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ESTIMATES FOR ERROR GENERATED BY INDETERMINANT ELASTICITY TENSOR

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Abstract. *In this paper, we discuss the influence of perturbed material behaviour in the context of linear elasticity problem. Of special interest is the relation between the data perturbations and the radius of the set of perturbated solutions. We apply functional a posteriori error estimates to establish bounds for the radius.*

1 INTRODUCTION

The problem of indeterminant data can be approached roughly in two ways, stochastic and non–stochastic. In stochastic approach the data is cosidered as a set of admissible values and some probability distribution defining the frequency by which they appear. For wider overview, we recommend[5]. One of the most popular non-stochastic approach is the so called "worst–case scenario"–method. From the engineering point of view, it is simple. The goal is to find the "worst" possible admissible solution (with respect to some criteria functional) among to the set of solutions. Extensive discussion about the method, study from the viewpoint of existence of solutions, and numerics can be found from[1]. Our study is related to the latter, but we have slightly different interest. Our aim is to study the relation between the set of admissible data and the set of solutions generated by it as in[2].

In continuum mechanics, an important source of indeterminacy is the constitutive relation between strains and stresses. It is one of the material properties and in practise, it is never complitely known.

In this paper, we consider the linear elasticity model. The model is as follows:

$$\begin{aligned} \sigma &= \mathbb{L}\epsilon, \quad \text{in } \Omega & (1)\\ \text{div}\sigma &= f, \quad \text{in } \Omega & (2)\\ u &= 0, \quad \text{on } \partial\Omega_1 & (3)\\ \sigma \cdot n &= F, \quad \text{on } \partial\Omega_2. & (4) \end{aligned}$$

The domain $\Omega \subset \mathbb{R}^3$ has Lipschitz continuous boundary. Here, we consider (linearized) theory of small displacements, where the strain–displacement relation $\epsilon : \mathbb{R}^3 \to \mathbb{M}^{3\times3}$ is defined as follows:

$$\epsilon(w) = \frac{1}{2}\left(\nabla w + (\nabla w)^T\right). \tag{5}$$

The linear operator $\mathbb{L} : \mathbb{M}^{3\times3} \to \mathbb{M}^{3\times3}$ defines the *constitutive relation*. The variational formulation of the problem is, find $u \in H^1_0(\Omega)$ s.t.

$$\int_\Omega \mathbb{L}\epsilon(u) : \epsilon(w)\, \mathrm{dx} = \int_\Omega f \cdot w\, \mathrm{dx} + \int_{\partial\Omega_2} F \cdot w\, \mathrm{dx}, \quad \forall w \in H^1_0(\Omega). \tag{6}$$

We write symmetric $\mathbb{M}^{3\times3}$ tensors using Voigts notation as follows:

$$\sigma = \begin{bmatrix} \sigma_x & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z \end{bmatrix}, \quad \sigma = \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{array} \right\}. \tag{7}$$

Our special interest is the constitutive relation, which is never complitely known. Isotropy is a typical assumption considering the material behaviour. However, this assumption does not take into accout inperfections, cracks etc, which cause local violations to the isotropy. Thus, we assume that the elasticity tensor $\mathbb{L}$ is not complitely known, but belongs to a set:

$$\mathbb{L} \in \Lambda := \{\mathbb{L} \in \mathbb{M}^{3\times 3} \to \mathbb{M}^{3\times 3} \,|\, \mathbb{L} = \mathbb{L}_0 + \delta\Psi\}. \tag{8}$$

The unperturbed "mean" tensor $\mathbb{L}_0$ is known. $\delta \geq 0$ is the magnitude of perturbation and Ψ is an arbitrary symmetric tensor, for which holds $|\Psi| \leq 1$. For example, $\mathbb{L}_0$ can be the standard isotropic elasticity tensor,

$$\mathbb{L}_0 = \frac{E}{(1+\nu)(1-2\nu)} \left\{ \begin{array}{cccccc} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & \hat{\nu} & & \\ & & & & \hat{\nu} & \\ & & & & & \hat{\nu} \end{array} \right\}, \tag{9}$$

where $\hat{\nu} := \frac{1}{2}(1-2\nu)$. Young's modulus E and Poisson ratio $\nu \in (0, \frac{1}{2})$ are material parameters. For the mean tensor, we assume that constants $\underline{c}$ and $\overline{c}$ of inequality

$$\underline{c} \leq \frac{\mathbb{L}_0\xi : \xi}{|\xi|^2} \leq \overline{c}, \quad \forall\, \xi \in \mathbb{M}^{3\times 3}, \tag{10}$$

are known or can be estimated. For example, for the isotropic tensor they are

$$\underline{c} := \frac{E_0}{2(1+\nu_0)} \quad \text{and} \quad \overline{c} := \frac{E_0}{1-2\nu_0}. \tag{11}$$

In order to guarantee the existence of solutions for any member of Λ, we assume that $\delta < \underline{c}$, so all problems are elliptic. Energy norm for the problem is

$$\| w \| := \left(\int_\Omega \mathbb{L}\epsilon(u) : \epsilon(w)\,\mathrm{dx} \right)^{1/2}, \tag{12}$$

we denote by $\| \cdot \|_0$ the norm generated by the mean tensor.

The solution mapping

$$\mathcal{S} : \Lambda \to H^1_0(\Omega) \tag{13}$$

defines the solution related to some particular tensor of set Λ. Consequently, solutions of the problem generated by every $\mathbb{L} \in \Lambda$ form the set $\mathcal{S}(\Lambda)$. We refer $\mathcal{S}(\Lambda)$ as the set of solutions.

The important quantity defining the accuracy limit for computations is the distance between the solution u_0 related to the "mean" data and the most distant member of the solution set (given in the $\|\!| \cdot |\!\|_0$-norm). This distance is called the radius of the solution set and is defined as follows:

$$r := \sup_{u \in \mathcal{S}(\Lambda)} \|\!| u - u_0 |\!\|_0 \tag{14}$$

and its normalized counterpart

$$\hat{r} := \sup_{u \in \mathcal{S}(\Lambda)} \frac{\|\!| u - u_0 |\!\|_0}{\|\!| u_0 |\!\|_0}. \tag{15}$$

The equivalence of norms generated by non–perturbed $\mathbb{L}_0$ and other members of the set Λ plays a crucial part. We assume that constants $\underline{K}$ and $\overline{K}$ in the inequality

$$\underline{K} \;\leq\; \frac{\mathbb{L}_0\xi : \xi}{\mathbb{L}\xi : \xi} \;\leq\; \overline{K}, \quad \forall\, \xi \in \mathbb{M}^{3\times 3},\; \mathbb{L} \in \Lambda \tag{16}$$

are known or can be estimated. These constants depend on δ and constants $\underline{c}$ and $\overline{c}$, and they are explicitely computable,

$$\overline{K} := \frac{1}{1-\theta} \quad \text{and} \quad \underline{K} := \max\left\{\frac{1}{\overline{c}/\underline{c} + \theta}, \frac{1-2\theta}{1-\theta}\right\}, \tag{17}$$

where

$$\theta := \frac{\delta}{\underline{c}} \tag{18}$$

is a normalized perturbation.

2 FUNCTIONAL A POSTERIORI ERROR ESTIMATES

Main tools of our analysis are functional a posteriori error estimators, discussed widely in[?,3]. They are denoted as minorant,

$$M^{\mathbb{L}}_{\ominus}(v,w) := -\int_{\Omega} \mathbb{L}\epsilon(w) : (\epsilon(w) + 2\epsilon(v))\,\mathrm{dx} + 2\int_{\Omega} f\cdot w\,\mathrm{dx} + 2\int_{\partial\Omega_2} F\cdot w\,\mathrm{dx}, \tag{19}$$

and majorant,

$$M^{\mathbb{L}}_{\oplus}(v,y) := \left(\int_{\Omega} (e(v) - \mathbb{L}^{-1}y) : (\mathbb{L}\epsilon(v) - y)\,\mathrm{dx}\right)^{1/2} + \frac{C_\Omega}{\sqrt{\underline{c}}}\left(\|\mathrm{div} y + f\|^2_{\Omega} + \|F - y\cdot n\|^2_{\partial_2\Omega}\right)^{1/2}, \tag{20}$$

where constant C_Ω is from the Korn's inequality,

$$\|w\|^2_{\Omega} + \|w\|^2_{\partial_2\Omega} \leq C^2_{\Omega} \,\|\!| w |\!\|^2, \quad \forall w \in H^1_0(\Omega). \tag{21}$$

The key properties of minorant and majorant are presented in the following Theorem.

Theorem 1. *Let u be the solution of the problem* (6)*, and $v \in H^1(\Omega)$ any conforming approximation. Then estimators* (19) *and* (20) *are* guaranteed *in the sense that*

$$M_{\ominus}^{\mathbb{L}}(v,w) \leq \| u - v \|^2 \leq M_{\oplus}^{\mathbb{L}}(v,y), \quad \forall w \in H^1(\Omega), y \in H(\mathrm{div},\Omega). \tag{22}$$

Moreover, they are sharp *in the sense that*

$$M_{\ominus}^{\mathbb{L}}(v,u-v) = \| u - v \|^2 = M_{\oplus}^{\mathbb{L}}(v,\mathbb{L}\epsilon(u)). \tag{23}$$

Proof. See[3].

3 APPLICATION OF FUNCTIONAL ERROR ESTIMATORS

The construction of two–sided estimates for the radius is based on the fact that functional error estimates depend explicitly on the problem data. In particular, they depend on elasticity tensor $\mathbb{L}$ (and depend also on the arbitrary perturbation tensor Ψ). Thus, we are eble to estimate radius of the solution set from above as follows:

$$r^2 \leq \overline{K} \sup_{u \in \mathcal{S}} \| u_0 - u \|^2 \tag{24}$$

$$= \overline{K} \sup_{|\Psi| \leq 1} \inf_{y \in H(\mathrm{div},\Omega)} M_{\oplus}(u_0,y) \tag{25}$$

$$\leq \overline{K} \inf_{y \in H(\mathrm{div},\Omega)} \sup_{|\Psi| \leq 1} M_{\oplus}(u_0,y). \tag{26}$$

Similarly, we may estimate the radius from below:

$$r^2 \geq \sup_{u \in \mathcal{S}(\Lambda)} \underline{K} \| u - u_0 \|_{\mathcal{L}}^2 \tag{27}$$

$$= \underline{K} \sup_{|Psi| \leq 1} \sup_{w \in V} \mathcal{M}_{\ominus}(u_0,w) \tag{28}$$

$$= \underline{K} \sup_{w \in V} \sup_{|\Psi| \leq 1} \mathcal{M}_{\ominus}(u_0,w), \tag{29}$$

Now, instead of being forced to compute solutions for all possible data candidates, we can take supremum with respect to indeterminant data (in this case, variation Ψ) directly over the estimator functional.

Moreover, the functions y and w are at our disposal. We may choose them in various ways to construct different estimates. For example, following procedure produces a priori estimates: If we select y as the exact stress of the mean solution, i.e. $y := \mathbb{L}_0\epsilon(u_0)$ and $w := \alpha u_0$, where $\alpha \in \mathbb{R}$ is arbitrary, we can compute following estimates for $\hat{r}$,

$$\sqrt{\underline{K}} \frac{\theta}{\sqrt{1-\theta}} \leq \hat{r} \leq \sqrt{\overline{K}} \frac{\theta}{\sqrt{1-\theta}}. \tag{30}$$

These estimates depend only on the problem data and perturbation magnitude.

4 CONCLUSIONS

Functional a posteriori error estimates can be applied to estimate the radius of the solution set, due to the properties presented in Theorem 1 and their explicite dependence on the problem data. We emphasize that similar treatment can be done for any model for which a posteriori error estimates are derived. Presented estimates provide the accuracy limit for computations and is motivated by development of adaptive and iterative schemes.

REFERENCES

[1] I. Hlaváček, J. Chleboun, and I. Babuška. Uncertain input data problems and the worst scenario method, *Elsevier*, (2004).

[2] O. Mali and S. Repin, *Estimates of the indeterminacy set for elliptic boundary–value problems with uncertain data, J. Math. Sci.* **150** 1869–1874 (2008).

[3] P. Neittaanmäki and S. Repin, Reliable Methods for Computer Simulation, Error Control and A Posteriori Estimates, *Elsevier*, (2004).

[4] S. Repin: A posteriori error estimation for variational problems with uniformly convex functionals, Math. Comput. **69** 481–500, (2000).

[5] G. I. Schuëller, A state-of-the-art report on computational stochastic mechanics *Prob. Engrg. Mech*, **12**, No.4. 197–321, (1997).