate choice for the stabilization parameter which results in a discontinuous Galerkin + LS (Least Square) at element level. Therefore, our formulation is weakly coercive, obeys the inf-sup condition in a norm for discrete space, is stable, and allows for an adequate error estimative, which contribute to overcome the main deficiencies of the current formulations. Numerical examples are presented in order to demonstrate the robustness and capability of the proposed method.*

1 INTRODUCTION

Originally derived for flows of incompressible liquids at small-length scales, a continuum-mechanical formulation based on a framework for fluid-dynamical theories involving gradient dependencies has been advanced, which provide a generalization of the Navier-Stokes equations through a higher-order spatial derivatives in the velocity field incorporating length-scales effects, and suitable boundary conditions on free and fixed boundaries surfaces (see Fried and Gurtin[1]). This second-gradient theory is suggested to capture effects at sufficiently small-length scales, being therefore an extension of the classical Navier-Stokes theory.

Additionally, viewing this framework involving gradiente dependencies as a means for generalization of the Navier-Stokes-α equations (NS-α), which combine Lagrangian averaged dispersive nonlinearity with Newtonian viscosity (see Chen et al.[2,3] , for instance), Fried and Gurtin developed an extended theory that provides an alternative continuum mechanical foundation for the NS-α equations[4,5]. In contrast to Lagrangian averaging, this new theory delivers both boundary conditions and thermodynamically-based Lyapunov relations, and accounts for a dependence of kinetic energy upon the velocity gradient, whose dependence suggests to alter the structure of the turbulent energy spectrum (see T.-Y. Kim et al.[6], for instance).

The slight generalized Navier-Stokes equations include fourth-order spatial gradients in the velocity field and, therefore, demand boundary conditions in addition to the classical non-slip condition, which involves a material length scale. The framework of the second-gradient theory provides the basis for numerical studies of flows in complex and realistic geometries of experimental importance.

These results encourage to advance new numerical methods to establish the range of predictions and applicability of this high-order continuum-mechanical formulation. Among the lines of inquiry that are encouraging by numerical studies of this theory include the applicability in the understanding of the liquid flow behavior at small-length scales, and elasticity problems with first and second deformation gradients dependencies on strain energy, as generalized by Gurtin[7] from the early Tupin's[8,9] results within a independent-constitutive equations framework. For flows through irregular geometries of experimental interest, the challenges include developing finite-element methods for fourth-order problems.

The standard variational form of the fourth-order differential equation holds spatial derivatives including order two in both the trial end admissible functions and, therefore, to deal with the difficulties associated with the continuity representation of the high-order derivative across the element boundaries, we adopt a discontinuous Galerkin approximation that allows fourth-order differential equations to be solved using standard C^0 finite element shape functions. This alternative approach avoids mixed methods or interpolation functions with a high degree of continuity, such as C^1 basis functions, employed to treat similar equations involving fourth-order differential operators.

The concept can be traced back to Engel et al.[10], who utilized an interior-penalty type formulation for solving thin beam, plates and strain gradient elasticity problems involving fourth-order elliptic operators. Continuity requirements for high-order derivatives are weakly satisfied by borrowing concepts from discontinuous Galerkin methods, which is achieved by including stabilized terms on the element boundaries. Wells et al.[11] followed a natural extension of this method to treat the hight-order differential operator appearing in the Carhn-Hilliard equation. Similarly, T.-Y. Kim et al.[12] developed a non-conforming discontinuous Galerkin method for the aforementioned gradient theory for incompressible flows closely related to a Nitsche's method for elliptic and parabolic problems[13]. See Arnold et al.[14], Brezzi et al.[15] and Angel et al.[10] for a detailed review of works and extensive literature surveys on several unconditionally stable methods for elliptic problems. Also, a family of discontinuous Galerkin finite elements methods for Stokes and Navier-Stokes problems is found in Girault et al.[16]. Also, see Cockburn et al.[17], Cockburn et al.[18] and Bey and Oden[19] for a additional background.

In this work, the focus is the advancement of a stabilized discontinuous finite element method for fourth-order problems base on the second-gradient theory for incompressible flows as discussed in Fried and Gurtin[1,4]. The proposed finite element formulation combines concepts from both stabilized and discontinuous Galerkin methods. Previously finite element approach for the treatment of fourth-order differential operators has been proposed by Engel et al.[10] and T.-Y. Kim et al.[11]. However, the weak coercivity or the inf-sup condition in a norm for the discrete problem has not been established yet. In the case of the fourth-order problem mentioned before, T.-Y. Kim et al.[11] has presented an approach in which the pressure is assumed to be continuous with the continuity imposed in a weak sense, which may compromise the convergence order of their method.

Here, the main challenge concerning the proper incorporation of the higher-order velocity gradients and pressure field is achieved, and we adopt an alternative approach in which the pressure may be discontinuous. Furthermore, we use a GLS formulation with an appropriate choice for the stabilization parameter which results in a discontinuous Galerkin + LS (Least Square) at element level. Therefore, our formulation is weakly coercive, obeys the inf-sup condition in a norm for discrete space, is stable, and allows for an adequate error estimative, which contribute to overcome the main deficiencies of the current formulations. We emphasize that for triangular and tetrahedral cubic elements the fourth-order operator is empty which implies that any calculation involving derivatives of order greater than two is trivial for these elements.

The approach followed to obtain the stabilization parameter is based on a recent procedure that find the optimum parameter of the GLS method for each element of the finite element mesh, see Carmo et al[20,21]. We defer identifying this parameter as dependent of the interpolation polynomial degree used for approximate the velocity and pressure field, the geometry of the element, as well as the diffusion tensor. The stability of the GLS method with this parameter is assured through a theorem that will be presented in a future communication. The organization of this paper is as follow: the fourth order problem is introduced in Section 2 and the associated variational problem is presented in Section 3. The discontinuous Galerkin formulation, along with a Galerkin/Least Square stabilization (GLS) of the finite element method is discussed in Section 4. Numerical examples are solved in Section 5, in order to show the capabilities of the method on some significante model problems. Concluding remarks and a discussion in Section 6 complete the paper.

2 THE STRONG FORM OF THE FOURTH-ORDER PROBLEM

We denote by $\Omega \subset R^n$ ($n = 2$ or 3) an open region in space, which is occupied by the body over some time interval. The boundary surface of Ω is assumed to be smooth, and is denoted by $\mathcal{S} = \partial\Omega$, and we write $\mathbf{n}$ for the outward unit normal on $\mathcal{S}$. Further, we consider a free-surface portion of $\mathcal{S}$, namely $\mathcal{S}_\mathrm{f}$, and the remainder as a fixed, impermeable surface without slip $\mathcal{S}_\mathrm{g}$, satisfying $\mathcal{S} = \mathcal{S}_\mathrm{f} \cup \mathcal{S}_\mathrm{g}$, such that $\mathcal{S}_\mathrm{f} \cap \mathcal{S}_\mathrm{g} = \emptyset$.

Working from a general framework for fluid-dynamical theories involving gradient dependencies, Fried and Gurtin[1,4] introduced the slight generalized incompressible flow problem:

$$\begin{cases} \rho\dot{\mathbf{u}} = \operatorname{div}(\mathbf{S} - p\mathbf{I}) + \operatorname{curl}\operatorname{div}\mathbf{G} & \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in} \quad \Omega, \\ \left.\begin{aligned} (\mathbf{S} - p\mathbf{I})\mathbf{n} + \operatorname{div}_\mathcal{S}(\mathbf{Gn}\times) + \mathbf{n}\times(\operatorname{div}\mathbf{G} - 2K\mathbf{Gn}) &= \mathbf{t}_\mathcal{S} \\ \mathbf{n}\times\mathbf{Gn} &= \mathbf{m}_\mathcal{S} \end{aligned}\right\} & \text{on} \quad \mathcal{S}, \end{cases} \tag{1}$$

in which $\mathbf{t}_\mathcal{S}$ and $\mathbf{m}_\mathcal{S}$ represent tractions on the bounding surface $\mathcal{S}$, and with depending on the constitutive relations for the extra stress $\mathbf{S}$ and the hyperstress $\mathbf{G}$ as a function of the form

$$\left.\begin{aligned} \mathbf{S} &= 2(\mu\mathbf{D} + \rho\alpha^2\overset{\circ}{\mathbf{D}}), \\ \mathbf{G} &= \mu\beta^2(\operatorname{grad}\omega + \gamma\operatorname{grad}\omega). \end{aligned}\right\} \tag{2}$$

Note that, to simplify our calculations, we use direct notation. However, for clarity, we also present key definitions and results in component form. We write ρ for the mass density, μ for the dynamic viscosity, $\mathbf{u}$ for the velocity field, p for pressure, $\mathbf{D} = \frac{1}{2}(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^\mathsf{T})$ for the stretching, $\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{DW} - \mathbf{WD}$ for the corotational rate of $\mathbf{D}$ with $\mathbf{W} = \frac{1}{2}(\operatorname{grad}\mathbf{u} - (\operatorname{grad}\mathbf{u})^\mathsf{T})$ the spin, $\mathbf{I}$ for the second-order identity tensor and $\omega = \operatorname{curl}\mathbf{u}$ for the vorticity. We use a superposed dot for the material time-derivative; $\dot{\mathbf{u}} = \frac{\partial\mathbf{u}}{\partial t} + (\operatorname{grad}\mathbf{u})\,\mathbf{u}$. In $(1)_3$, $\mathbf{a}\times$ denotes the axial tensor defined by $(\mathbf{a}\times)\mathbf{w} = \mathbf{a}\times\mathbf{w}$ for every vector $\mathbf{w}$, so that $(\mathbf{a}\times)_{ij} = \varepsilon_{ikj}a_k$, then given any *skew* tensor $\mathbf{A}$ there is a unique vector $\mathbf{a}$ such that $\mathbf{A} = (\mathbf{a}\times)$, cf. Fried and Gurtin[4]. The operator $\operatorname{div}_\mathcal{S}$ defines the surface divergence on

$\mathcal{S}$ and $K = -\frac{1}{2}\text{grad}_{\mathcal{S}}\,\mathbf{n}$ is the mean curvature of $\mathcal{S}$, while μ, α, β are nonnegative scalar moduli with $|\gamma| \leq 1$.

Using $(2)_{1,2}$ in $(1)_1$, and stipulating that the moduli μ, α, β are constant, we arrive at the flow equation

$$\rho\dot{\mathbf{u}} = -\text{grad}\,p + \mu\nabla^2(\mathbf{u} - \beta^2\nabla^2\mathbf{u}) + 2\rho\alpha^2\text{div}\,\mathring{\mathbf{D}}, \tag{3}$$

which envolves two length scales, the modulus α of energetic origin and the other one of dissipative origin, β. Remark that the constitutive dimensionless parameter γ does not enter the flow equation, it would be present in the boundary conditions prescribing the hypertraction; consider the condition $(5)_2$.

In addition to the flow equation, the theory delivers boundary conditions. On $\mathcal{S}_\text{f}$ the classical condition $(\mathbf{S} - p\mathbf{I})\mathbf{n} = \sigma K\mathbf{n}$ is replaced by the conditions

$$(\mathbf{S} - p\mathbf{I})\mathbf{n} + \text{div}_{\mathcal{S}}(\mathbf{Gn}\times) + \mathbf{n}\times\text{div}\,\mathbf{G} = 2\sigma K\mathbf{n} \quad \text{and} \quad \mathbf{n}\times\mathbf{Gn} = \mathbf{0}, \tag{4}$$

where σ denotes the surface tension of the free surface, while on $\mathcal{S}_\text{g}$ the classical non-slip boundary condition is replaced by the generalized adherence conditions or *wall-eddy condition*[4]

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad \mathbf{n}\times(\mathbf{Gn} - \mu\ell\omega) = \mathbf{0} \quad \text{on} \quad \mathcal{S}_\text{g}. \tag{5}$$

where ℓ is the *wall-eddy length* which carries dimensions of length.

Here, the corotational rate of $\mathbf{D}$ in the extra stress $(2)_1$ is also neglected, so that

$$\mathbf{S} = 2\mu\mathbf{D}, \tag{6}$$

and as a result, the last term on the right-hand side of (3) is empty. For sake of simplicity, we neglected time dependency and advection whereby restricting our attention to a *fourth-order Stokes-like problem* of (3).

In order to lay down a strong form of the fourth-order problem, we consider the following spaces as defined in Adams[22]. Let $L^2(\Omega)$ be a Hilbert space equipped with the scalar product $(\cdot,\cdot)_{L^2(\Omega)}$ and norm $\|\cdot\|_{L^2(\Omega)}$ on Ω. Also, let $H^m(\Omega)$ $(m \geq 1)$ be the classical Sobolev space with distributional derivatives up to order m equipped with scalar product $(\cdot,\cdot)_{H^m(\Omega)}$ and norm $\|\cdot\|_{H^m(\Omega)}$. The corresponding vector (product) spaces are likewise denoted by $H^1(\Omega)^n = H^1(\Omega)\times ...\times H^1(\Omega)$, with scalar product $(\cdot,\cdot)_{H^1(\Omega)^n}$ and norm $\|\cdot\|_{H^1(\Omega)^n}$.

Finally, we consider the following spaces

$$\begin{aligned} H^1_{\text{div}}(\Omega,\mu) = \{\eta \in H^1(\Omega);\ &\text{div}\,(\mu\,\text{grad}\,\eta) \in L^2(\Omega), \\ &\nabla^2\big(\text{div}\,(\mu\beta^2\,\text{grad}\,\eta)\big) \in L^2(\Omega)\} \end{aligned}$$

and

$$H^1_{\text{div}}(\Omega,\mu)^n = \{\mathbf{u} \in H^1(\Omega)^n;\ u_i \in H^1_{\text{div}}(\Omega,\mu),\ (i = 1,...,n)\},$$

for which we suppose that $0 < \mu_0 < \mu < \mu_1$ in Ω, where μ_0, μ_1 are nonnegative real constants.

In this work, we focus on the following boundary-value problem for the fourth-order flow equation that reads: find $(\mathbf{u}, p) \in H^1_{\text{div}}(\Omega, \mu)^n \times L^2(\Omega)$ such that

$$\begin{cases} \operatorname{grad} p - \operatorname{div}(\mu \operatorname{grad} u_i) + \nabla^2\big(\operatorname{div}(\mu\beta^2 \operatorname{grad} u_i)\big) = 0 & \text{in} \quad \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in} \quad \Omega, \\ \left.\begin{matrix} (2\mu\mathbf{D} - p\mathbf{I})\mathbf{n} + \operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times) + \mathbf{n} \times \operatorname{div} \mathbf{G} = \sigma K \mathbf{n} \\ \text{and} \quad \mathbf{n} \times \mathbf{G} = \mathbf{0} \end{matrix}\right\} & \text{on} \quad \mathcal{S}_{\text{f}}, \\ u_i = 0 \quad \text{and} \quad \mathbf{n} \times (\mathbf{Gn} - \mu\ell\omega) = \mathbf{0} & \text{on} \quad \mathcal{S}_{\text{g}}. \end{cases} \tag{7}$$

See Fried and Gurtin[1,4,5] for a complete justification and discussion of the constitutive equations and boundary conditions. The quantities $\mu\beta^2$ and $\mu\ell$ can be interpreted as a hyperviscosity and a boundary viscosity, respectively, and we refer β as the gradient lenth

3 THE ASSOCIATED VARIATIONAL PROBLEM

As a result of working within a framework based on the principe of virtual power[1,4], we recall that the weak formulation of the flow equation and the boundary conditions of problem (7) is straightforward to derive, and consists in finding $(\mathbf{u}, p) \in S_u \times S_p$ such that

$$A(\mathbf{u}, p, \mathbf{v}, q) = l(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in (V_u \times S_p), \tag{8}$$

where

$$\begin{aligned} A(\mathbf{u}, p, \mathbf{v}, q) \quad &= \quad \int_\Omega \big(\mathbf{S}(\mathbf{u}) : \operatorname{grad} \mathbf{v} \ + \mathbf{G}(\mathbf{u}) : \operatorname{grad} \operatorname{curl} \mathbf{v} - p \operatorname{div} \mathbf{v}\big)\, d\Omega \\ &+ \quad b(\mathbf{u}, p, \mathbf{v}, q) \ + \int_{\mathcal{S}_{\text{g}}} \mu\ell\omega \times \mathbf{n} \cdot \frac{\partial \mathbf{v}}{\partial n}\, d\mathcal{S}, \end{aligned}$$

$$b(\mathbf{u}, p, \mathbf{v}, q) = \int_\Omega q \operatorname{div} \mathbf{u}\, d\Omega,$$

and

$$l(\mathbf{v}, q) = \int_{\mathcal{S}_{\text{f}}} \sigma K \mathbf{n} \cdot \mathbf{v}\, d\mathcal{S}.$$

Regularity and boundary conditions are established by the solution set $S_u = \{\mathbf{u} \in H^2(\Omega)^n;\ u_i = g_i \text{ on } \mathcal{S}_{\text{g}}\}$ and the admissible variation space $V_u = \{\mathbf{v} \in H^2(\Omega)^n;\ v_i = 0 \text{ on } \mathcal{S}_{\text{g}}\}$. Also, we consider the set $S_p = \{p \in L^2(\Omega); \int_\Omega p\, d\Omega = 0\}$ and $V_p = \{q \in L^2(\Omega); \int_\Omega q\ d\Omega = 0\}$ as the spaces of square-summable functions with zero mean in Ω. Using the relations (6) and $(2)_2$, we obtain correspondence with the flow equation $(7)_1$.

4 THE ASSOCIATED DISCONTINUOUS GALERKIN/LEAST-SQUARE FORMULATION

We present the discontinuous Galerkin formulation proposed. To this end, let us consider a partition $\Omega^h = \{\Omega_1, ..., \Omega_{ne}\}$ of the region Ω into non degenerated finite elements Ω_e, each with boundary $\Gamma_e = \partial\Omega_e$, such that Ω_e can be mapped in standard elements by isoparametric mappings: $\Omega_e \cap \Omega_{e'} = \emptyset$ and $\Gamma_e \cap \Gamma_{e'} = \emptyset$ if $e \neq e'$; $\Omega \cup \mathcal{S} = \cup_{e=1}^{ne}(\Omega_e \cup \Gamma_e)$. The following terminology is also used in the discontinuous Galerkin formulation: $\Gamma_{ee'} = \cup_{e=1}^{ne}\Gamma_e - \mathcal{S}$ for the union of interelement boundaries, where $\mathcal{S} = \mathcal{S}_\mathrm{f} \cup \mathcal{S}_\mathrm{g}$ with $\mathcal{S}_\mathrm{f} \cap \mathcal{S}_\mathrm{g} = \emptyset$. The jump $[\![\cdot]\!]$ and mean-values $\{\cdot\}$ operators on a vector- or scalar-function across either side of each interelement boundary are definite as usual.

In what following we introduce the finite element spaces

$$H^{h,k} = \{\eta \in H^1(\Omega); \eta_e \in \mathcal{P}^k(\Omega_e)\},$$

$$H^{h,k,n} = \{\eta = (\eta_1, \ldots, \eta_n); \eta_i \in H^{h,k}\},$$

$$L^{h,l} = \{\eta \in L^2(\Omega); \eta_e \in \mathcal{P}^l(\Omega_e)\},$$

with $\mathcal{P}^m$ being the space of the polinomial shape functions on local coordinates defined on element Ω_e, where $m \geqslant 0$ is the polinomial order; $k \geqslant 0$ and $k \geqslant l \geqslant 0$ are integers. Also, we consider the solution sets

$$S_u^{h,k} = \{\mathbf{w}_h \in H^{h,k,n};\ w_i^h = g_i^h \quad \text{on} \quad \Gamma_g\} \quad \text{and} \quad S_p^h = S_p \cap L^{h,l},$$

where g_i^h $(i = 1, \ldots, n)$ is the usual interpolating of g_i, and the spaces

$$V_u^{h,k} = V_u \cap H^{h,k,n} \quad \text{and} \quad V_p^h = V_p \cap L^{h,l}.$$

We remark that these function spaces possess less regularity than would be required in a conventional approach for fourth-order equation, which would seek solutions in a subspace of $H^2(\Omega)$, rather than in subspaces of $H^1(\Omega)$.

Following the ideas presented in Engel et al.[10], the associated discontinuous Galerkin/least-square formulation to the problem (8) then reads: let τ be any positive constant, find $(\mathbf{u}^h, p^h) \in S_u^{h,k} \times S_p^h$ such that, for all $(\mathbf{v}^h, q^h) \in S_u^{h,k} \times V_p^h$,

$$\begin{aligned} A_h(\mathbf{u}^h, p^h, \mathbf{v}^h, q^h) &= A_{cd}(\mathbf{u}^h, p^h, \mathbf{v}^h, q^h) \\ &+ \sum_{e=1}^{ne} A_{LS}^e(\mathbf{u}^h, p^h, \mathbf{v}^h, q^h) \ = l_{tot}(\mathbf{v}^h, q^h), \end{aligned} \tag{9}$$

where

$$l_{tot}(\mathbf{v}^h, q^h) = \int_{\mathcal{S}_\mathrm{f}} \sigma K \mathbf{n} \cdot \mathbf{v}^h \, d\mathcal{S},$$

$$
\begin{aligned}
A_{cd}(\mathbf{u}^h, p^h, \mathbf{v}^h, q^h) &= A(\mathbf{u}^h, p^h, \mathbf{v}^h, q^h) \\
&+ \sum_{e=1}^{ni} \Big(- \int_{\Gamma_{ee'}} \{\mathbf{G}(\mathbf{u}^h)\mathbf{n}\} \cdot [\![\operatorname{curl} \mathbf{v}^h]\!] \, d\Gamma \\
&+ \int_{\Gamma_{ee'}} \{\mathbf{G}(\mathbf{v}^h)\mathbf{n}\} \cdot [\![\operatorname{curl} \mathbf{u}^h]\!] \, d\Gamma \; + \int_{\Gamma_{ee'}} \frac{\tau}{h_{ee'}} [\![\operatorname{curl} \mathbf{v}^h]\!] \cdot [\![\operatorname{curl} \mathbf{u}^h]\!] \, d\Gamma \Big),
\end{aligned}
$$

$$
\begin{aligned}
A(\mathbf{u}^h, p^h, \mathbf{v}^h, q^h) &= \\
& \int_\Omega \big(\mathbf{S}(\mathbf{u}^h) : \operatorname{grad} \mathbf{v}^h \; + \mathbf{G}(\mathbf{u}^h) : \operatorname{grad} \operatorname{curl} \mathbf{v}^h - p^h \operatorname{div} \mathbf{v}^h \big) \, d\Omega \\
& + b(\mathbf{u}^h, p^h, \mathbf{v}^h, q^h) \; + \int_{\mathcal{S}_{\mathrm{g}}} \mu \ell \omega \times \mathbf{n} \cdot \frac{\partial \mathbf{v}^h}{\partial n} \, d\mathcal{S},
\end{aligned}
$$

$$
b(\mathbf{u}^h, p^h, \mathbf{v}^h, q^h) = \int_\Omega q^h \operatorname{div} \mathbf{u}^h \, d\Omega,
$$

and

$$
A^e_{LS}(\mathbf{w}, q, \delta\mathbf{w}, \delta q) = \int_{\Omega_e} \delta(h_e^2) \left(\sum_{i=1}^{n} R_i(\mathbf{w}, q) \, R_i(\delta\mathbf{w}, \delta q) \right) d\Omega,
$$

in which

$$
R_i(\mathbf{w}, q) = -\operatorname{div} (\mu \operatorname{grad} w_i) + \nabla^2 \big(\operatorname{div} (\mu \beta^2 \operatorname{grad} w_i) \big) + \frac{\partial q}{\partial x_i},
$$

with $(\mathbf{w}, q) \in H^1_{\mathrm{div}}(\Omega_e, \mu)^n \times L^2(\Omega_e)$ and $(\delta\mathbf{w}, \delta q) \in H^1_{\mathrm{div}}(\Omega_e, \mu)^n \times L^2(\Omega_e)$. Here, $h_{ee'}$ denote a measure of the interelement boundary, while h_e is a mesh parameter usually adopt as a measure of the element e.

It is well-known that the determination of the stabilized parameter $\delta(h_e^2)$ is one of the key issues in the development of stabilized finite element methods. See Franca and Frey[23], Zienkiewicz and Taylor[24] and Xia and Yao[25] for a review concerning these parameters. Here, we will adopt the following expression for these parameters: $\delta(h_e^2) = \alpha_e (h_e)^2/\mu$, where the dimensionless constant α_e depends on the degree of interpolation polinomial, the geometry of the element and the viscosity μ, according to methodology introduced by Carmo et al.[20,21].

5 Numerical examples

Here, we demonstrate the potentials of the stabilized finite element method described above when applied to the fourth-order problem (7). We solved the classical lid-driven cavity flow problem in which the domain is $\Omega = \big[0,1\big] \times \big[0,1\big]$ and boundary conditions are set to $\mathbf{u} = (0,0)$ and $\mathbf{n} \times (\mathbf{Gn} - \mu\ell\omega) = \mathbf{0}$ on $x = 0$ and $0 < y < 1$, $\mathbf{u} = (0,0)$ and $\mathbf{n} \times (\mathbf{Gn} - \mu\ell\omega) = \mathbf{0}$ on $x = 1$ and $0 < y < 1$, $\mathbf{u} = (0,0)$ and $\mathbf{n} \times (\mathbf{Gn} - \mu\ell\omega) = \mathbf{0}$ on $y = 0$ and $0 < x < 1$ and $\mathbf{u} = (1,0)$ and $\mathbf{n} \times \mathbf{Gn} = \mathbf{0}$ on $y = 1$ and $0 < x < 1$. Note that on the fixed boundary surface, no-slip boundary conditions are considered along with generalized adherence boundary conditions. Here, we only examine the limiting case of $\ell \to 0$ for the gradient theory, which correspond to weak adherence conditions on the cavity walls. One nodal pressure is also set to zero to avoid the constant mode and the Reynolds number was 1. We used a cartesian uniform mesh with 20×20 second order quadrilateral elements.

First, we compare numerical approximations for the the components of the velocity field and pressure field for the classical Stokes flow and the fourth-order Stokes-like flow obtained by the second gradient theory. As examined by Fried and Gurtin[1], the theory yields the classical solution when gradient length is negligibly small in comparison to the representative length scale of the flow domain and the adherence length is negligibly small in comparison to the gradient length, and although the analise has been confined to the problem of plane Poiseuille flow subject to weak adherence conditions, it is carried over general flows.

Figures 1, 2 and 3 show a comparison of the velocity component profiles and pressure profiles across the cavity centerlines $x = 0.5$ and $y = 0.5$, respectively, for the classical theory and weak-adherence boundary condition on the cavity flow, and different values of gradient length β. The numerical result for the classical Stokes theory were performed with a quasi-optimum α parameter for the GLS stabilization, $\alpha = 0.1$, as obtained by Carmo et al.[20,21]. For the gradient theory we adopted and $\alpha = 0.25$. Figures 4 and 5 provide the pressure comparison between classical and gradient theories.

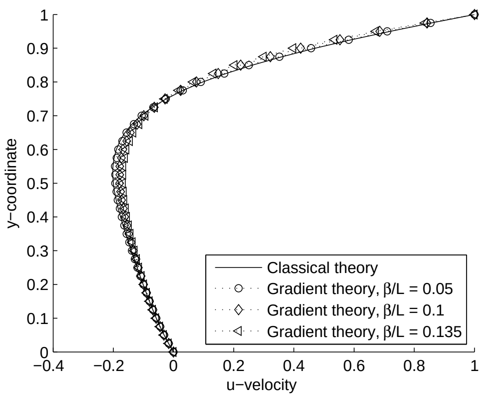

Figure 1: Velocity profiles across $x = 0.5$.

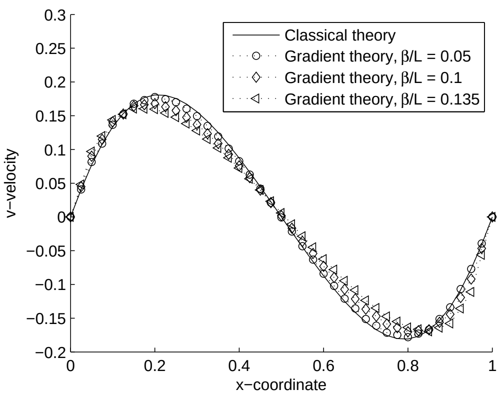

Figure 2: Velocity profiles across $y = 0.5$.

As expected, the results reveal a qualitative difference between classical and gradient theory. Remark that where the ratios β/L, determined by the gradient and flow domain lengths, are sufficiently large would the gradient effects due to fourth-order terms appear-

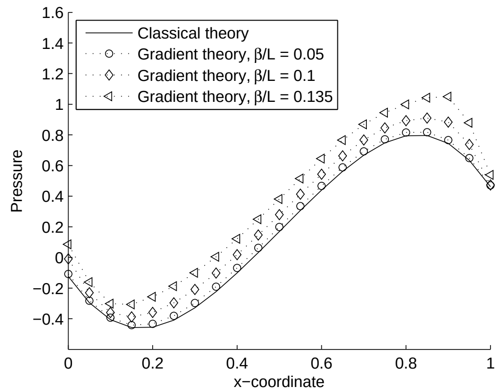

Figure 3: Pressure profiles across $y = 0.5$.

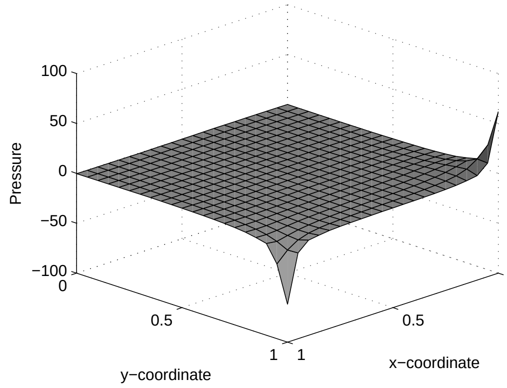

Figure 4: Pressure field for classical Stokes theory and stabilization parameter $\alpha = 0.1$.

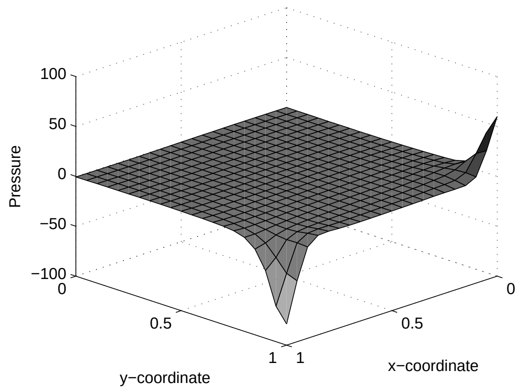

Figure 5: Pressure field for gradient theory and stabilization parameter $\alpha = 0.25$.

ing in the flow equation (7) and generalized boundary condition be of importance. Hence, the gradient effects are important only for flow domain length scale sufficiently small and, therefore, the classical Stokes equation perhaps loses validity.

We next examine the sensibility of our stabilized method with different α parameters. The results were obtained using a uniform mesh of 20×20 elements and ratio $\beta/L = 0.135$. Shown in figures 6, 7 and 8 are the velocity profiles and pressure fields across the cavity centerlines $x = 0.5$ and $y = 0.5$, respectively. As expected, the sensibility of the results with the α stabilization parameter is evident. Results for the pressure field are depicted in figures 9, 10 and 11, indicating again the dependence of the pressure with the α parameter.

We remark that $\alpha = 0.25$ is a good guess for the case investigated here. However, the optimum and quasi-optimum stabilization parameters for the discontinuous Galerkin/least-square method proposed for the fourth-order incompressible problem will be automatic computed and should be preferred, whenever available.

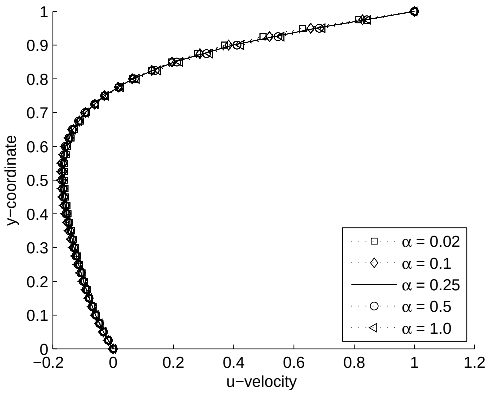

Figure 6: Velocity profiles across $x = 0.5$.

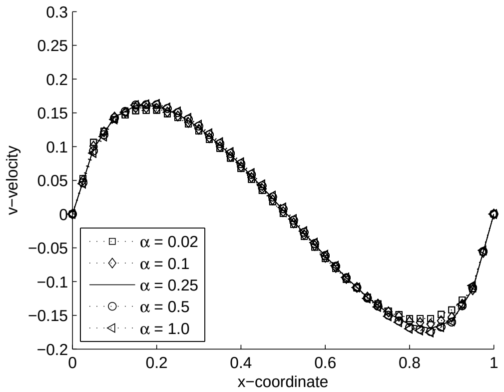

Figure 7: Velocity profiles across $y = 0.5$.

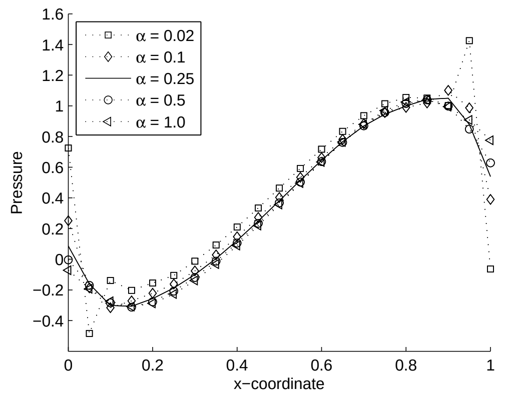

Figure 8: Pressure profiles across $y = 0.5$.

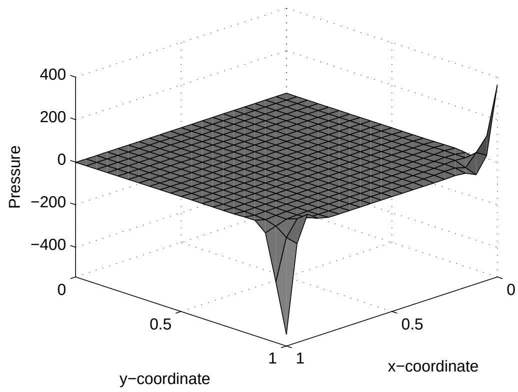

Figure 9: Pressure field for gradient theory and stabilization parameter $\alpha = 0.02$.

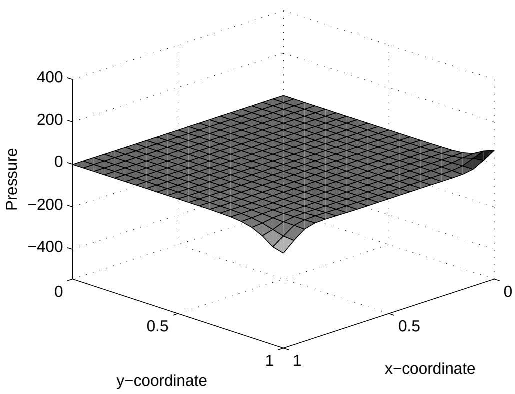

Figure 10: Pressure field for gradient theory and stabilization parameter $\alpha = 0.22$.

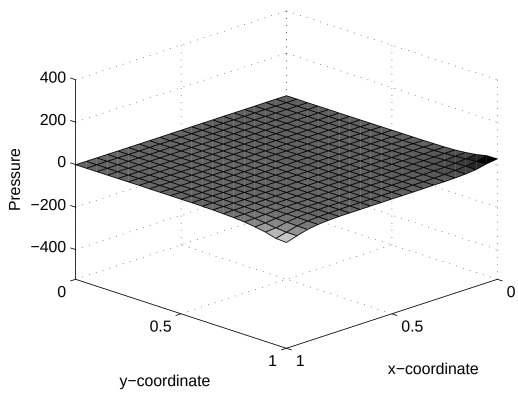

Figure 11: Pressure field for gradient theory and stabilization parameter $\alpha = 1.0$.

6 CONCLUSIONS

In this communication, we advanced a stabilized finite element method for a fourth order flow equation originated from a second-gradient theory for fluid. The proposed finite element method combines concepts from both stabilized and discontinuous Galerkin methods. We use a GLS formulation with a suitable choice for the stabilization parameter which result in a discontinuous Galerkin + Least Square at element level. Here, we based our approach on second order quadrilateral elements, so that the fourth-order operator is empty which implies that any calculation involving derivatives of order greater than two is trivial for these elements; the same is true for triangular and tetrahedral cubic elements. The quasi-optimum stabilization parameter for each element of the mesh depends on the polinomial used to approximate the velocity and pressure fields, the geometry of the element and the viscosity tensor. The numeric results suggest that the finite element method purposed here is robust for modeling fourth order incompressible flow at sufficiently small scale. The authors understand that this work can be naturally extended to developing finite-element methods based on the generalization of the Navier-Stokes-$\alpha\beta$ equation[4] for turbulent flows.

ACKNOWLEDGEMENT

The support of the Brazilian governmental foundation CAPES is gratefully acknowledged.

REFERENCES

[1] E. Fried, M.E. Gurtin, Tranctions, balances, and boundary conditions for nonsimple matrials with applications to liquid flow at small length scales, *Arch. Ration. Mech. Anal*, **182**, 513–554 (2006).

[2] S. Chen, C. Foias, D.D. Holm, E. Olson, E.S. Titi, S. Wynne, The CamassaHolm equations as a closure model for turbulent channel and pipe flow, *Phys. Rev. Lett.*, **81**, 5338-5341 (1998).

[3] S. Chen, C. Foias, D.D. Holm, E. Olson, E.S. Titi, S. Wynne, The CamassaHolm equations and turbulence, *Phys. D*, textbf133, 49-65 (1999).

[4] E. Fried and M.E. Gurtin, A continuum mechanical theory for turbulence: a generalized NavierStokes-α equation with boundary conditions, *Theor. Comput. Fluid Dyn.*, **22**, 433-470 (2008) .

[5] E. Fried and M.E. Gurtin, Turbulent kinetic energy and a possible hierarchy of length scales in a generalization of the Navier-Stokes-α theory, *Phys. Rev. E*, **75** (2007).

[6] T.-Y. Kim, M. Cassiani, L.D. Albertson, J.D. Dolbow, E. Fried, M.E. Gurtin, Impact of the inherent separation of scales in the Navier-Stokes-$\alpha\beta$ equations, *Phys. Rev. E*, **79** (2008).

[7] M.E. Gurtin, A gradient theory of single-cristal viscoplasticity that accounts for geometrically necessary dislocations, *J. Mech. Phys. Solids*, **84**, 809–819 (2001).

[8] R.A. Toupin, Elastic matrials with couple stresses, *Arch, Ration. Mech. Anal.*, **11**, 385–414 (1962).

[9] R.A. Toupin, Elastic matrials with couple stresses, *Arch, Ration. Mech. Anal.*, **17**, 85–112 (1964).

[10] G. Engel, K. Garikipati, T.J.R. Hughes, M.G. Larson, L. Mazzei, R.L. Taylor, Continuous/discontinouos finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity, *Comput. Meth. Appl. Mech. Eng.*, **191**, 3669–3750 (2002).

[11] G.N. Wells, E. Kuhl, K. Garikipati, A discontinuous Galerkin method for the Cahn-Hilliard equation, *J. Comput. Phys.* **218**, 860-0877 (2006).

[12] T.-Y. Kim, J.D. Dolbow, E. Fried, A numerical method for a second-gradient theory of incompressible fluid flow, *J. Comput. Phys.*, **223**, 551–570 (2007).

[13] J.A. Nistche, Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind, *Abhandlunhgen Seminar der Univesität Hanburg*, **36**, 9–15 (1970/1971).

[14] D.N. Arnold, F. Brezzi, B. Cockburn, L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Number. Anal.*, **39**, 1749–1779 (2002).

[15] F. Brezzi, G. Manzini, D. Marini, P. Pietra, A. Russo, Discontinuous Galerkin methods for elliptic problems, *Numer. Meth. Partial Differ. Equations*, **16**, 365-378 (2000).

[16] V. Girault, B. Rivière, M.-F. Wheeler, A discontinuous Galerkin method with nonoverlapping domain decomposition for For Stokes and Navier-Stokes problems, *Math. Comput.*, **74**, 53–84 (2005).

[17] B. Cockburn, G. Kanschat, D. Schötzau, The local discontinuous Glerkin method for linear incompressible fluid flow: A review, **34**, 491–506 (2005).

[18] B. Cockburn, C.W. Shu, The local discontinuous Galerkin finite element method for convectivediffusion systems, *SIAM J. Numer. Anal.*, **35**, 2440-2463 (1998).

[19] K.S. Bey, J.T. Oden, hp-version discontinuous Galerkin methods for hyperbolic conservations laws, *Comput. Meth. Appl. Mech. Eng.*, **133**, 259-286 (1996).

[20] E.G.D. Carmo, J.P.L. Santos, W.J. Mansur, Uma determinação sistematíca do parâmetro de calibração do método Galerkin Least Square (GLS), In Proceedings of the *30th edition of the Iberian-Latin-American Congress on Computational Methods in Engineering*, Armação dos Búzios, Brazil (2009).

[21] E.G.D. Carmo, J.P.L. Santos, W.J. Mansur, A systematic determination at element level of the parameter of stabilization of the method GLS for Stokes flow and nearly incompressible linear elasticity, *Applied Mathematical Modelling*, submited.

[22] R.A. Adams, Sobolev spaces, New York, Academic Press, 1975.

[23] L.P. Franca, S.L. Frey, Stabilized finite element methods: II. The incompressible NavierStokes equations, *Comput. Methods Appl. Mech. Engrg.*, **99**, 209-233 (1992). 233.

[24] O.C. Zienkiewicz, R.L. Taylor, Finite element method: its basis and fundamentals, Londres, Published Elsevier (2005).

[25] K. Xia, H. Yao, A Galerkin/least-square finite element formulation for nearly incompressible elasticity/stokes flow, *Appl. Math. Modelling*, **31**, 513–529 (2007).