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REGULARIZATIONS OF TURBULENT FLOW

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Abstract. *The Navier-Stokes equations accurately describe turbulent flows. Yet, they do not provide a tractable model. The mathematical analysis of the Navier-Stokes equations is extremely hard, and most turbulent flows cannot be computed numerically from the Navier-Stokes equations because the solution possess far too many scales of motion. At the crossroad of theory and numerical simulation new tractable models for turbulence – based on regularizations of the Navier-Stokes equations – start to develop. This results into flow fields that contain less scales than the actual flow, whereas the large scales of motion are predicted well. To that end, the nonlinear (convective) term in the Navier-Stokes is altered. This can be done in various ways, yielding different regularization models. We consider regularizations that preserve the conservation and symmetry properties of the convective operator. The underlying idea is to restrain the convective production of small scales in an unconditional stable manner, meaning that the solution of the regularized system cannot blow up in the energy norm. Existence and uniqueness of regularized solutions can be proven. Additionally it can be shown that the solution of the regularized system - on a periodic box in dimension three - actual has a range of scales with wavenumber k for which the rate at which energy is transferred (from scales $> k$ to those $< k$) is independent of k. In this so-called inertial subrange the energy behaves like $k^{-5/3}$. Compared to Navier-Stokes, the inertial subrange is shortened yielding a more amenable problem to solve numerically. The numerical method used to solve the regularized system preserves the symmetry and conservation properties too. The resulting simulation method is successfully tested for channel flow (Re_τ = 180)*

1 Navier-Stokes equations and turbulence

The understanding and prediction of turbulence by means of its numerical simulation is one of the most elusive and important goals in science and engineering [1]. The Navier-Stokes (NS) equations provide an appropriate model for turbulent flow [2]. In the absence of compressibility ($\nabla \cdot u = 0$), the equations are

$$\partial_t u + \mathcal{C}(u, u) + \mathcal{D}(u) + \nabla p = 0, \tag{1}$$

where u denotes the velocity field, p stands for the pressure. The diffusive and convective terms are given by $\mathcal{D}(u) = -\frac{1}{\mathrm{Re}}\nabla \cdot \nabla u$, Re is the Reynolds number, and

$$\mathcal{C}(u, v) = (u \cdot \nabla)v, \tag{2}$$

respectively. Attempts at simulating turbulence directly from (1) are limited to very low Reynolds numbers [3]. The reason for this limitation is rooted in the physics of turbulence: at moderate Reynolds numbers flows simply possess far too many scales of motion [4]-[6]. The general notion is that energy enters the turbulence at the largest scales. Since these scales cannot reach a near equilibrium between the rate at which energy is supplied and the rate at which energy is dissipated (by the action of viscosity), they break up, transferring their energy to somewhat smaller scales. These smaller scales undergo a similar break-up process, and transfer their energy to yet smaller scales. This energy cascade continues until the scale becomes so small that energy can be dissipated. The entire range of scales is to be resolved when turbulence is computed from (1). Yet, in most applications we can just solve the dynamics of the large scales, and certainly not the small scales where the dissipation takes place. This poses a challenge for a coarse-grained description of turbulence [7].

2 Large-eddy simulation

In large-eddy simulation (LES) [8]-[10] the coarse-grained description is obtained by applying a spatial filter, say $u \mapsto \overline{u}$, to Eq. (1):

$$\partial_t \overline{u} + \mathcal{C}(\overline{u}, \overline{u}) + \mathcal{D}(\overline{u}) + \nabla \overline{p} = \mathcal{C}(\overline{u}, \overline{u}) - \overline{\mathcal{C}(u, u)}. \tag{3}$$

The right-hand side represents the effects of the residual scales on the 'large eddies' (the part of the fluid motion with velocity $\overline{u}$). It depends on both u and $\overline{u}$. To remove the dependence on u (i.e., to close the system in terms of $\overline{u}$) the commutator of $\mathcal{C}$ and the filter is replaced by a model:

$$\partial_t v + \mathcal{C}(v, v) + \mathcal{D}(v) + \nabla q = \mathcal{M}(v). \tag{4}$$

Here, the variable name is changed from $\overline{u}$ to v to stress that the solution of Eq. (4) differs from that of Eq. (3), because the closure model is not exact. The key idea is that the spectral support of $v \approx \overline{u}$ is much smaller than that of u, which enables us to solve (4) numerically when it is not feasible to solve (1). Because turbulence is so far from being completely understood, there is a wide range of closure models, mostly based on heuristic, *ad hoc* arguments that cannot be derived from the NS-equations, see for example [10] and the references therein.

3 Regularization modelling

Recently, it has been proposed to reduce the computational complexity by modifying the nonlinear term in the NS-equations in such a way that the large scales of motion remain unaltered, whereas the tail of the modulated spectrum falls off much faster than the NS-spectrum [11]. Here scales of motions are defined with the help of the diffusive operator in the NS-equations. This self-adjoint operator possesses an orthogonal, complete basis of eigenvectors w_j: $\mathcal{D}w_j = \lambda_j w_j$, where the eigenvalues can be ordered, $0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq$ ..., and $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$ [12]. The κ-th mode of u is given by $u_\kappa = \sum_{\lambda_j=\kappa^2} \hat{u}_j w_j$. The number of dynamically significant modes u_κ is much too large to be computed directly from the NS-equations. Regularization is one of the mechanisms by which the dynamics can be reduced. Basically, the nonlinearity in (1) is altered to restrain the convective energetic exchanges:

$$\partial_t u_\epsilon + \widetilde{\mathcal{C}}(u_\epsilon, u_\epsilon) + \mathcal{D}u_\epsilon + \nabla p_\epsilon = 0, \tag{5}$$

where ϵ denotes a parameter introduced by the regularization procedure. The basis functions w_j are obviously not altered by the regularization. The general idea is that the low modes of the solution u_ϵ of the regularized system (5) should approximate the corresponding low modes of the Navier-Stokes solution u, whereas the high modes should vanish much faster (so that they need not be computed). This can be achieved in various ways, yielding different regularization models. These regularization principles are appealing as they generate a mathematics-based procedure to confine the dynamics, i.e., it can be proven rigorously that properties of the NS-equations are retained, that the resulting set of PDE's is well-posed, etc. [10].

3.1 Leray regularization

The first outstanding result which involves filtering of the nonlinearity is the proof of existence of weak solutions to the NS-equations by Leray [13]. He considered:

$$\partial_t u_\epsilon + \mathcal{C}(\overline{u}_\epsilon, u_\epsilon) + \mathcal{D}(u_\epsilon) + \nabla p_\epsilon = 0. \tag{6}$$

Leray proved that (6) has a unique C^∞ solution for any filter length $\epsilon > 0$, which is bounded and has a subsequence that converges to the weak NS-solution as $\epsilon \rightarrow 0$. The striking result here is that any filtering is sufficient to guarantee uniqueness of C^∞ solutions. Stated in physical terms, Leray's result ascertains that the energy cascade stops at a certain scale of motion, everywhere in the spatial domain and for all times. Cheskidov *et al.* [14] have showed that the complexity of the 3D Leray model (6) lies between that of the 2D and 3D Navier-Stokes.

4 Symmetry and conservation properties

In the absence of diffusion ($\mathcal{D} = 0$), the NS-equations conserve particular quantities, like the energy, enstropy (in 2D) and helicity. These conservation properties are a crucial factor in determining how solutions behave. Therefore we want to preserve these invariants under regularization. The energy is defined by $|u|^2 = (u, u)$, where the innerproduct is defined in the usual way,

$$(u, v) = \int_\Omega u \cdot v \, dx.$$

Here, Ω denotes the flow domain. Differentiating (u, u) with respect to time yields the convective contribution $(\mathcal{C}(u, u), u)$. As a result of the skew-symmetry [12],

$$(\mathcal{C}(u, v), w) = -(v, \mathcal{C}(u, w)), \tag{7}$$

we have $(\mathcal{C}(u, v), v) = 0$ for any pair u, v. Hence, the energy equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \tfrac{1}{2}|u|^2 = -\tfrac{1}{\mathrm{Re}}|\nabla u|^2 = -\tfrac{1}{\mathrm{Re}}|\omega|^2. \tag{8}$$

Taking the curl of the NS-equations gives

$$\partial_t \omega + \mathcal{C}(u, \omega) = \mathcal{C}(\omega, u) - \mathcal{D}(u), \tag{9}$$

where $\omega = \nabla \times u$ is the vorticity. Consequently, the enstrophy $|\omega|^2$ is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t} \tfrac{1}{2}|\omega|^2 = (\mathcal{C}(\omega, u), \omega) - \tfrac{1}{\mathrm{Re}}|\nabla \omega|^2. \tag{10}$$

The trilinear form in (10) vanishes in 2D. This property is extensively used in the proof of the existence and uniqueness of (weak and strong) 2D NS-solutions [16]. In 3D, however, the term $(\mathcal{C}(\omega, u), \omega)$ does not vanish and the question of existence and uniqueness is still open. One of the main open problems is to find sharp bounds for $(\mathcal{C}(\omega, u), \omega)$. Presently, no bound is known that guarantees that the viscous term $\frac{1}{\mathrm{Re}}|\nabla \omega|^2$ can stop the vorticity cascade. In mathematical terms, it cannot yet be proven that a strong solution, that is a velocity field for which the enstrophy remains finite for all times, exists. It can be proven that as long as a strong solution exists, it is unique. *A priori*, however, it cannot yet be excluded that in a rare event the vorticity ω bursts driving the energy to extreme small scales. The existence of weak solutions is proven in 3D, but it is not yet proven that they are unique. For more details, see the monograph [16]. Eqs. (8)-(10) show that the energy and enstrophy (in 2D) are conserved (in case $\mathcal{D} = 0$). The evolution of the helicity (ω, u) follows from Eq. (1) and Eq. (9). The resulting convective contribution $(\mathcal{C}(u, u), \omega) + (\mathcal{C}(u, \omega), u) - (\mathcal{C}(\omega, u), u)$ vanishes as an immediate consequence of the skew symmetry (7). Thus, the helicity is also conserved.

5 Symmetry-preserving regularization

As the full energy cascade cannot be computed, we aim to reduce the computational complexity by modifying the nonlinear operator $\mathcal{C}$ in a such a manner that its invariants (energy, enstrophy and helicity) are preserved. The following class of regularizations

$$\partial_t u_\epsilon + \mathcal{C}_n(u_\epsilon, u_\epsilon) + \mathcal{D}(u_\epsilon) + \nabla p_\epsilon = 0 \tag{11}$$

($n = 2, 4, 6$) in which the convective term is given by

$$\begin{aligned}
\mathcal{C}_2(u,v) &= \overline{\mathcal{C}(\overline{u},\overline{v})} && (12)\\
\mathcal{C}_4(u,v) &= \mathcal{C}(\overline{u},\overline{v}) + \overline{\mathcal{C}(\overline{u},v')} + \overline{\mathcal{C}(u',\overline{v})} && (13)\\
\mathcal{C}_6(u,v) &= \mathcal{C}(\overline{u},\overline{v}) + \mathcal{C}(\overline{u},v') + \mathcal{C}(u',\overline{v}) + \overline{\mathcal{C}(u',v')} && (14)
\end{aligned}$$

and the primes denote residuals, *e.g.* $u' = u - \overline{u}$, preserves the skew-symmetry:

$$(\mathcal{C}_n(u,v), w) = -(v, \mathcal{C}_n(u,w)) \tag{15}$$

for $n = 2, 4, 6$ (and any self-adjoint filter). Additionally, the vorticity equation becomes

$$\partial_t \omega + \mathcal{C}_n(u,\omega) = \mathcal{C}_n(\omega,u) - \mathcal{D}(u) \tag{16}$$

and in 2D: $(\mathcal{C}_n(\omega,u),\omega) = 0$. Now as for the NS-equations, it can be shown that the energy, enstrophy and helicity are preserved [15]. The difference between $\mathcal{C}(u,u)$ and $\mathcal{C}_n(u,u)$ is of the order ϵ^n with $n = 2, 4, 6$ (where ϵ is the width of the filter). Note: Leray's model is second-order accurate, preserves the energy, but not the enstrophy/helicity.

6 Vortex stretching mechanism

To see how the approximations given by (12)-(14) restrain the production of small scales of motion, we consider the vortex stretching mechanism (in this section) and triad interactions (in the next section), respectively. If it happens that the source term $\mathcal{C}_n(\omega,u)$ in Eq. (16) is so strong that the dissipative term $\mathcal{D}(\omega)$ cannot prevent the intensification of vorticity, smaller and smaller vortical structures may be produced locally. The Navier-Stokes equations lead to the source term

$$\mathcal{C}(\omega,u) = S(u)\omega, \tag{17}$$

where $S(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$ is the deformation tensor. The trace of this symmetric tensor is zero. Consequently, $S(u)$ has at least one non-negative eigenvalue. If ω is aligned with an eigenvector associated with a positive eigenvalue, then the source term $\mathcal{C}(\omega,u)$ in the vorticity equation is positive, which may lead to an increase of the vorticity magnitude. As the angular momentum is conserved (in the absence of viscous dissipation) an increase of the vorticity magnitude implies that fluid elements are stretched along the direction of the eigenvector associated with the positive eigenvalue. This phenomenon, called vortex stretching, implies a transfer of energy from large scales of motion to smaller ones, *i.e.* drives the energy cascade. Here, it may be noted that the evolution of a short material line element δr is given by $\partial_t \delta r + \mathcal{C}(u, \delta r) = \mathcal{C}(\delta r, u)$. Thus it is as if the vorticity behaves like a line material element coinciding instantaneously with a portion of the vortex line. The source in the dynamics of $|\delta r|^2$ is given by $\mathcal{C}(\delta r, u) \cdot \delta r = \delta r \cdot \mathcal{S}(u) \delta r$.

The approximations (12)-(14) alter the vortex stretching mechanism in 3D ($\mathcal{C}_n(\omega, u)$ is identically zero in 2D). The vortex stretching term becomes:

$$\mathcal{C}_2(\omega, u) = \overline{\overline{S}\overline{\omega}} \quad (18)$$

$$\mathcal{C}_4(\omega, u) = \overline{S}\overline{\omega} + \overline{\overline{S}\omega'} + \overline{S'\overline{\omega}} \quad (19)$$

$$\mathcal{C}_6(\omega, u) = \overline{S}\overline{\omega} + \overline{S}\omega' + S'\overline{\omega} + \overline{S'\omega'} \quad (20)$$

In the Navier-Stokes dynamics, vortex stretching leads to the production of smaller and smaller scales; hence to a continuous, local increase of both S' and ω'. Consequently, at the positions where vortex stretching occurs, the terms with S' and ω' will eventually amount considerably to $S\omega = \overline{S}\overline{\omega} + \overline{S}\omega' + S'\overline{\omega} + S'\omega'$. Since these terms are diminished in (18)-(20), the symmetry-preserving approximations $\mathcal{C}_n$ of the convective term counteract the production of smaller and smaller scales by means of vortex stretching and may eventually stop the continuation of the vortex stretching process.

7 Triadic interactions

To study the interscale interactions in more detail, we continue in the spectral space. The spectral representation of the convective term in the Navier-Stokes equations is given by

$$\mathcal{C}_k(\hat{u}, \hat{v}) = \mathrm{i}\Pi(k) \sum_{p+q=k} \hat{u}_p q \hat{v}_q, \quad (21)$$

where $\Pi(k) = I - kk^{\mathrm{T}}/|k|^2$ denotes the projector onto divergence-free velocity fields in the spectral space. Taking the Fourier transform of (11)-(14), we obtain the evolution of each Fourier-mode $\hat{u}_k(t)$ of u_ϵ for the approximation $\mathcal{C}_n$:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{|k|^2}{\mathrm{Re}} \right) \hat{u}_k + \mathrm{i}\Pi(k) \sum_{p+q=k} f_n(\hat{g}_k, \hat{g}_p, \hat{g}_q)\, \hat{u}_p q \hat{v}_q = 0. \quad (22)$$

The mode $\hat{u}_k(t)$ interacts only with those modes whose wave vectors p and q form a triangle with the vector k. Compared with (21), every triad interaction is multiplied by

$$f_2(\hat{g}_k, \hat{g}_p, \hat{g}_q) = \hat{g}_k \hat{g}_p \hat{g}_q$$

$$f_4(\hat{g}_k, \hat{g}_p, \hat{g}_q) = \hat{g}_k \hat{g}_p + \hat{g}_k \hat{g}_q + \hat{g}_p \hat{g}_q - 2\hat{g}_k \hat{g}_p \hat{g}_q$$

$$f_6(\hat{g}_k, \hat{g}_p, \hat{g}_q) = \hat{g}_k + \hat{g}_p + \hat{g}_q - \hat{g}_k \hat{g}_p - \hat{g}_k \hat{g}_q - \hat{g}_p \hat{g}_q + \hat{g}_k \hat{g}_p \hat{g}_q$$

where $\hat{g}_k$ denotes the k-th Fourier-mode of the kernel of the convolution filter, *i.e.*, $\overline{\hat{u}}_k = \hat{g}_k \hat{u}_k$. The functions f_n satisfy $f_n(1,1,1) = 1$ and $f_n(0,0,0) = 0$. Furthermore, all the first-order partial derivatives of $f_n(\hat{g}_k, \hat{g}_p, \hat{g}_q)$ are strictly positive for $0 < \hat{g}_k, \hat{g}_p, \hat{g}_q < 1$. Hence, the factor $f_n(\hat{g}_k, \hat{g}_p, \hat{g}_q)$ by which every Navier-Stokes interaction is multiplied is a monotone function of $\hat{g}_k$, $\hat{g}_p$, and $\hat{g}_q$.

A generic, symmetric convolution filter satisfies

$$\hat{g}_k = 1 - \tfrac{\epsilon^2}{24}\ |k|^2 + \mathcal{O}(\epsilon^4).$$

Consequently,

$$\begin{aligned} f_2 &\approx 1 - \tfrac{\epsilon^2}{24}\ (|k|^2 + |p|^2 + |q|^2), \\ f_4 &\approx 1 - (\tfrac{\epsilon^2}{24})^2\ (|k|^2|p|^2 + |k|^2|q|^2 + |p|^2|q|^2), \\ f_6 &\approx 1 - (\tfrac{\epsilon^2}{24})^3\ |k|^2|p|^2|q|^2, \end{aligned}$$

respectively. In other words, the interactions between large scales of motion (short wave vectors, $\epsilon|k| < 1$) approximate the Navier-Stokes dynamics up to $\mathcal{O}(\epsilon^n)$, with $n = 2, 4, 6$, respectively. Hence, the triadic interactions between large scales of motion are only slightly altered. All interactions involving longer wave vectors (smaller scales of motion) are reduced. The amount by which the interactions between the wave vector triple (k, p, q) are lessened depends on the length of the legs of the triangle $k = p + q$. In case $n = 4$, for example, all triadic interactions for which at least two legs are (much) longer than $1/\epsilon$ are (strongly) attenuated, whereas interactions for which two legs are (much) shorter than $1/\epsilon$ are reduced to a small degree only.

8 Mathematical basis

The symmetry-preserving regularizations (11)-(14) yield uniqueness and the expected regularity properties: for all initial velocities in $H = \{u \in \mathrm{L}^2(\Omega), \nabla \cdot u = 0\}$ where the spatial domain is given by $\Omega = (0, 2\pi)^3$ and periodic boundary conditions are enforced, and $\epsilon > 0$, Eq. (11), with $\mathcal{C}_n$ given by (12)-(14), has a unique C^∞ solution. This solution is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$, where the time $t \in (0,T)$, with $T > 0$ arbitrary, and $V = \{u \in \mathrm{H}^1(\Omega), \nabla \cdot u = 0\}$). One subsequence converges weakly in $L^2(0,T;V)$ to a weak NS-solution as $\epsilon \to 0$. The proof is in fact a copy of Leray's proof in [13]. So as for Leray's model, any filtering in Eqs. (12)-(14) is sufficient to guarantee that the energy cascade stops at a certain scale of motion.

Fifty years after Kolmogorov's landmark papers on the cascade-concept [4]-[5], Foias et al. [6] proved Kolmogorov's results in a mathematically rigorous manner. They proved that the solution (existence is assumed) of the NS-equation - on a periodic box in dimension three - actual has a range of scales with wavenumber κ for which the rate at which energy is transferred (from scales $> \kappa$ to those $< \kappa$) is independent of κ. In this range the energy behaves like $\kappa^{-5/3}$. The proofs by Foias et al. are also applicable to the regularized system (11), because the regularization preserves symmetry and conservation properties of the nonlinearity. So, the leading part of the inertial subrange is approximated properly.

9 Inertial subrange

The regularized system (11) should be more amenable to solve numerically than the Navier-stokes equations, while its solution has to approximate the low wavenumbers of the Navier-Stokes solution. Therefore, the leading part of inertial subrange of the energy cascade is to be retrieved. In this section we consider the projection of (11) (here, with periodic or no-slip boundary conditions) on the space of divergence free vector fields,

$$\partial_t u_\epsilon + \mathcal{C}_n(u_\epsilon, u_\epsilon) + \mathcal{D}u_\epsilon = f, \tag{23}$$

where f is the forcing term; $\mathcal{C}_n$ and $\mathcal{D}$ represent the projection of the convective and diffusive term, respectively. We proof - using the reasoning of Foias *et al.* [6] - that any skew-symmetric regularization of the Navier-Stokes equations possesses an inertial subrange for $\kappa_f < \kappa << \kappa_\tau$, where κ_f denotes the highest wavenumber of the forcing f and κ_τ is the Taylor wavenumber of the solution of the regularized system (11).

The κ-th mode of the regularized solution is denoted by $u_\kappa = \sum_{\lambda_j=\kappa^2} \hat{u}_{\epsilon,j} w_j$ where w_j denote the eigenfunctions of the dissipative operator $\mathcal{D}$. We introduce the notations $u_{<s} = \sum_{\kappa<s} u_\kappa$ and $u_{ls} = u_{<s} - u_{<l}$, with $0 \leq l \leq s < \infty$. The evolution of u_{ls} follows straightforwardly from Eq. (11). Taking the inner product with u_{ls} yields

$$\mathrm{d}_t \tfrac{1}{2}|u_{ls}|^2 + \tfrac{1}{\mathrm{Re}}|\nabla u_{ls}|^2 - (f_{ls}, u_{ls}) = T_l - T_s, \tag{24}$$

where $T_\kappa = T_\kappa(u) \stackrel{\text{def}}{=} (\mathcal{C}_n(u,u), u_{<\kappa})$ represents the (regularized) flux of kinetic energy through the wavenumber κ. Eq. (24) resembles the energy budget equation that follows from the Navier-Stokes equations (1). The only difference is that the energy flux T_κ is defined using $\mathcal{C}_n$ instead of $\mathcal{C}$. The skew-symmetry (15) has a number of important consequences. First, it ensures that the flux into the opposite direction is given by $(\mathcal{C}_n(u,u), u - u_{<\kappa}) = -(\mathcal{C}_n(u,u), u_{<\kappa}) = -T_\kappa(u)$. Secondly, it enables to separate the flux in a manner similar to the Navier-Stokes equations: $T_\kappa(u) = (\mathcal{C}_n(u_{\geq\kappa}, u_{\geq\kappa}), u_{<\kappa}) - (\mathcal{C}_n(u_{<\kappa}, u_{<\kappa}), u_{\geq\kappa})$. Thus the net energy flux equals the difference between the inertial effects by the high modes on the low ones and the inertial effects by low modes on the high ones. Thirdly, for a regular solution with $u_\kappa \to 0$ as $\kappa \to \infty$, we have

$$T_\kappa \to 0 \quad \text{as} \quad \kappa \to \infty, \tag{25}$$

if and only if $\mathcal{C}_n$ is skew-symmetric. To proof this, we suppose (for the sake of the argument) that $\mathcal{C}_n$ contains a symmetric part, say $\mathcal{S}_n$ with $(\mathcal{S}_n(u,v), w) = (v, \mathcal{S}_n(u,w))$. Then $T_\kappa(u) = (\mathcal{C}_n(u, u_{\geq\kappa}), u_{<\kappa}) + (\mathcal{S}_n(u, u_{<\kappa}), u_{<\kappa})$. Taking the limit $\kappa \to \infty$ for a regular solution gives $T_\kappa(u) \to (\mathcal{S}_n(u,u), u)$ as $\kappa \to \infty$, which demonstrates that (25) holds if and only if $\mathcal{S}_n = 0 \Leftrightarrow$ (15) holds.
The time average of the energy budget equation (24) becomes

$$\tfrac{1}{\mathrm{Re}} < |\nabla u_{ls}|^2 > = < (f_{ls}, u_{ls}) > + < T_l > - < T_s >, \tag{26}$$

where the average operator $<\cdot>$ is defined as in [6]. Here, the forcing f_κ vanishes for $\kappa > \kappa_f$. The flux T_s vanishes for $s \to \infty$, too, because we consider skew-symmetric regularizations, see Eq. (25). Consequently, taking $l > \kappa_f$ and $s \to \infty$ in Eq. (26) gives

$$<T_l> \;=\; \tfrac{1}{\mathrm{Re}}<|\nabla u_{l\infty}|^2>, \tag{27}$$

which shows that $<T_\kappa)>$ is nonnegative for $\kappa > \kappa_f$. The difference $<T_l> - <T_s>$ is also nonnegative for $\kappa_f < l < s$; hence, $<T_\kappa>$ is monotone decreasing.

Furthermore, the difference of the two fluxes can be bounded with the help of the average dissipation rate $\epsilon = \frac{1}{\mathrm{Re}}<|\nabla u|^2>$:

$$\begin{aligned} <T_l> - <T_s> \;&=\; \tfrac{1}{\mathrm{Re}}<|\nabla u_{ls}|^2> \\ &\le\; \tfrac{1}{\mathrm{Re}}s^2<|u_{ls}|^2> \;\le\; \tfrac{1}{\mathrm{Re}}s^2<|u|^2> \;=\; \left(\frac{s}{\kappa_\tau}\right)^2\epsilon, \end{aligned} \tag{28}$$

where $\kappa_f < l < s$ and the Taylor wavenumber (of the regularized solution) is given by $\kappa_\tau = \sqrt{<|\nabla u|^2>/<|u|^2>}$.
On the other hand, for $s \to \infty$ and $l > \kappa_f$ we have

$$<T_l> \;=\; \tfrac{1}{\mathrm{Re}}<|\nabla u_{l\infty}|^2> \;=\; \epsilon - \tfrac{1}{\mathrm{Re}}<|\nabla u_{0l}|^2> \;\ge\; \left(1 - \left(\frac{l}{\kappa_\tau}\right)^2\right)\epsilon. \tag{29}$$

By reading this inequality from right to left we obtain an upper bound for the dissipation rate ϵ. Thus, with the help of (29), the right-hand side in (28) can be bounded in terms of the energy flux $<T_l>$. When the resulting inequality is divided by $<T_l>$ we get, for $\kappa_f < l < s$,

$$1 - \frac{s^2}{\kappa_\tau^2 - l^2} \;\le\; \frac{<T_s>}{<T_l>} \;\le\; 1. \tag{30}$$

So, in conclusion, for $\kappa_f < l \le s << \kappa_\tau$ we have $<T_s> \;\approx\; <T_l>$, that is the energy flux through the wavenumber κ is nearly constant (independent of Re and κ) for κ in the range $\kappa_f, \kappa << \kappa_\tau$. In other words, the energy cascade of any skew-symmetric regularization possesses an inertial subrange. The length of this range is controlled by the Taylor wavenumber of the regularized system (11).

Finally, it may be remarked that the energy flux in the inertial subrange is nearly equal to the dissipation rate ϵ. Indeed, Eq. (27) leads to $<T_l> \;\le\; \epsilon$. Combining this result with (29) gives $<T_l> \;\approx\; \epsilon$ (provided $l << \kappa_\tau$). These rigorous results show that the conditions prevailing in the inertial subrange of the energy cascade are strictly satisfied for $\kappa_f < \kappa << \kappa_\tau$. A simple dimensional analysis gives then Kolmogorov's -5/3 spectrum. Thus the leading part of the inertial subrange of any skew-symmetric regularization closely resembles the corresponding part of the energy cascade in a turbulent flow governed by the NS-equations.

10 Regularization parameter

Unfortunately, the mathematical basis of the regularization model does not provided a theorem which tells us how to take ϵ given the smallest scale of motion that we are able to solve numerically. Moreover, it is not very likely that such a theorem will be proven soon, because the underlying problem is directly related to the Clay Navier-Stokes Millennium Problem, i.e., requires a substantial unlocking of the secrets hidden in the NS-equations. Yet, we know from mathematical theory that the vorticity cascade stops at wavenumber κ_c (note: the concept 'wavenumber' is based on the eigenvalues of the projection of the diffusive operator $\mathcal{D}$ onto the space of divergence-free vector fields) if for all wavenumbers $\kappa \geq \kappa_c$ the right-hand side of Eq. (10) is nonpositive. In fact, the essentially difficulty is that the existence of such a κ_c cannot be proven. We can, however, determine the regularization parameter ϵ (for a given value of κ_c) such that that the right-hand side of the regularized vorticity equation is nonpositive:

$$(\mathcal{C}_n(\omega, u)_\kappa, \omega_\kappa) - \tfrac{1}{\mathrm{Re}}|\nabla \omega_\kappa|^2 \leq 0 \tag{31}$$

for all $\kappa \geq \kappa_c$. From a physical point of view, this requirement is based on the scenario of the energy/vorticity cascade as described in [17], e.g. Condition (31) excludes that the vorticity ω bursts driving the energy to small scales. In order to formulate (31) in physical space, we consider an arbitrary part Ω_Δ with diameter Δ of the flow domain, where $\kappa_c = \pi/\Delta$. In a numerical simulation Ω_Δ will typically be a grid cell. Furthermore, we suppose that Ω_Δ is a periodic box. The underlying reason for this assumption is that boundary terms resulting from integration by parts vanish. Poincaré's inequality states that there exists a constant C_Δ, depending only on Ω_Δ, such that for every function u

$$\int_{\Omega_\Delta} |u - \overline{u}_\Delta|^2 \, dx \ \leq \ C_\Delta \int_{\Omega_\Delta} |\omega|^2 \, dx, \tag{32}$$

where $\overline{u}_\Delta$ is the average value of u in Ω_Δ. The residual field $u - \overline{u}_\Delta$ contains eddies of size smaller than Δ. The regularization must keep them from becoming dynamically significant. Poincaré's inequality (32) shows that the $L^2(\Omega_\Delta)$ norm of the residual field is bounded by a constant (independent of u) times the $L^2(\Omega_\Delta)$ norm of ω. Consequently, we can confine the dynamically significant part of the motion to scales $\geq \Delta$ by controlling the $L^2(\Omega_\Delta)$ norm of the vorticity. According to Eq. (9) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\Delta} \tfrac{1}{2}|\omega|^2 \, dx \ = \ \int_{\Omega_\Delta} \left(\mathcal{C}(\omega, u) \cdot \omega - \tfrac{1}{\mathrm{Re}} |\nabla \omega|^2 \right) \, dx$$

cf. Eq.(10). In Ref. [18] it is proven that the vortex stretching term can be bounded in terms of the middle eigenvalue λ_2 of the deformation tensor $S(u)$. In particular, it is shown that

$$\int_{\Omega_\Delta} \mathcal{C}(\omega, u) \cdot \omega \, dx = \int_{\Omega_\Delta} \omega \cdot S(u)\omega \, dx \ \leq \ \max \lambda_2^+ \int_{\Omega_\Delta} |\omega|^2 \, dx,$$

where the maximum is to be taken over Ω_Δ and $\lambda_2^+ = \frac{1}{2}(\lambda_2 + |\lambda_2|)$. Consequently, the vortex-stretching term is dominated by the dissipative term, that is

$$\int_{\Omega_\Delta} \left(\mathcal{C}(\omega, u) \cdot \omega - \tfrac{1}{\mathrm{Re}}|\nabla\omega|^2\right) \, dx \;\leq\; 0$$

for $\kappa \geq \kappa_c$, if

$$\max \lambda_2^+ \;\leq\; \tfrac{1}{\mathrm{Re}}\,\kappa_c^2$$

If this condition is satisfied, the regularization need not be applied. Because the viscous damping at scale κ_c is then sufficient to stop the production of smaller scales. So in that case we can take $\epsilon = 0$. The condition above, however, is often not satisfied; hence the dynamics is to be regularized. The symmetry-preserving regularizations given by (12)-(14) reduce the vortex-stretching term according to Eqs. (18)-(20). In spectral space, the stretching term is reduced by the factor $f_n(\hat{g}_k, \hat{g}_p, \hat{g}_q)$ where $|k| = \kappa_c$ and $k = p + q$, see Section 7. Now, by supposing that the local interactions are dominant, we get the condition

$$f_n(\hat{g}_k, \hat{g}_k, \hat{g}_k)\; \lambda_2^+ \;\leq\; \tfrac{1}{\mathrm{Re}}\,\kappa_c^2 \tag{33}$$

for $|k| = \kappa_c$. For the $\mathcal{C}_4$-regularization, for example, the damping function is given by $f_4 = 3\hat{g}_k^2 - 2\hat{g}_k^3$. This function depends on the (spectral) characteristics of the filter as well as on the filter length ϵ. So, for a given filter, Eq (33) can be used to determine the filter length ϵ. For more details, see Ref. [19].

11 Numerical simulation

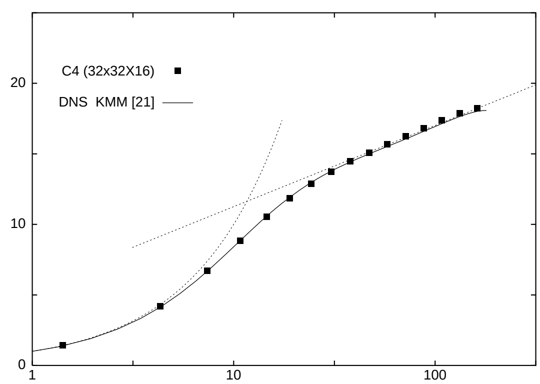
To actually benefit from the mathematical basis of regularization the discretization has to preserve the invariants (energy, enstrophy and helicity) too. In the past years, we have made substantial progress by discretizing differential operators in such a way that their fundamental conservation or dissipation properties are preserved [20]. In particular we have developed discretizations of the convective operator that preserve the skew-symmetry (15). The resulting discretizations differ essentially from the common approaches that minimize local truncation error. Concretely, we note that a symmetry-preserving discretization is stable on any grid. The close connection between the proposed regularization and the discretization method forms, of coarse, a decisive point in the choice of the numerical method. For instance, the proof of the existence of an inertial range with a -5/3 spectrum can be can be translated directly to the discrete system.

12 Turbulent channel flow

The regularization $\mathcal{C}_4$ has been tested for turbulent channel flow by means of a comparison with direct numerical simulations (DNS-data). In wall-coordinates the Reynolds number is given by $Re_\tau = 180$. This flow forms a prototype for near-wall turbulence: virtually every LES has been tested for it. We consider two, coarse, computational grids consisting of 16×16×8 and 32×32×16 grid points, respectively. Details about the numerics (grid-stretching, time-stepping, etc.) can be found in [20]. The results will be compared to the DNS data of Kim *et al.* [21]. The extend of the computational domain in the periodical directions is identical with that of the DNS in Ref. [21].

In the present test, we have applied a discrete, three-point filter in each spatial direction. In one direction the filter reads $(1 + 2c^2)\overline{u}_i = c^2 u_{i-1} + u_i + c^2 u_{i+1}$ where the parameter c is related to the filter length. The value of c is determined on the fly such that Eq. (33) holds, that is the minimum value of c satisfying (33) is taken.

At the considered grids we were not able to compute good results with the help of Leray's regularization. Yet, overall good agreement between the $\mathcal{C}_4$-calculation and the DNS is observed for both the first- and second-order statistics, see Fig. 1-2. The least



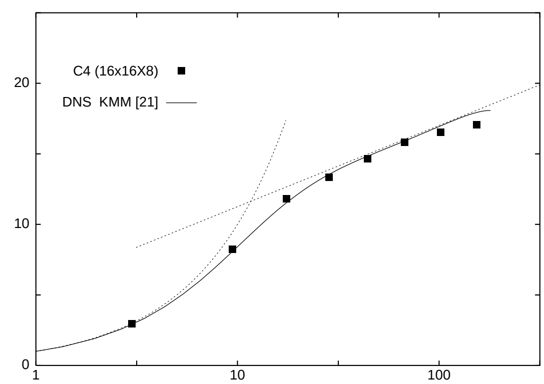


Figure 1: The mean velocity (in wall coordinates) for 32×32×16 gridpoints (left) and 16×16×8 gridpoints (right).

to be expected from numerical simulations of turbulence is a good prediction of the mean flow. Fig. 1 (right-hand picture) shows that $\mathcal{C}_4$ predicts the mean flow already at the very coarse 16×16×8 grid. Notice that the first discrete streamwise velocity lies at $y^+ \approx 3$. The next is located at $y^+ \approx 10$. The location of the other points can be observed too: each symbol in Fig. 1 corresponds to a point of the grid.

Our approach is based on the idea that the low modes of the solution u_ϵ of Eq. (11) approximate the corresponding low modes of the solution u of the Navier-Stokes equations, whereas the high modes of u_ϵ vanish faster than those of u. In order to investigate this basic idea, we consider the one-dimensional, streamwise energy spectra at $y^+ \approx 3$, *i.e.* at the first point of the 16×16×8 grid (counted from the wall) Fig. 2 (right-hand figure) displays a near-wall energy spectra. As can be seen, the energy spectrum of the solution of (11)+(13) follows the DNS for large scales of motion, whereas a much steeper (numerically speaking: more gentle) power law is found for smaller scales, which is precisely what a regularization is ought to do. More results can be found in [19], [22], [23], e.g.

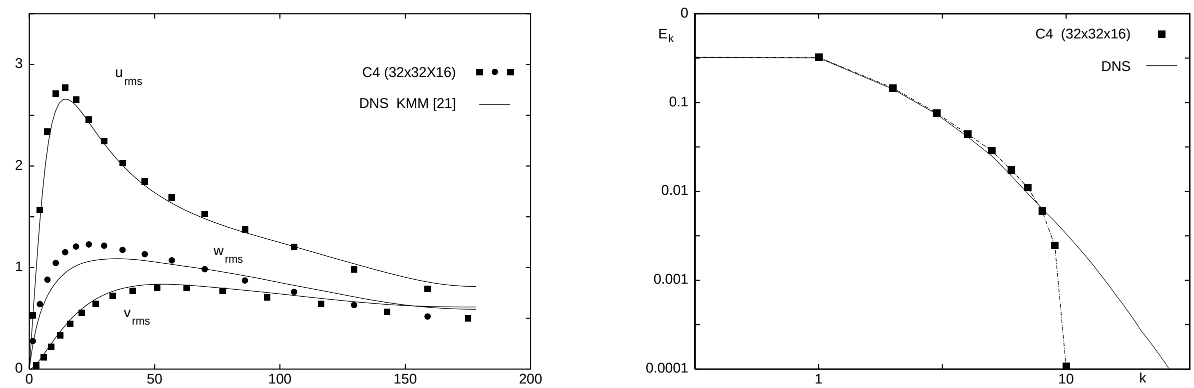


Figure 2: The left-hand figure displays the root-mean-square of the fluctuating velocities (32×32×16 gridpoints). The right-hand figure displays a near-wall energy spectrum (at $y^+ \approx 3$).

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