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FINITE-VOLUME METHOD FOR MODELLING SHALLOW WATER FLOWS IN PRESENCE OF EXTERNAL FORCES

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Abstract. *The numerical method for study of hydrodynamic flows over an arbitrary bed profile in the presence of external force is proposed in this paper. This method takes into account the external force effect, it use the quasi-two-layer model of hydrodynamic flows over a stepwise boundary with consideration of features of the flow near the step. There are performed the numerical simulations based on proposed algorithm of various physical phenomena, such as a breakdown of the rectangular fluid column over an inclined plane, the large-scale motion of fluid in the gravitational field in the presence of Coriolis force over an underlying mountain-like surface. Computations made for two dimensional dam-break problem on slope precisely conform to laboratory experiments.*

1 INTRODUCTION

The shallow water equations are derived from the non-stationary three-dimensional Euler equations by averaging over a vertical coordinate and taking into consideration hydrostatic pressure distribution [1]. It is expected that solutions of depth-averaged equations have similar properties as depth-averaged solutions of initial fluid equations. The obtained equations are also rather complicated for constructing general analytical solutions because of their nonlinearity and bed complexity [2, 3, 4]. However, they can be successfully integrated numerically [5, 6, 7].

The main difficulty in numerical simulation of nonhomogenous shallow water equations consists in its non-divergence property determined by nonhomogeneity of the right-hand side of the momentum conservation equations due to their nonlinearity and bed complexity. The numerical methods which have been developed and effectively used in studying of shallow water flows suppose that the external force effects are insignificant. The other problem consists in compatibility of solutions of traditional depth-averaged equations with depth-averaged solutions of Euler equations due to significant role of shallow flows dependences on vertical coordinate [8].

The main problem in numerical integration of hyperbolic balance equations is in developing finite-difference schemes satisfying the conditions of conservation of stable states. In the context of Saint-Venant's equations this means the equilibrium of resting water. The schemes which satisfy such properties are called well-balanced. In [9] it was suggested to use Roe modified scheme for satisfying the well-balanced conditions. This idea was developed in papers [10, 11]. The turbulence phenomena and mass sources in problems comprising nonhomogeneous bed are considered in papers [12, 13], and the improved reconstruction in [14, 15]. It was suggested in [16, 17] to use either ordinary or modified Godunov-type schemes (Riemann-solvers). This approach is well-proved and this idea has been widely applied to solve the problems with various source terms. In [18, 19] was suggested the method where the bed effects are compensated by changing of the values of fluid depths. In [5, 20, 21, 22] was proposed the method that uses the scheme of hydrostatic reconstruction of the flows on cells interfaces. The phenomena appearing because of presence of the Coriolis force were simulated using this scheme, particularly the geostrophic adaptation phenomenon. The same phenomena were considered in paper [23] in which LeVeque method is used [16, 17]. The propagation of tsunami waves and flooding processes were simulated in [24, 25]. In [26] was proposed the surface gradient method of interpreting source terms in shallow water equations, it is based on the accurate reconstruction of conservative variables on cell interfaces. The method of separating flow quantities which keeps exact balance between the flow gradients and source terms was proposed in paper [27].

The method of high-order accuracy and its improving reconstruction was proposed in [28]. The exact partial solutions of the problem including a step-wise boundary on a bed are presented in [29]. The exact partial solutions for the problem including nonhomogeneous bed are presented in [30]. Also we have to mention the works [6, 25, 31, 32, 33, 34, 35, 36, 37, 38, 39] which made a significant contribution to solution of problems of simulating hydrodynamic flows over a nonhomogeneous bed with source terms of various origin. The main difficulty arising in applying such methods is the lack of a unique analytical solution of the one-dimensional problem for nonlinear bed [40]. However, in all aforementioned works this problem was successfully solved by introducing some additional assumptions.

The presence of external forces, bed nonhomogeneities often causes vertical nonhomogeneity of horizontal shallow flows [8] that changes the values of hydrodynamic quantities averaged over the depth. It is necessary to take into consideration the vertical structure of depth-averaged fluid flows near the peculiarities of the bed to describe the indicated effects more adequately. In this paper we propose to use the Riemann-solver which is adapted to the flow parameters for calculating shallow water flows over an arbitrary bed in the presence of external force. The proposed method belongs to the family of methods based on the solution of the dam-break problem. The method consists in reducing of the problem to successive solutions of classical shallow water equations on the flat plane using Godunov's method with allowance for the vertical nonhomogeneity effect in calculating the flows through the boundaries of cells adjoining to stepwise boundaries. The vertical nonhomogeneity leads to the Riemann problem solution on a step based on the quasi-two-layer shallow water model developed in [41, 42, 43, 44, 45, 46] it is applied here for a generalized non-stationary bed which represents an external force.

The main difficulty in modeling of fluid flows over a complex bed consists in that both partly and complete flooded domains may take place. Partly flooded domains may appear in flows when fluid depth is related to the value of bed gradient when approximated by steps. In this case the step wall is partly wetting and the algorithm suggested in present work considers exactly mechanical work done only by this part of the step. Partly flooded domains present a real challenge for most finite-difference schemes and require special efforts need to be made to capture such domains. Algorithm suggested in our work avoids this difficulty in a natural way. In present work we extend algorithms developed in [41, 42, 43, 44, 45, 46] to the case of multiply-connected domains including partly flooded and dry regions.

Section 2 presents the shallow water model over an arbitrary bed profile in the presence of external force. The corresponding shallow water equations are written and the idea of the suggested Riemann-solver adaptable to flow parameters is described. Section 3 presents the quasi-two-layer model and the developed finite-difference scheme. Section 4 reviews the well known finite-difference schemes for calculating shallow water flows accounting for the external force effect in the form of fictitious bed, and compares them with our scheme. Section 5 presents the results of test computations with a constant external force corresponding to an inclined bed. The simulation of large-scale motion of heavy fluid (gas) is performed with allowance for the Coriolis force over a complex trigonometric function shape bed. Calculations and comparisons with laboratory experiment of two-dimensional dam-break problem on a slope are performed. The main results of the work are formulated in section 6.

2 MODEL OF SHALLOW WATER OVER AN ARBITRARY BED PROFILE IN THE PRESENCE OF EXTERNAL FORCE

We consider two-dimensional shallow water equations for fluid flows over an arbitrary nonhomogeneous bed with the external force in the divergence form [1, 7]:

$$\begin{cases} \dfrac{\partial h}{\partial t}+\dfrac{\partial (hu)}{\partial x}+\dfrac{\partial (hv)}{\partial y}=0 \\ \dfrac{\partial (hu)}{\partial t}+\dfrac{\partial \left(hu^2+gh^2/2\right)}{\partial x}+\dfrac{\partial huv}{\partial y}=-gh\dfrac{db}{dx}+Eh\,, \\ \dfrac{\partial (hv)}{\partial t}+\dfrac{\partial \left(hv^2+gh^2/2\right)}{\partial y}+\dfrac{\partial hvu}{\partial x}=-gh\dfrac{db}{dy}+Eh \end{cases} \tag{1}$$

where $h(x,y,t)$ is the depth of fluid, $u(x,y,t)$ is the depth-averaged horizontal velocity in the x direction, $v(x,y,t)$ is the depth-averaged horizontal velocity in the y direction, $b(x,y)$ is the function describing the bed shape, $E = E(h,u,v,x,y,t)$ is the external force, *g* is the gravity acceleration. The first equation in system (1) represents the mass conservation law; the second and third equations represent the momentum conservation laws for corresponding velocity vector components.

The corresponding one-dimensional problem, obtained by reducing the system (1) over one of spatial coordinates with an accuracy of projection of the momentum equation on a non-reduced spatial direction, corresponds to the one-dimensional spatial problem of SWE over a non-stationary bed. Neglecting in (1), for the sake of definiteness, the derivatives with respect to y we get the following equations:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial (hu)}{\partial x} = 0 \\ \dfrac{\partial (hu)}{\partial t} + \dfrac{\partial \left(hu^2 + gh^2/2 \right)}{\partial x} = -gh\dfrac{db}{dx} + Eh \end{cases} \tag{2}$$

Here $u = u(x,t)$, $b = b(x)$ and $E = E(h,u,x,t)$. The problem becomes essentially two-dimensional if the projection of exterior force E onto one of the coordinate axes depends either on the full velocity vector, or on the projection of a velocity vector onto other coordinate axis. Neglecting one of the spatial variables during the solving an essentially two-dimensional problem results in violating the momentum conservation law. This, in turn, makes necessary to introduce some fictitious work to compensate these violations, in spite of all non-physical character of such compensation. Such situation exists, for example in modeling of rotating shallow water, when Coriolis force acts as an external force. Use of quasi-two-layer model allows improving representation of transversal component of a velocity vector under the assumption of its stationarity on each time step that in particular leads to improvement of conservation of the potential vorticity vector. We will use this advantage of developed scheme in future developments. If *E* depends only on *h, u*, *x* and *t*, splitting of system (1) into a set of subsystems (2) is physically justified.

Existence of well-developed and tested numerical methods makes especially attractable the reduction of a problem in the presence an external force to solution of the problem of shallow water flows over a composed non-stationary bed. Actually, to obtain a formal coincidence of the types of equations, it must be presented in the following form:

$$g\frac{dk}{dx} = -E, \tag{3}$$

Where $k = k(x,t)$ is the fictitious bed. As it was mentioned above E depends on *h*, *u*, *x* and *t*. Now *k* depends only on *x* and *t* because *h* and *u* are also depend only on *x* and *t*. Then system (2) can be re-written as:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial (hu)}{\partial x} = 0 \\ \dfrac{\partial (hu)}{\partial t} + \dfrac{\partial \left(hu^2 + gh^2/2\right)}{\partial x} = -gh\dfrac{d(b+k)}{dx} \end{cases}. \quad (4)$$

As it can easily be seen, such a presentation essentially narrows the class of possible solutions, excluding from consideration spatially non-integrable functions specifying the external force *E*. This condition is generally needed to write down formally system (4), and may not be significant in finite difference representations of all functions defined in discrete points.

If we introduce the difference grid with a spatial mesh size *X* in the *x* direction and consider two adjacent cells in *x*, then, according to formula (3) we have in the left cell after averaging the fictitious bed with a slope tangent E_l/g, and with the tangent E_r/g in the right cell. Therefore, the corresponding difference of average heights of a fictitious bed on the boundary of the two adjacent cells is:

$$\frac{\left|E_r X/g + E_l X/g\right|}{2} \quad (5)$$

The possibility of simulating the shallow water in the presence of external force by introducing the fictitious non-stationary bed has been used for a long time for constructing numerical models [5, 16, 17, 21, 23]. The fact that the methods developed for composed bed can be useful in solving the problems with an external force, apparently, causes no doubts. In spite of the fact that the influence of fictitious bed, obviously, directly leads to accounting for bed work performed over the fluid flow, the external force, by itself, could not commit any work. The method proposed in this paper makes it possible to provide the physical interpretation of the mentioned formal adaptation of the methods. This makes it possible to better understand the limits of applicability of the given approach for splitting finite-difference schemes, being not limited to a particular applied model. The method proposed in this work allows to visualize the features of the splitting approach to numerical simulation as a whole and, thus, provides physical ideas for improving stability criteria in the finite-difference implementation.

The underlying problem for all Godunov-type methods is the Riemann problem that is the one-dimensional formulation of the Cauchy problem for two semi-infinite domains, in each of which the values of all hydrodynamic parameters are constant at the initial time. It is clear that in the presence of the external *y*-dependent force such a problem is physically meaningful for a rather small time intervals only, since its general solution is sought as a set of partial self-similar solutions.

As a result, the problem of calculating of the shallow water flows in the presence of external force becomes identical to the problem of calculating the shallow water flows over a non-stationary nonhomogeneous bed. Papers [41, 42, 43, 44, 45, 46] have offered the quasi-two-layer method for calculating shallow water flows over an nonhomogeneous bed. In present work we show how quasi-two-layer method can be applied in case of a non-stationary bed. The effective bed profile is approximated by a piecewise constant function separating it into a finite number of domains with a stepwise boundary.

3 QUASI-TWO-LAYER MODEL AND FINITE-DIFFERENCE SCHEME FOR THE EQUATIONS OF SHALLOW WATER OVER AN ARBITRARY BED PROFILE IN THE PRESENCE OF EXTERNAL FORCE

A distinctive feature of the suggested model is a separation of a studied flow into two layers in calculating flow quantities near each step, and improving the approximation of depth-averaged solutions of the initial three-dimensional Euler equations. The uniqueness of such a separation into two layers is provided by the uniqueness of solution of the Dirichlet problem for defining this boundary. One of the advantages of the developed method is that it allows one to take into account both flow velocity and the height of the step at each point of space and at any time instant. We are solving the shallow-water equations for one layer, introducing the fictitious lower layer only as an auxiliary structure in setting up the appropriate Riemann problems for the upper layer. Actually quasi-two-layer approach leads to appearance of additional terms in one-layer finite-difference scheme. These terms provide the mechanical work made by nonhomogeneous bed interacting with flow. Suggested approach modifies properly values of hydrodynamic fluxes in one-layer finite-difference scheme for nonhomogeneous bed as well by adjusting flow to that for upper layer in really two-layer model in the vicinity of discrete steps.

Below we develop finite-difference representation of the external force in numerical Godunov-type models for flows of shallow water based on quasi-two-layer model [41, 43, 45]. The basic idea of the quasi-two-layer model and model application for development of finite-difference scheme for shallow water equations over fully flooded bed is described in [42, 44, 46]. In this section we extend these ideas to the case of existence partially inundated areas and take into account the external force influence on the flow. We consider a shallow non-viscous flow over a step of height b_0 turned to the left with no loss of generality (Fig. 1).

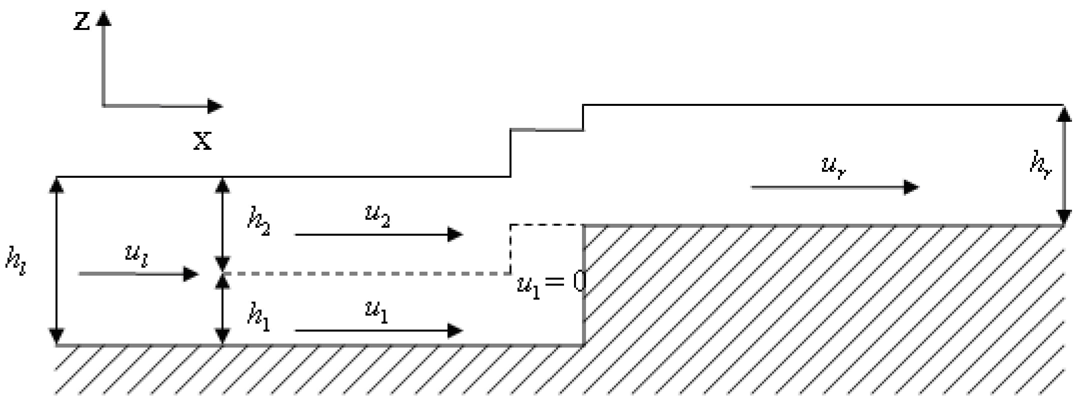

Figure 1: External force effect on the fluid flow.

If the fluid height is lower than the step height, the step would act as an impermeable boundary for the lower layer. Thus, if the height of a fluid is higher than that of the step, we may consider two fluid layers: the lower layer with the flow parameters h_1 and u_1 for which the step is an impermeable boundary and the upper one with the flow parameters h_2 and u_2 for which there is no direct influence of the step, obviously, $h_l = h_1 + h_2$. Equations for the two-layer system are following:

$$\begin{cases} \dfrac{\partial h_1}{\partial t} + \dfrac{\partial}{\partial x}(h_1 u_1) = 0, \\ \dfrac{\partial h_2}{\partial t} + \dfrac{\partial}{\partial x}(h_2 u_2) = 0, \\ \dfrac{\partial}{\partial t}(h_1 u_1) + \dfrac{\partial}{\partial x}(h_1 u_1^2 + \dfrac{1}{2} g h_1^2) + g h_1 \dfrac{\partial h_2}{\partial x} = 0, \\ \dfrac{\partial}{\partial t}(h_2 u_2) + \dfrac{\partial}{\partial x}(h_2 u_2^2 + \dfrac{1}{2} g h_2^2) + g h_2 \dfrac{\partial h_1}{\partial x} = 0, \end{cases} \qquad (6)$$

with the corresponding boundary condition $u_1 = 0$ for $x = 0$, $t \geq 0$ (the step affects the height and velocity of the lower layer near the step) and with the initial conditions for $t = 0$:

$$h_1 = h^*,\ h_2 = h_l - h^*,\ u_1 = u_l,\ u_2 = u_l \quad \text{for} \quad x \leq 0,$$
$$h_2 = h_r,\ u_2 = u_r \qquad \text{for} \quad x > 0. \qquad (7)$$

To solve system (6) we split it into two so that the variables h_1 and u_1 can be calculated independently of the variables h_2 and u_2. Assuming that the wave pattern in the lower layer is formed much faster than that in the upper one, the layers are separated so that the lower layer interacting with the step is at rest and forms a common horizontal plane with the step. This makes it possible to disregard the term $g h_1 \partial h_2 / \partial x$ in the third equation of system (6). We suppose that the upper layer does not appreciably affect the lower one, i.e., we disregard the term $g h_2 \partial h_1 / \partial x$. Then the effect of the lower layer on the upper one is taken into account by changing the initial condition $h_2 = h_l - h^*, x \leq 0$, which is determined by the change in the lower layer depth due to the step drag. As confirmed below by numerical calculations, these assumptions allow us to adequately describe the resulting flow. The adequacy in this case should imply the compatibility between the numerical results and theoretically possible ones when analytically solving the problem of arbitrary discontinuity decay over the step for shallow water equations.

Thus, under the above assumptions, we have the quasi-two-layer model described by the equations where the variables h_1, u_1 and h_2, u_2 are now connected only by the initial parameter h^* which should be chosen so that the lower layer height directly at the step coincides with the step height $h_1|_{x=0} = b_0$ for $t \geq 0$. Consequently, the problem of finding h^* reduces to the solution of the Dirichlet inverse problem:

$$\begin{cases} \dfrac{\partial h_1}{\partial t} + \dfrac{\partial}{\partial x}(h_1 u_1) = 0, \\ \dfrac{\partial}{\partial t}(h_1 u_1) + \dfrac{\partial}{\partial x}(h_1 u_1^2 + \dfrac{1}{2} g h_1^2) = 0, \end{cases} \qquad (8)$$

$h_1 = h^*$ for $x < 0$, $t = 0$; $u_1 = u_l$ for $x \leq 0$, $t = 0$; $u_1 = 0$ for $x = 0$, $t \geq 0$; $h_1 = b_0$ for $x = 0$, $t \geq 0$.

The solution of system (8) depends on the fluid flow direction, that is, on velocity u_l:

a) $u_l > 0$, the flow has a form of the shock wave reflected to the left, on the left side of which the flow parameters are h^* and $u_l > 0$. On the right side the fluid is at rest, that is, $h = b_0$ and $u = 0$, and the closing depth, determined by the step effect, is calculated as follows:

$$u_l = (b_0 - h^*)\sqrt{\frac{g}{2}\frac{(b_0 + h^*)}{b_0 h^*}}; \tag{9}$$

The solution is determined directly from the algebraic formula (9) that follows from Hugoniot's relations on a hydraulic jump [19];

b) $u_l < 0$, the flow has a form of rarefaction wave moving to the left, on the left side of which the flow parameters are h^* and $u_l < 0$, and on the right side as $h = b_0$ and $u = 0$, Therefore, the solution is determined from the corresponding Riemann's invariant, which describes the rarefaction wave:

$$h^* = \frac{1}{g}\left(\sqrt{gb_0} - \frac{1}{2}u_l\right)^2. \tag{10}$$

Expression (10) determines, in the explicit form, the depth of the lower part of a flow that is completely stopped by the step.

The situation is essentially different in the case when h^* is outside interval $(0, h_l]$. Physically, it is possible in two situations. First, h^* may be equal to zero. This implies absence of the lower layer, and consequently absence of the fluid and/or changes of the height of the bed. Second possibility is $h^* > h_l$. It corresponds to the full deceleration of the layer at the left side, i.e. in such situation the bed is a non-leaking boundary for all fluid at the left and therefore, it is necessary to accept that $h_2 = 0$ for $x < 0$. Then value h_1 for $x = 0$ can be found as a solution of the problem of interaction of all fluid at the left with a non-leaking boundary.

After evaluation of h^* one can find the values of flow quantities as the solution of classical Riemann problem for shallow water over a smooth bed with corresponding initial conditions. Using the quasi-two-layer method, one can find the flow quantities on cell faces, which determine, in the difference scheme, the values of all hydrodynamic parameters in the next time layer.

Thus, depending on flow parameters near the stepwise bed, the depths of fluid's lower and upper layers are calculated, and for calculating the flow parameters on a face the classical Riemann problem for upper layer's parameters is solved. The proposed method is adapted to the flow parameters and allows one to take into account the fluid flow features at each point of space and at each time instant. It should be noted that these features are considered in suggested model only in two-layer approximation in the steps vicinity.

For obtaining the difference scheme we integrate the system of equations of two-layer shallow water with the external force over a nonhomogeneous bed. After that we passed to surface integrals. And then we apply the integral conservation laws to each cell. Then applying the mean value theorem and supposing that at cell boundary the values of all hydrodynamic parameters represent the solutions of a corresponding one-dimensional problem during the whole integration time step of, we obtain the difference scheme:

$$\frac{H_{x,y}^{t+1}-H_{x,y}^{t}}{\tau}=\frac{H_{x-1/2,y}^{t}U_{x-1/2,y}^{t}-H_{x+1/2,y}^{t}U_{x+1/2,y}^{t}}{X}+\frac{H_{x,y-1/2}^{t}V_{x,y-1/2}^{t}-H_{x,y+1/2}^{t}V_{x,y+1/2}^{t}}{Y},$$

$$\frac{H_{x,y}^{t+1}U_{x,y}^{t+1}-H_{x,y}^{t}U_{x,y}^{t}}{\tau}=\left(\begin{array}{l}\frac{g\left(H_{x-1/2,y}^{t}+i_x\times\left(\left(B_{x,y}^{t}-B_{x-1,y}^{t}\right)+\left(K_{x,y}^{t}+K_{x-1,y}^{t}\right)/2\right)\right)^2}{2}-\\ -\frac{g\left(H_{x+1/2,y}^{t}+i_x\times\left(\left(B_{x+1,y}^{t}-B_{x,y}^{t}\right)+\left(K_{x+1,y}^{t}+K_{x,y}^{t}\right)/2\right)\right)^2}{2}+\\ +H_{x-1/2,y}^{t}\left(U_{x-1/2,y}^{t}\right)^2-H_{x+1/2,y}^{t}\left(U_{x+1/2,y}^{t}\right)^2\end{array}\right)/X+ \quad (11)$$

$$+\left(H_{x,y-1/2}^{t}U_{x,y-1/2}^{t}V_{x,y-1/2}^{t}-H_{x,y+1/2}^{t}U_{x,y+1/2}^{t}V_{x,y+1/2}^{t}\right)/Y$$

$$\frac{H_{x,y}^{t+1}V_{x,y}^{t+1}-H_{x,y}^{t}V_{x,y}^{t}}{\tau}=\left(\begin{array}{l}\frac{g\left(H_{x,y-1/2}^{t}+i_y\times\left(\left(B_{x,y}^{t}-B_{x,y-1}^{t}\right)+\left(K_{x,y}^{t}+K_{x,y-1}^{t}\right)/2\right)\right)^2}{2}-\\ -\frac{g\left(H_{x,y+1/2}^{t}+i_y\times\left(\left(B_{x,y+1}^{t}-B_{x,y}^{t}\right)+\left(K_{x,y+1}^{t}+K_{x,y}^{t}\right)/2\right)\right)^2}{2}+\\ +H_{x,y-1/2}^{t}\left(U_{x,y-1/2}^{t}\right)^2-H_{x,y+1/2}^{t}\left(U_{x,y+1/2}^{t}\right)^2\end{array}\right)/Y+$$

$$+\left(H_{x-1/2,y}^{t}U_{x-1/2,y}^{t}V_{x-1/2,y}^{t}-H_{x+1/2,y}^{t}U_{x+1/2,y}^{t}V_{x+1/2,y}^{t}\right)/X$$

Here τ is the time step; *X* and *Y* are spatial mesh sizes; *H* is the depth of fluid; *U* is the velocity in the *x* direction; *V* is the velocity in the *y* direction. Subscripts *x*, *y* designate the values of function related to the center of masses of a cell with number *(x, y).* Half-integers $x\pm1/2, y\pm1/2$ designate the values of quantities at the boundary between cells with numbers *x*, $x\pm1$, and *y*, $y\pm1$, respectively. Superscript *t* designates the number of a step in time, $K_{x,y}$ is the height of a fictitious bed and $B_{x,y}$ is the height of a bed. Respectively, according to formula (5):

$$\frac{K_{x+1,y}+K_{x,y}}{2}=\frac{\left|E_{x+1,y}X/g+E_{x,y}X/g\right|}{2},\frac{K_{x,y}+K_{x-1,y}}{2}=\frac{\left|E_{x,y}X/g+E_{x-1,y}X/g\right|}{2},$$

$$\frac{K_{x,y+1}+K_{x,y}}{2}=\frac{\left|E_{x,y+1}X/g+E_{x,y}X/g\right|}{2},\frac{K_{x,y}+K_{x,y-1}}{2}=\frac{\left|E_{x,y}X/g+E_{x,y-1}X/g\right|}{2}.$$

The variables i_x, i_y assume either the value 0 – in the case of negative difference of corresponding heights $\left(B_{x+1,y}-B_{x,y}\right)+\left(K_{x+1,y}+K_{x,y}\right)/2$, $\left(B_{x,y+1}-B_{x,y}\right)+\left(K_{x,y+1}+K_{x,y}\right)/2$, $\left(B_{x,y}-B_{x-1,y}\right)+\left(K_{x,y}+K_{x-1,y}\right)/2$,

$\left(B_{x,y}-B_{x,y-1}\right)+\left(K_{x,y}+K_{x,y-1}\right)/2$ of a effective bed (is equal sum of bed and fictitious bed), or the value of $0\le i_x,i_y\le 1$ – in the case of positive difference. Variables i_x, i_y takes value 1 in case if h^* is determined by equations (9,10) for a corresponding face of a cell and does not exceed value of depth inside the cell. Otherwise, variables i_x, i_y on a corresponding face are equal to the ratio of the depth formed under the total retardation of the flow on the indicated face to the corresponding difference of heights of the effective bed. Thus unlikely algorithms developed in [41, 42, 43, 44, 45, 46], in finite-difference scheme (11) party flooded and dry steps of effective bed are accounted. The values $H^t_{x\pm1/2,y\pm1/2}$, $U^t_{x\pm1/2,y\pm1/2}$, $V^t_{x\pm1/2,y\pm1/2}$ are calculated on the faces by solving the corresponding Riemann problems considering that transversal flow velocity is transported convectively. It is clear that developed finite-difference scheme is well-balanced type since it is balanced in case of zero velocities and free surface is horizontal.

The studied equations are hyperbolic type. Therefore, by the analogy with the shallow water equations, in which the external force effects is disregarded [19], the developed method uses the standard Courant – Friedrichs condition as the stability condition. This condition, however, should take into consideration the velocity of propagation of perturbations, including those in the fictitious lower layer. That is, the time step cannot exceed the minimal time during which the perturbations in each layer pass the half of a cell. With regard to correction for two-dimensional statement we have:

$$\tau=R\frac{\Delta t_x\Delta t_y}{\Delta t_x+\Delta t_y}, \qquad (12)$$

where $\Delta t_x,\Delta t_y$ are the minimum times of propagation of perturbations along the abscissa and ordinate axes, respectively; $R<1$ is the additional factor for increasing the reliability. In calculations presented below this factor is taken equal to 0.4. The success of selecting the stability criterion in this form is confirmed by test calculations presented below.

For improving accuracy of developed finite-difference scheme we use 2nd order accuracy computations in some tests (Subsections 4.2, 4.3, 4.4). Increase of accuracy on spatial coordinate is reached by applying the piecewise-linear reconstruction to the distribution of functions value in a cell with the use of the minmod limiter suggested for the first time by Kolgan [19, 47] for the solutions of accuracy problems of Godunov type methods in gas-dynamics:

$$W^t_x=\min\mathrm{mod}\left(\frac{F^t_{x+1}-F^t_x}{\Delta x},\frac{F^t_x-F^t_{x-1}}{\Delta x}\right)\qquad \text{where } F^t_x\equiv\begin{pmatrix}H^t_x\\U^t_x\\V^t_x\end{pmatrix},\ \alpha=0.72.$$

$$\min\mathrm{mod}(a,b)=\alpha\frac{1}{2}\left(signa+signb\right)\min\left(|a|,|b|\right)$$

The second order of accuracy in time is reached by application of two-step-by-step algorithm predictor - corrector. At a stage predictor there are found auxiliary values of required sizes for the whole step on time with the help of quasi-two-layer algorithm of the first order of the accuracy. These auxiliary values are used to find the values on an intermediate step in time by using arithmetic averaging with values of the previous time step. On the corrector step the given sizes are reconstructed in space: $F_x^{t+\frac{\tau}{2}}+\frac{1}{2}\Delta xW^t_x$,

$F_{x+1}^{t+\frac{\tau}{2}} - \frac{1}{2}\Delta x W_{x+1}^{t}$, accordingly at the left and at the right sides of the face $x+1/2$. Next, the values of fluid variables on borders of cells corresponding to an intermediate time layer are found.

4 RESULTS OF NUMERICAL CALCULATIONS

In order to verify the efficiency of the proposed method a number of the modeling problems corresponding to different types of nonhomogeneities of the bed and several types of external forces has been solved.

4.1 Dam-break problem of fluid column of parallelepiped shape on a sloping plane

Let us assume that at the center of an infinite plane, situated at some angle to the horizon, *k*, exist the fluid column of parallelepiped shape. Thus, one considers the effect of the constant external force that is equal to:

$$E = gk \tag{13}$$

For obtaining the numerical solution, we approximate the inclined plane by the system of ledges with a constant step in space. The selected problem of the fluid column breakdown is most indicative for checking the method efficiency, since it combines in itself all possible types of solutions simultaneously.

The quasi-two-layer method of the first order accuracy is applied for numerical modeling. Comparisons were performed for the cases of resting fluid and fluid flowing upwards along the inclined plane. The spatial step in our computation is taken 0,012 m per cell. The following initial data were taken for comparisons: the size of a discontinuity of the fluid column with height of 1.5 m and square base with side of 2 m over the inclined plane (the inclination value is 1:20), covered with fluid. In numerical experiment the fluid was flowing onto the plane at velocity of 2 m/s. The initial depth was 1 m everywhere outside the region occupied by a breaking column of fluid. The initial flow parameters are shown by a dash-doted line. The dashed line on plots indicates the flow depth and velocity obtained by using the proposed hydrodynamic model, the black line – solutions of combination of two Riemann problems on slope [48] (Fig. 2).

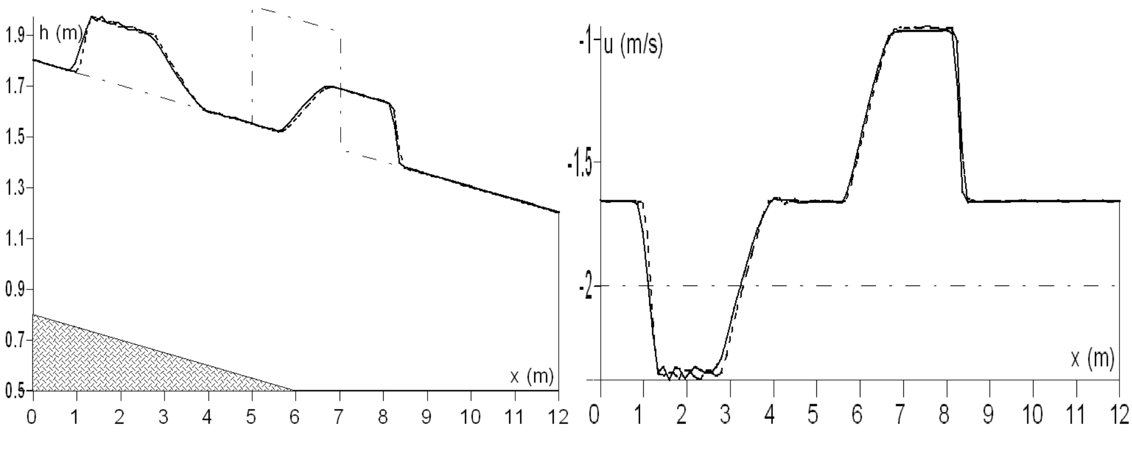

Figure 2: Depth and velocity of fluid flow over an inclined bed with fluid leakage, t = 0.7 s in the cross section by the plane of symmetry, y = 0. Dash-doted line - initial flow parameters; dashed line - flow depth and velocity obtained by using the proposed hydrodynamic model; black line – flow depth and velocity obtained by decision of Riemann problems on slope.

The presented plots (Fig. 2) reveal almost complete analogy between the results obtained by means of the quasi-two-layer method and solution of combination of two Riemann problems on slope [48]. This demonstrates the model efficiency in description of such natural phenomena. The oscillations arising behind shock waves' fronts are within method error and do not result in violating stability with due time.

4.2 Rotating fluid flow. The classical Rossby problem

The quasi-two-layer method of the second order accuracy is applied for numerical modeling on space and in time. The rectangular grid of the size 200 X 10 cells is used. The classical Rossby problem is simulated as the test one [49]. The initial disturbance considered was

$$\begin{cases} h(x,0) = h_0 \\ u(x,0) = 0 \\ v(x,0) = Vv_{jet}(x) \end{cases}, \tag{14}$$

where h_0 is the initial depth, V is the characteristic scale of velocity, $v_{jet}(x)$ is the normalized profile specified as follows: $v_{jet}(x) = \dfrac{(1+\tanh(4x/L+2))(1-\tanh(4x/L-2))}{(1+\tanh(2))^2}$.

L is the characteristic scale of disturbance. Characteristic parameters g, h_0, f are fixed. The characteristic scale of velocity V and the characteristic scale of disturbance L are calculated from two dimensionless parameters: the Rossby-Kibel (Ro) and Burgers (Bu) numbers: $Ro = \dfrac{V}{fL}$, $Bu = \dfrac{R_d^{\,2}}{L^2}$, where R_d is the deformation radius: $R_d = \dfrac{\sqrt{gh_0}}{f}$. The characteristic time scale is specified by the following formula: $T_f = \dfrac{2\pi}{f}$. The results of evolution of depth h_0, in the case of $Ro = 1$, $Bu = 0.25$, are presented below.

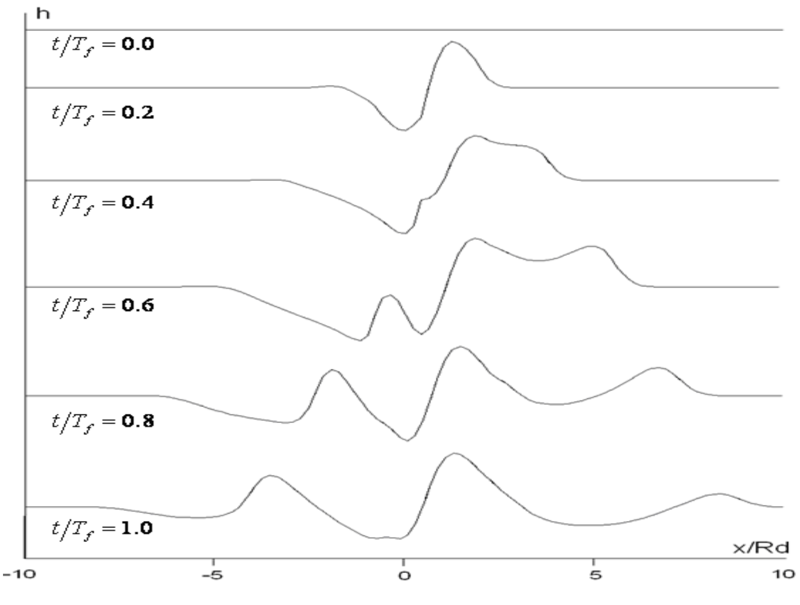

Figure 3: Evolution of acoustic-gravitational waves propagation, as a result of effect of the initial disturbance $Vv_{jet}(x)$, using the quasi-two-layer method.

Figure 3 shows the evolution obtained by using of the quasi-two-layer model. One can see good coincidence of characteristic peaks of running-away acoustic-gravitational waves and the central balanced part with results presented in paper [5]. This testifies to the efficiency of using the quasi-two-layer model in the description of large-scale geophysical phenomena.

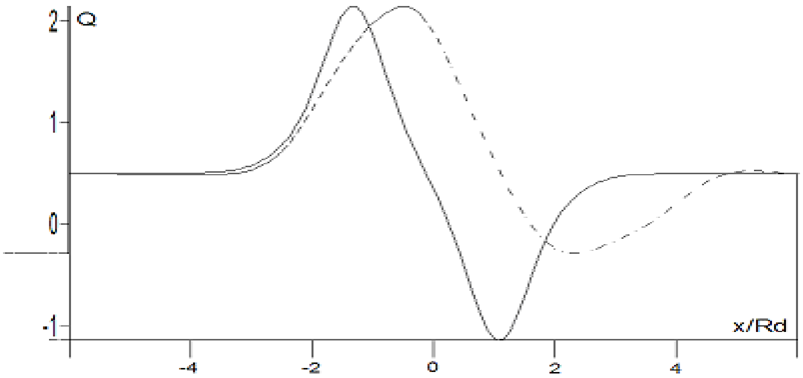

Figure 4: Potential vorticity at the initial (black line, $t = 0, 2T_f$) and final (dashed line, $t = 16T_f$) time instants.

Figure 4 shows the comparison of potential vorticity values at the initial ($t = 0T_f$) and final ($t = 16T_f$) time instants for the classical Rossby problem and $Ro = 1$, $Bu = 0.5$. The potential vorticity is defined the following formula:

$$Q = \frac{\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} + f}{h} \tag{15}$$

One can see that the invariant Q – the potential vortex is conserved with time. Note that the real time of the process equals to 12 days, approximately. It is seen from the presented plot that the maximum of a function is shifted to the anticyclonic region, and the minimum of the potential vorticity increases with time. The given results are determined by purely nonlinear effects and match well with those obtained in paper [50].

4.3 Flow of a rotating fluid over mounted parabolic profile

Problem was simulated, which contained both the nonhomogeneous bed, and the external force – the Coriolis force. The quasi-two-layer method of the second order accuracy is applied for numerical modeling on space and time. The shallow water equations in this case are as follows:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial(hu)}{\partial x} + \dfrac{\partial(hv)}{\partial y} = 0 \\ \dfrac{\partial(hu)}{\partial t} + \dfrac{\partial\left(hu^2 + gh^2/2\right)}{\partial x} + \dfrac{\partial huv}{\partial y} = -gh\dfrac{db}{dx} + fvh \\ \dfrac{\partial(hv)}{\partial t} + \dfrac{\partial\left(hv^2 + gh^2/2\right)}{\partial y} + \dfrac{\partial hvu}{\partial x} = -gh\dfrac{db}{dy} - fuh \end{cases} \tag{16}$$

where f is the Coriolis parameter. According to (16), we get the external forces: $E_x = fv$ and $E_y = -fu$. The large-scale motion of fluid in the gravitational field in the presence of Coriolis force over the mountain-like bed is considered. Typical parameters of the problem are: the linear dimensions 10^6 by 10^6 m, the mountain height is $1,2 \cdot 10^3$ m, the fluid (depth is $2 \cdot 10^3$ m, and the Coriolis parameter is 0.00001452 s^{--1}. The initial wind parameters are $u = 0$ m / s, $v = 20$ m / s. The boundary free-slip conditions are satisfied. The computational domain contains 3600 cells (60×60 cells).

As a result, we found that the characteristic time of one revolution of a system as a whole is 24 hours, which corresponds to the natural phenomenon (the characteristic time of one revolution of a system as a whole for the geophysical dynamics problems equals to one day [51]). Figure 5 shows the flow evolution during 24 hours.

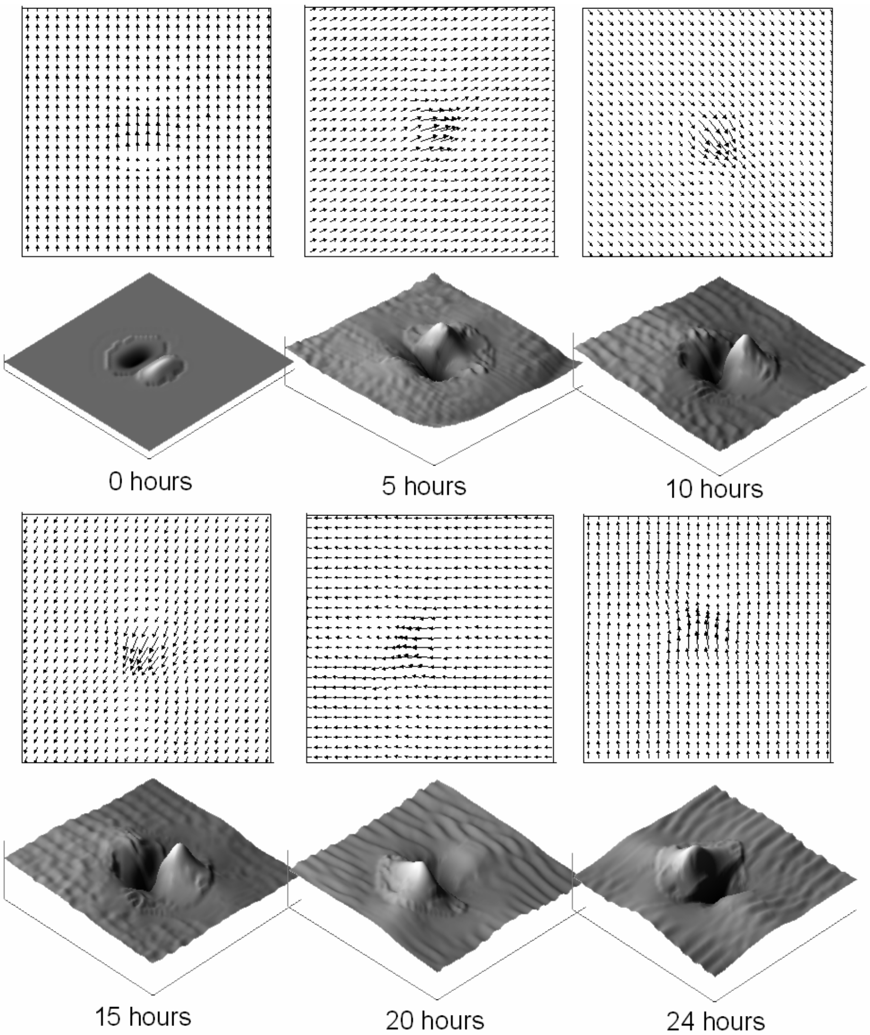

Figure 5: Evolution of fluid (gas) flow under the Coriolis force effect over a mountain; upper plots – fields of velocities, lower plots – the free surface.

4.4 Two-dimensional dam-break problem taking into account a hydraulic friction

The computational domain represents a rectangle of size 3 X 2 m, restricted by non-leaking walls on three sides. There is a wall apart 1 m from the left non-leaking boundary with a hole of width 0.40 m which is symmetrical with respect to the abscissa axis. The thickness of the wall is a negligible quantity and it is not used in calculations.

A hole in a wall is initially closed and all the fluid rests on the left of it. The fluid can freely leak through the right boundary of the computational domain. Define the Cartesian coordinate system in the following manner: the wall belongs to the ordinate axis, and the abscissa axis divides the computational domain in two symmetrical parts. The origin of coordinates coincides with the centre of the hole in a wall. All requirements of the performed numerical modeling are taken according to requirements of laboratory and numerical experiments from [52]. The slope angle of a bed in the performed numerical and laboratory experiments varied from 0 % to 10 %. Laboratory installation is made from plexiglass and supplied by the mechanism for opening a shutter of the hole in the wall at a velocity sufficient that dynamics of upraise of the shutter was not reflected in a pattern of formed flow. Detailed description of laboratory installation and the numerical experiments can be fined in the paper of Fraccarollo and Toro [52].

The shallow water equations in this case look as follows:

$$\begin{cases} \dfrac{\partial h}{\partial t}+\dfrac{\partial(hu)}{\partial x}+\dfrac{\partial(hv)}{\partial y}=0 \\ \dfrac{\partial(hu)}{\partial t}+\dfrac{\partial\left(hu^2+gh^2/2\right)}{\partial x}+\dfrac{\partial huv}{\partial y}=-gh\dfrac{db}{dx}+\dfrac{1}{2}\lambda u|u|h \\ \dfrac{\partial(hv)}{\partial t}+\dfrac{\partial\left(hv^2+gh^2/2\right)}{\partial y}+\dfrac{\partial hvu}{\partial x}=-gh\dfrac{db}{dy}+\dfrac{1}{2}\lambda v|v|h \end{cases} \tag{17}$$

where $\lambda = 2gn^2h^{-\frac{4}{3}}$, $n = 0.007$. External force for system (17) have the form of hydraulic friction: $E_x = \frac{1}{2}\lambda u|u|$, $E_y = \frac{1}{2}\lambda v|v|$.

The quasi-two-layer method of the second order accuracy is applied for numerical modelling on space and time. Results for the case of horizontal and sloping beds are shown below. The slope angle is 6.3 degrees. Depth of a fluid to the left of a wall is 0.6 м at initial time moment, the fluid was at rest for all the results given below. The computational domain is divided by regular grid of size 150 × 50.

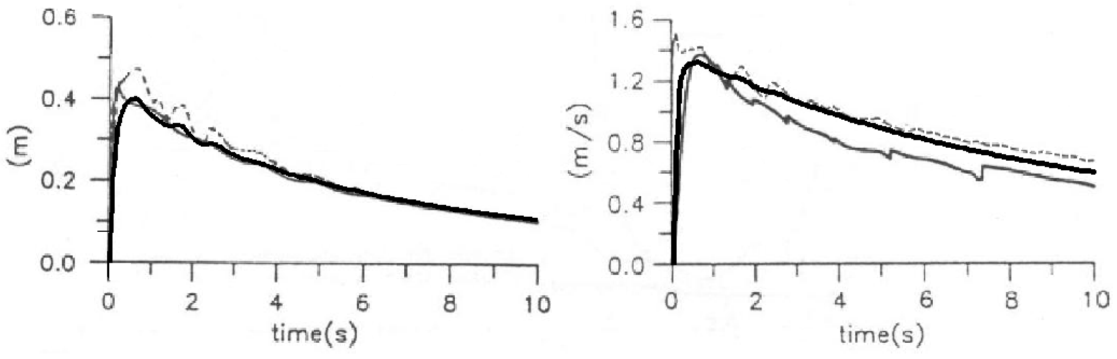

Figure 6: Left: fluid depths on time at point O. Right: fluid velocities on time at point O. Thin grey line - data obtained in laboratory experiment, dashed grey line - numerical results of WAF method, heavy black line – numerical results of proposed quasi-two-layer method.

Plots of fluid depths and velocities in a point 0 corresponding to the centre of dam-break are shown on Fig 6. The left plot corresponds to fluid depths, right – fluid velocities. Thin grey line corresponds to the data obtained in laboratory experiment

[52], dashed grey line - to numerical results of WAF method [52], heavy black line – to numerical results of proposed quasi-two-layer method.

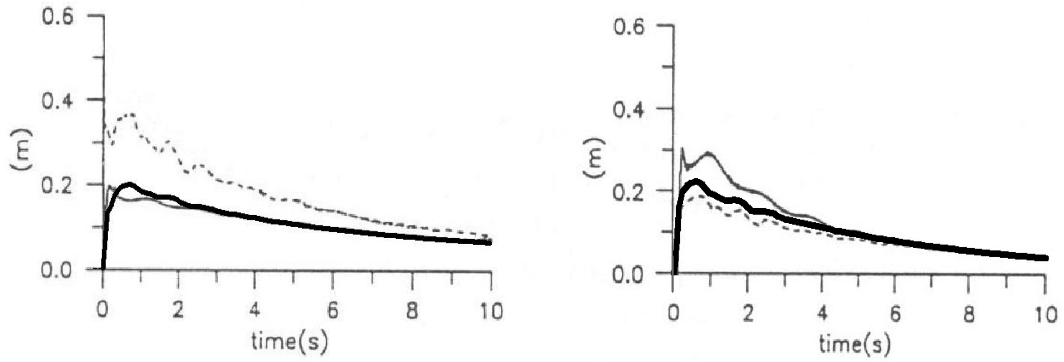

Figure 7: Left: Fluid depths on time at point P1. Right: Fluid depths on time at point P3. Thin grey line - data obtained in laboratory experiment, dashed grey line - numerical results of WAF method, heavy black line – numerical results of proposed quasi-two-layer method.

Dynamics of fluid depth for a horizontal bed in control points P1 and P3 are shown on Fig. 7: obtained experimentally – thin grey line, and numerically on the basis of WAF method [52] – dashed grey line, heavy black line – on the basis of proposed quasi-two-layer method.

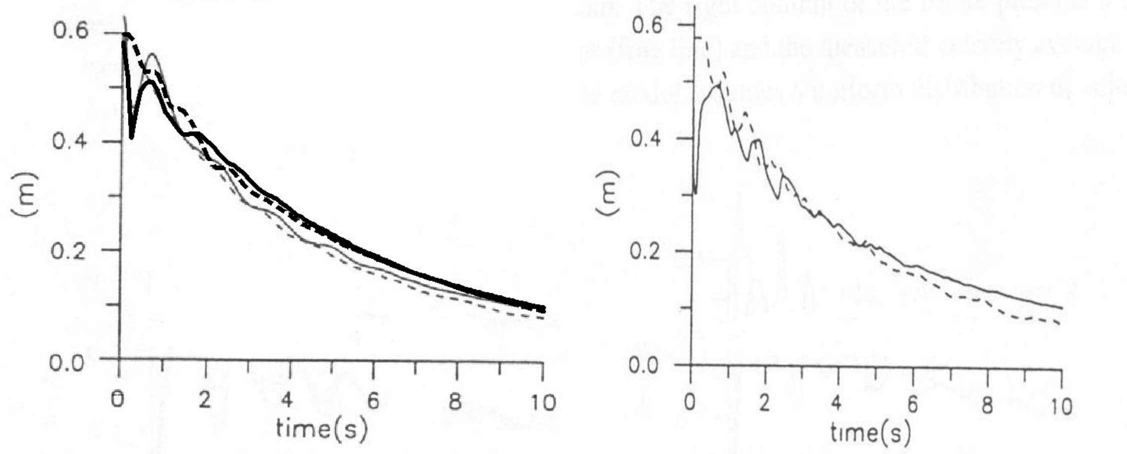

Figure 8: Left: Fluid depths on time for experiment with a sloping bed. Thin lines correspond to the data obtained in laboratory experiment: dashed – point P2, solid – point 0. Heavy lines correspond to numerical results of proposed quasi-two-layer method: dashed – point P2, solid – point 0.
Right: Fluid depths on time for experiment with a sloping bed. Numerical results of WAF method: solid line – point 0, dashed line – point P2

Dynamics of fluid depth for a sloping bed in two control points are shown on Fig. 8 (Left side). Thin lines correspond to the data obtained in laboratory experiment: dashed – point P2, solid – point 0. Numerical results of proposed quasi-two-layer method shown by heavy lines: dashed – point P2, solid – point 0. On Fig. 8 (Right side) numerical results of WAF method for the same points are shown. One can see that proposed quasi-two-layer method is in good agreement with experimental data and improves accuracy of results in comparison with the numerical results of WAF method obtained in article [52].

5 CONCLUSIONS

New numerical method for simulating hydrodynamic flows over an arbitrary bed profile in the presence of external force is proposed in this paper. The presentation of arbitrary external force by a fictitious bed, which, along with the bed relief features, is approximated by a non-stationary stepwise bed, allows one to apply the methods based on the Riemann problem solution for determining the flow quantities. For solving the Riemann problem we used an advanced algorithm which is based on the two-layer presentation of fluid flow near the step. Such representation allowed to obtain in suggested finite difference scheme the additional terms providing mechanical work made by nonhomogeneous bed interacting with flow. This also modifies properly values of hydrodynamic fluxes in steps vicinity. The calculation of flows in the proposed algorithm is performed on the basis of the quasi-two-layer shallow water model, which represents the generalization of the classical single-layer model in relation to the initial system of Euler equations. It is shown that the developed finite-difference scheme belongs to the well-balance class. The developed approach allows one to restore the flow structure throughout the space-time domain with consideration for its vertical peculiarities near the step. The calculation of essentially two-dimensional problems with a corrected transversal velocity component will be accomplished in a separate work. The applicability of the method is demonstrated by the example of solution of the problem of breakdown of a moving fluid column on an inclined plane. The motion of fluid in the Coriolis force presence over a bed of the given profile was simulated. Reliability and validity of our results is shown on the basis of the solution of two-dimensional dam-break problem taking into account a hydraulic friction and comparison of the obtained solutions with available results of laboratory experiments.

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