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REGULARIZATION MODELING OF COMMUTATOR-ERRORS IN LARGE-EDDY SIMULATION OF WALL-BOUNDED TURBULENCE

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Abstract. *Commutator errors arise in large-eddy simulation of incompressible turbulent flow from the application of non-uniform filters to the continuity – and Navier-Stokes equations. Control over the commutator errors compared to the turbulent stress fluxes can only be obtained by appropriately restricting the spatial variations of the filter-width and filter-skewness. For situations in which the dynamical consequences of the commutator errors are significant, e.g., near solid boundaries, explicit similarity modeling for the commutator errors is proposed, including Leray regularization. The performance of this commutator error parameterization is illustrated for the one-dimensional Burgers equation. The Leray approach is found to capture the filtered flow with higher accuracy than conventional similarity modeling, which is particularly relevant for large filter-width variations.*

1 INTRODUCTION

The extension of the large-eddy approach to spatially heterogeneous flows provides a natural motivation for the use of filters with space-dependent filter-widths. The application of such filters to the Navier-Stokes equations seriously complicates the turbulent closure problem, as both contributions from the turbulent stress tensor as well as from so-called commutator errors emerge. The former are related to spatial filtering of the non-linear convective terms, while the latter arise from interchanging the filtering operator and spatial derivatives. Although the commutator errors are of limited dynamical importance in many bulk turbulence configurations, their relevance in near-wall turbulence is significant, especially at high Reynolds number. We propose explicit closure models for the commutator-errors, derived from regularization principles for the Navier-Stokes equations. We consider in particular the Leray model for the commutator-error. The quality of the explicit commutator-error models is assessed for Burgers flow.

In large-eddy simulation (LES) of turbulence one aims to predict the primary features of an unsteady flow without explicitly resolving all dynamically relevant length-scales [1]. The modeling of turbulent flow in large-eddy simulation starts from the introduction of a spatial, low-pass filter with externally specified filter-width Δ. This allows one to locally distinguish flow-features with a length-scale larger than Δ from flow-features with length-scale smaller than Δ. In the large-eddy simulation context, the former are referred to as 'resolved' while the latter class of flow-structures is identified as 'subgrid' or 'sub-filter'. During a simulation, the time-dependent resolved scales are explicitly calculated while the dynamic effects of the subgrid scales on the evolution of the resolved scales are represented through the introduction of an explicit 'subgrid' model [2].

The desire to extend large-eddy simulation to flows in complex domains generally implies that one is confronted with strongly varying turbulence intensities within the flow-domain. In certain regions a nearly laminar flow may exist while a lively, fine-scale turbulent flow can be present simultaneously in other regions. As an example, one may think of flow over a backward facing step [3, 4] which displays very small turbulent boundary layer scales characteristic of a separated shear layer, while at other locations in the domain an unsteady but large-scale flow may be observed. Various other examples from nature and technology come to mind, which emphasize the need to incorporate heterogeneous smaller scales into the large-eddy approach in order to consistently address turbulent flows in complex situations.

In the filtering approach to large-eddy simulation, strongly non-uniform turbulence can be accommodated efficiently using a filter operator with non-uniform filter-width [5, 6]. The use of such filters, however, complicates the subgrid closure problem through the appearance of additional commutator errors [7]. These terms arise because non-uniform filtering does not commute with spatial differentiation. That is, $\overline{\partial_x u} \neq \partial_x \overline{u}$ where $\partial_x u$ denotes differentiation of the solution u with respect to x and the overline indicates the filter operation. In this paper we discuss the additional commutator error closure terms and propose and illustrate explicit similarity modeling and Leray regularization [8, 9]. These findings encourage the further extension of the large-eddy approach to turbulence under realistic flow conditions and in complex flow domains.

To provide an illustration of explicit dynamical consequences of commutator errors and their similarity models, the evolution of running 'ramp-cliff' waves in the viscous Burgers equation is studied in one dimension. It is shown that the 'Leray-regularized' formulation provides a better representation of the non-uniformly filtered velocity field than the extended Bardina similarity model for this situation. In particular, the Leray approach simultaneously captures turbulent stress fluxes as well as commutator errors without increasing computational costs compared to the traditional case of uniform filters. The extension toward non-uniformly filtered turbulent flow in three dimensions is the subject of ongoing research.

The organization of the paper is as follows. In section 2 we consider how commutator errors arise from applying non-uniform spatial filters to the Navier-Stokes equations. For cases in which the variations in the properties of the filter are sufficiently abrupt, explicit modeling of the dynamic effects of the commutator errors is required. Similarity and Leray models are introduced in section 3 and their performance for the Burgers equation is illustrated numerically in section 4. Finally, concluding remarks are collected in section 5.

2 COMMUTATOR ERRORS IN THE FILTERING APPROACH TO LES

This section introduces non-uniform filters with compact support and applies them to the equations governing incompressible flow. The application of a non-uniform filter generates turbulent stress fluxes as well as commutator errors. These closure terms will be written as the commutator bracket of the filter operator and either the product operator, or the derivative operator, establishing its shared algebraic properties with the Poisson-bracket in classical mechanics [15, 16]. The filtered velocity field is shown to acquire a non-zero divergence as a consequence of spatial variations in the filter properties. Finally, the effects of the commutator errors are described in terms of their contributions to the kinetic energy evolution. This description expresses the additional energy-transfer and interaction mechanisms associated with non-uniform filtering.

A general compact-support filter, whose application in one spatial dimension is denoted by ℓ, can be written as:

$$\overline{u}(x,t) = \ell(u)(x,t) = \int_{x-\Delta_-(x,t)}^{x+\Delta_+(x,t)} \frac{H(x,\xi,t)}{\Delta(x,t)} u(\xi,t)\; d\xi \qquad (1)$$

where $H(x,\xi,t)$ is the 'characteristic' filter function and $\Delta_\pm \geq 0$ denote the upper – and lower bounding functions which define the filter-width $\Delta = \Delta_+ + \Delta_-$. The filter ℓ is assumed to be normalized, i.e., $\ell(1) = 1$. This class of filters can readily be extended to product-filters in three spatial dimensions by defining the composition $L = \ell_1 \circ \ell_2 \circ \ell_3$ where ℓ_j with $j = 1, 2, 3$, represents filtering in the x_j-direction only, as in (1). The application of such filters gives rise to a number of additional closure terms, to which we turn next.

As is well-known, incompressible flow is governed by the principles of conservation of mass and momentum. These can be expressed in terms of the continuity equation and Navier-Stokes equations as

$$\begin{aligned} &\partial_j u_j = 0 \\ &\partial_t u_i + \partial_j(u_j u_i) + \partial_i p - \frac{1}{Re}\partial_{jj} u_i = 0 \quad ; \quad i = 1, 2, 3 \end{aligned} \qquad (2)$$

where u_j is the component of the velocity $\mathbf{u}$ in the x_j-direction, t denotes time and ∂_t, ∂_j are the partial derivative operators with respect to t and x_j respectively. Moreover, p is the pressure and $Re = (u_r \lambda_r)/\nu_r$ denotes the Reynolds number in terms of reference velocity u_r, length-scale λ_r and kinematic viscosity ν_r [17]. Throughout, the summation convention is adopted, implying summation over repeated indices.

If one applies the filter L to the incompressible flow equations, commutator errors may arise, e.g., if $\overline{\partial_x f} - \partial_x \overline{f} = L(\partial_x f) - \partial_x(L(f)) = [L, \partial_x](f) \neq 0$. Here, the commutator error is written in terms of the commutator bracket $[L, \partial_x]$ of L and the derivative operator ∂_x. One may show that $[L, \partial_j](f) = 0$ for $j = 1, 2, 3$, if and only if the filter L is a convolution filter, which, by definition is spatially uniform. As for the continuity equation we may formally write

$$\partial_j \overline{u}_j = -[L, \partial_j](u_j) \tag{3}$$

Hence, the divergence of the non-uniformly filtered velocity differs from zero, i.e., $\overline{u}_j$ is not solenoidal, and the corresponding continuity equation is no longer in local conservation form. The term on the right-hand side corresponds to apparent local creation and annihilation of 'resolved' mass as a consequence of variations in $\Delta_\pm$ and H.

Likewise, filtering the Navier-Stokes equations yields the following system of equations:

$$\begin{aligned} \partial_t \overline{u}_i + \partial_j(\overline{u}_j \overline{u}_i) &+ \partial_i \overline{p} - \frac{1}{Re} \partial_{jj} \overline{u}_i = -\Big\{ [L, \partial_t](u_i) \\ &+ \partial_j([L, S](u_i, u_j)) + [L, \partial_j](S(u_i, u_j)) \\ &+ [L, \partial_i](p) - \frac{1}{Re}[L, \partial_{jj}](u_i) \Big\} \end{aligned} \tag{4}$$

We observe that commutator brackets emerge involving the filter L and the product operator $S(f, g) = fg$, as well as commutator brackets of L and first – or second order partial differentiation. Filtering a linear term such as $\partial_t u_i$ gives rise to a 'mean-flow' term $\partial_t \overline{u}_i$ and a corresponding commutator error $[L, \partial_t](u_i)$. Filtering the nonlinear convective terms leads to two different types of closure terms. First, as in the case of uniform filtering, the divergence of the turbulent stress tensor $\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j = [L, S](u_i, u_j)$ arises. The divergence of τ_{ij} will be referred to as the turbulent stress flux. Second, an associated commutator error $[L, \partial_j](S(u_i, u_j))$ emerges from filtering the convective fluxes. The local conservation form of the Navier-Stokes equations is no longer maintained as a result of the non-uniform filtering, similar to what was observed in (3) for the continuity equation.

Spatial filtering of the incompressible flow equations gives rise to an 'LES-template' [1] in which the 'Navier-Stokes' operator on the left hand side of (4) acts on the filtered solution $\{\overline{u}_i, \overline{p}\}$. In addition, several unclosed terms arise. Of these, only the parameterization of the turbulent stress fluxes $\partial_j([L, S](u_i, u_j)) = \partial_j \tau_{ij}$ has attracted much attention in the literature. However, the subgrid modeling problem associated with non-convolution filters entails various additional commutator errors. These terms require explicit modeling in case the spatial and temporal properties of the filter are sufficiently variable.

The effects of the commutator errors can be effectively quantified by considering the turbulent kinetic energy equation. Multiplying (4) by $\overline{u}_i$ and summing over i yields after some rewriting

$$\begin{aligned} \partial_t(k) + \partial_j(\overline{u}_j k) &= \frac{1}{Re} \overline{u}_i \partial_{jj} \overline{u}_i - \overline{u}_i \partial_j \tau_{ij} - k \partial_j \overline{u}_j - \overline{u}_i \partial_i \overline{p} \\ -\overline{u}_i[L, \partial_t](u_i) &- \overline{u}_i[L, \partial_j](u_i u_j) - \overline{u}_i[L, \partial_i](p) + \frac{1}{Re} \overline{u}_i [L, \partial_{jj}](u_i) \end{aligned} \tag{5}$$

where $k = \overline{u}_i\overline{u}_i/2$ and we have used the identity $\partial_j(\overline{u}_j\overline{u}_i\overline{u}_i) = 2\overline{u}_i\partial_j(\overline{u}_j\overline{u}_i) - \overline{u}_i\overline{u}_i\partial_j\overline{u}_j$. On the right-hand side of (5) one identifies contributions due to the viscous terms and the turbulent stresses. Moreover, since $\partial_j\overline{u}_j \neq 0$ a specific commutator error contribution arises from the continuity equation (3) in addition to a pressure related term. The last four terms on the right-hand side of (5) represent effects of commutator errors in the momentum equations (4).

The resolved kinetic energy in a flow domain Ω of size $|\Omega|$ is defined as

$$E = \frac{1}{|\Omega|}\int_\Omega d\mathbf{x}\ \frac{1}{2}\overline{u}_i\overline{u}_i \tag{6}$$

In a flow domain with periodic boundary conditions, the evolution of E can be written in terms of the commutator error contributions as

$$\begin{aligned} |\Omega|\frac{dE}{dt} &= -\int_\Omega d\mathbf{x}\ \Big(\frac{1}{Re}\partial_j\overline{u}_i\partial_j\overline{u}_i - \tau_{ij}\partial_j\overline{u}_i\Big) - \int_\Omega d\mathbf{x}\ (\overline{p}-k)[L,\partial_j](u_j) \\ &- \int_\Omega d\mathbf{x}\ \overline{u}_i\Big([L,\partial_t](u_i) + [L,\partial_i](p)\Big) - \int_\Omega d\mathbf{x}\ \overline{u}_i[L,\partial_j](u_iu_j) \\ &+ \frac{1}{Re}\int_\Omega d\mathbf{x}\ \overline{u}_i[L,\partial_{jj}](u_i) \end{aligned} \tag{7}$$

after some partial integrations. One observes the usual dissipation of kinetic energy E due to the viscous terms, as well as the transport term involving the subgrid stress tensor: $\tau_{ij}\partial_j\overline{u}_i$. Moreover, one notices a contribution arising from the fact that the filtered velocity field is no longer solenoidal involving $(\overline{p}-k)[L,\partial_j](u_j)$. Finally, four terms emerge characterizing the effects of the commutator errors in the momentum equations. The magnitude of the various terms and commutator errors can be quantified by explicitly evaluating the different integrals during a simulation or by post-processing direct numerical simulation data-bases (see also [11]).

3 Similarity and regularization modeling of commutator errors

This section considers explicit modeling of the commutator errors. We include similarity modeling and Leray regularization. Specifically, we will extend Bardina's approach [12] to include commutator errors and derive the implied subgrid models for the turbulent stresses and the commutator error, arising from non-uniform Leray regularization [8, 9].

Bardina's similarity model for the turbulent stress tensor arises by applying the definition of $\tau_{ij} = [L,S](u_i,u_j)$ to the filtered solution $\overline{u}_i$, i.e.,

$$\tau_{ij} \to m^B_{ij} = [L,S](\overline{u}_i,\overline{u}_j) = \overline{\overline{u}_i\overline{u}_j} - \overline{\overline{u}}_i\overline{\overline{u}}_j \tag{8}$$

Extending this idea to the commutator error suggests the following parameterization:

$$[L,\partial_j](u_iu_j) \to [L,\partial_j](\overline{u}_i\overline{u}_j) \tag{9}$$

In a turbulent boundary layer flow the model contributions (8) and (9), were shown to provide a high correlation with the actual turbulent stress tensor and commutator errors respectively [6]. The extended Bardina model may be motivated in an alternative manner. Instead of distinguishing separate closure problems for the commutator error and the turbulent stress fluxes, we may consider the full non-uniformly filtered convective flux, i.e., $\overline{\partial_j(u_iu_j)} - \partial_j(\overline{u}_i\overline{u}_j)$ [18]. One may verify that

$$\begin{aligned}\overline{\partial_j(u_iu_j)} &= \partial_j(\overline{u}_i\overline{u}_j) + \{\overline{\partial_j(u_iu_j)} - \partial_j(\overline{u}_i\overline{u}_j)\} \\ &= \partial_j(\overline{u}_i\overline{u}_j) + [L, \partial_j \circ S]\,(u_i, u_j) \end{aligned} \qquad (10)$$

The similarity closure arises as before, i.e., $[L, \partial_j \circ S](u_i, u_j) \rightarrow [L, \partial_j \circ S](\overline{u}_i, \overline{u}_j)$, where:

$$[L, \partial_j \circ S](\overline{u}_i, \overline{u}_j) = [L, \partial_j] \circ S(\overline{u}_i, \overline{u}_j) + \partial_j \circ [L, S]\,(\overline{u}_i, \overline{u}_j) \qquad (11)$$

In rewriting this model, use was made of the Leibniz rule, applied to the operators L, ∂_j and S. The similarity modeling of the separate closure problems for the commutator error, cf. (9), and the turbulent stress fluxes, cf. (8), is hence re-obtained directly from the similarity modeling of the full convective-flux. By adopting the same modeling assumptions for both the turbulent stress tensor and the commutator error the combined model can be implemented at reduced computational cost.

Recently, the Leray regularization principle [8] was revisited in the context of large-eddy simulation [9]. In this approach the convective fluxes $u_j\partial_j u_i$ are replaced by $\overline{u}_j\partial_j u_i$, i.e., the solution $\mathbf{u}$ is convected with a smoothed velocity $\overline{\mathbf{u}}$. The governing Leray equations for incompressible flow are given by

$$\partial_j u_j = 0 \quad ; \quad \partial_t u_i + \overline{u}_j\partial_j u_i + \partial_i p - \frac{1}{Re}\partial_{jj}u_i = 0 \qquad (12)$$

This formulation can be written in terms of $\{\overline{u}_i, \overline{p}\}$ by assuming the existence of a (formal) inverse L^{-1} of L, i.e., $u_j = L^{-1}(\overline{u}_j)$. After some calculation, one obtains the filtered momentum equation as

$$\begin{aligned}\partial_t\overline{u}_i &+ \partial_j(\overline{u}_i\overline{u}_j) + \partial_i\overline{p} - \frac{1}{Re}\partial_{jj}\overline{u}_i = -\Big([L, \partial_t](u_i) \\ &+ \Big\{\partial_j(m^L_{ij}) + \overline{u_i\partial_j\overline{u}_j}\Big\} + [L, \partial_j](S(u_i, \overline{u}_j)) \\ &+ [L, \partial_i](p) - \frac{1}{Re}[L, \partial_{jj}](u_i)\Big) \end{aligned} \qquad (13)$$

The divergence of the turbulent stress tensor $\partial_j\tau_{ij}$ in (4) is represented in terms of the Leray model $m^L_{ij} = \overline{\overline{u}_j u_i} - \overline{u}_j\overline{u}_i$ and an additional term associated with the divergence of the filtered velocity field:

$$\partial_j\tau_{ij} \rightarrow \partial_j(m^L_{ij}) + \overline{u_i\partial_j\overline{u}_j} \qquad (14)$$

in which the commutator error is expressed as $[L, \partial_j](u_i u_j) \to [L, \partial_j](\overline{u}_j u_i)$. The other commutator errors are identical to those in (4) with the understanding that in actual simulations every occurrence of an unfiltered flow-variable implies the application of L^{-1} to the smoothed field. The Leray model is known to provide good predictions of three-dimensional turbulent mixing at arbitrarily high Reynolds number when a uniform filter is used [9].

4 Commutator-error dynamics in Burgers flow

In this section we compare the extended Bardina and Leray models for the commutator-error in a simple but illustrative situation by considering 'ramp-cliff' solutions of the viscous Burgers equation running across a region of strong filter non-uniformity. While the dynamics of the viscous Burgers equation is clearly different from that of the Navier-Stokes equations, both models for fluid flow are quadratically nonlinear and exhibit the same commutator errors, except for the pressure term. First, we describe the numerical method, then some *a priori* analysis is given to establish the magnitude of the commutator errors for specific non-uniformities in filter width and finally we illustrate the performance of the explicit similarity and regularization models for the commutator errors.

We consider the one-dimensional viscous Burgers equation

$$\partial_t u + \partial_x(\frac{1}{2}u^2) - \frac{1}{Re}\partial_{xx}u = 0 \tag{15}$$

and adopt $Re = 500$. The numerical treatment of this equation follows the standard method of lines [1]. Introducing $u_j(t) = u(x_j, t)$ as the discrete solution at location x_j at time t, the method of lines implies the semi-discretization

$$\frac{du_j}{dt} + f_j = 0 \quad ; \quad u_j(0) = u_0(x_j) \tag{16}$$

where f_j denotes the numerical flux at x_j and u_0 the initial condition. We adopt periodic boundary conditions and obtain the numerical flux by second order accurate finite differencing on a non-uniform grid. The approximation of the convective term $(1/2)\partial_x(u^2)$ follows from

$$\partial_x(u^2) \to \delta_x(u^2)_j = \frac{u_{j+1}^2 - u_{j-1}^2}{x_{j+1} - x_{j-1}} \tag{17}$$

and the approximation of the viscous term $\partial_{xx}u$ results from

$$\partial_{xx}u \to \delta_{xx}(u)_j = a_{j,j+1}u_{j+1} - a_{j,j}u_j + a_{j,j-1}u_{j-1} \tag{18}$$

where

$$\begin{aligned} a_{j,j+1} &= \frac{2}{(x_{j+1}-x_j)(x_{j+1}-x_{j-1})} \\ a_{j,j} &= \frac{2}{(x_{j+1}-x_j)(x_j-x_{j-1})} \\ a_{j,j-1} &= \frac{2}{(x_j-x_{j-1})(x_{j+1}-x_{j-1})} \end{aligned} \tag{19}$$

On a uniform grid with grid-spacing h these weights reduce to $a_{j,j-1} = a_{j,j+1} = 1/h^2$ and $a_{j,j} = 2/h^2$ which is recognized as the second order accurate finite difference scheme for the second order derivative.

The system of ordinary differential equations in (16) is integrated in time using explicit time-integration, restricted by stability time-steps. In fact, we use time-steps associated with local stability that follow from the Courant-Friedrich-Lewy condition (CFL). This implies

$$\Delta t = \min_j(\Delta t_j) \quad ; \quad \Delta t_j = \Gamma \frac{\Delta x_j}{|u_j|} \quad ; \quad \Delta x_j = \frac{x_{j+1}-x_{j-1}}{2} \tag{20}$$

where the CFL-number Γ is chosen consistent with the stability requirements of the adopted time-integration method. Use was made of either the explicit Euler forward method at a low value $\Gamma = 0.1$ or the compact storage, four-stage explicit Runge-Kutta method with $\Gamma = 1.5$ [1].

The implementation of the Bardina model follows the standard LES-template (4), applied to the one-dimensional Burgers equation. For the Leray model we used (12) as the basis for the implementation. Since the numerical illustration is in one spatial dimension only, computational resources do not represent a limiting factor. Typically, we show results in which the Burgers equation is discretized using grids with $N = 2048$ intervals. This is more than adequate for obtaining the nearly grid-independent, unsteady solution, as was verified independently by comparing results on different grids. An example of a developing ramp-cliff solution is shown in figure 1. The initial velocity profile consists of a small positive value to which a Gaussian profile is added. In the examples shown, this Gaussian profile is centered around $x = -5$ and its width was selected as 5.

The explicit filtering required in the Bardina and Leray models, or when filtering direct numerical simulation data, was implemented using the trapezoidal rule for evaluating the top-hat filter. Specifically, we define

$$\overline{u}_j(t) = \frac{1}{x_{j+n_1}-x_{j-n_2}} \int_{x_{j-n_2}}^{x_{j+n_1}} u(\xi,t)d\xi \tag{21}$$

which covers $n_1 + n_2$ grid-cells. In terms of the grid, a non-uniform filter-width $\Delta_j = x_{j+n_1} - x_{j-n_2}$ is obtained together with a normalized skewness

$$\sigma_j = \frac{x_{j+n_1} - 2x_j + x_{j-n_2}}{x_{j+n_1} - x_{j-n_2}} \tag{22}$$

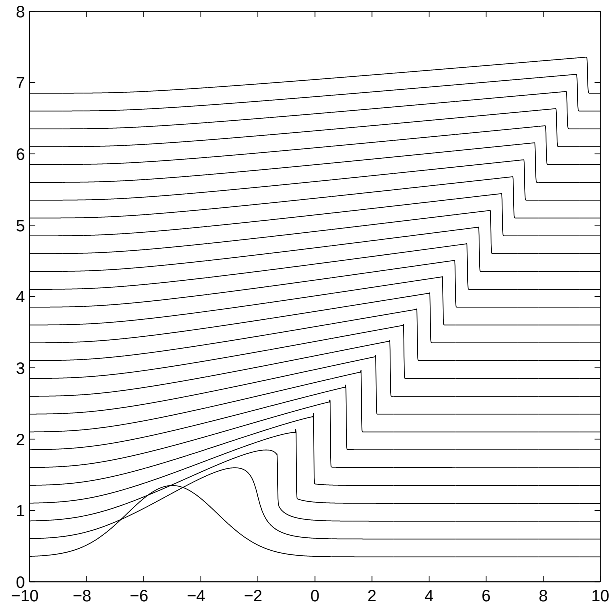

Figure 1: Solution to the Burgers equation initiated with a Gaussian profile, developing into a ramp-cliff structure at $Re = 500$. The solution is shown at times $t = 0, 1, 2, \ldots$.

The integration over ξ in (21) is evaluated using the composite trapezoidal rule. This rule is sufficiently accurate, because in one dimension we can allow for large numbers of intervals, i.e., take n_1 and n_2 large enough to imply negligible discretization errors. This setting allows one to approximate the grid-independent large-eddy solution corresponding to a fixed filter-width distribution.

We consider a non-uniform grid with grid-spacing $h_i = (\ell/N)(1 + g_i)$ where ℓ is the length of the domain, which was set equal to 20 in our simulations. The grid is non-uniform around $i = N/2$ with

$$g_i = A \sin\Big(2\pi \frac{(i - N/2)}{(N(m - 2q)/m)}\Big) \quad ; \quad \frac{qN}{m} \leq i \leq \frac{(m - q)N}{m}$$

and 0 otherwise. We use $q = 3$, $m = 8$ and $N = 2^n$ with n sufficiently large. The parameter $A < 1$ controls the ratio between largest and smallest intervals $(1 + A)/(1 - A)$.

In figure 2 we collected the contributions to the total convective flux for a representative uniform and non-uniform case. The solution and the filtered solution both display the 'ramp-cliff' structure. The total flux in figure 2(a) is piecewise linear and the turbulent stress flux is localized in the cliff-region. In figure 2(b) the filter-width non-uniformity strongly influences the mean flux on the 'ramp side' near $x = -3$. The commutator error compensates for this such that the total flux remains nearly linear in x. The commutator error and the turbulent stress flux have comparable magnitude in the filtered cliff-region $2 \lesssim x \lesssim 4$, in which these contributions are seen to partially counteract each other.

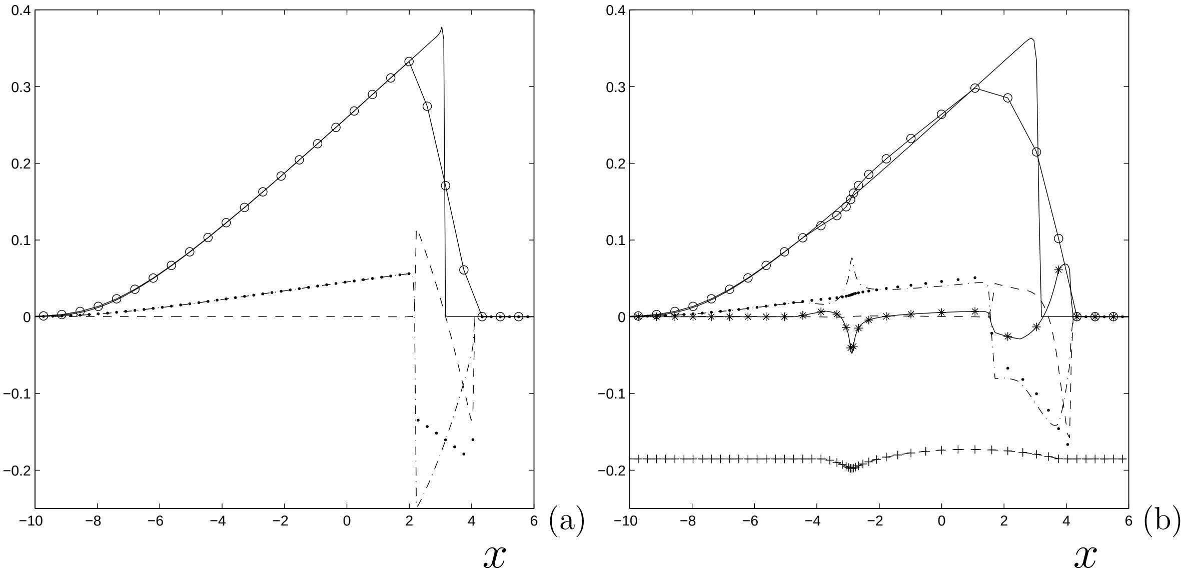

Figure 2: Snapshot of the solution (multiplied by 1/2) (solid) and filtered solution (solid; markers o) obtained at $N = 2048$. Convective flux: total (dots), mean (dash-dotted), turbulent stress (dashed), commutator error (solid with $*$). In (a) we use $\Delta = \ell/16$, i.e., computational filter-width $n_1 = n_2 = 64$ and in (b) the non-uniform case with $A = 1/2$ and the same values of n_1 and n_2 is shown. Underneath in (b), the grid-spacing (minus 0.2) as a function of x is presented.

In figure 3(a) we show the locations of the front and back of the ramp-cliff solution as a function of time. These locations are defined to be where $|u|$ equals $\max(|u|)/20$. Upon comparing filtered Burgers results with predictions from the Leray and Bardina parameterizations, one finds the Leray results are more accurate. The L_2-norm of the fluxes show that the commutator error is about 1/3-1/2 the value of the turbulent stress flux in this case. The Leray model also preserves better the qualitative properties of the filtered Burgers solution cf. figure 3(b). The Bardina parameterization creates additional variations in the solution, which do not correspond to the physics of the filtered Burgers equation in this rather extreme non-uniform case.

5 Concluding remarks

In this paper the commutator errors associated with non-uniform filtering in large-eddy simulation were studied. For a general class of non-uniform filter operators the filtered, incompressible Navier-Stokes equations were derived and all closure terms were identified. Besides the turbulent stress contributions, commutator errors were shown to arise. An independent control over the commutator errors cannot be obtained through the application of a general high-order filter.

A more detailed analysis of the commutator errors and turbulent stress fluxes shows that the commutator errors may be reduced in size by explicitly restricting the variations in the filter-width and skewness of the filter. This suggests employing only gradually

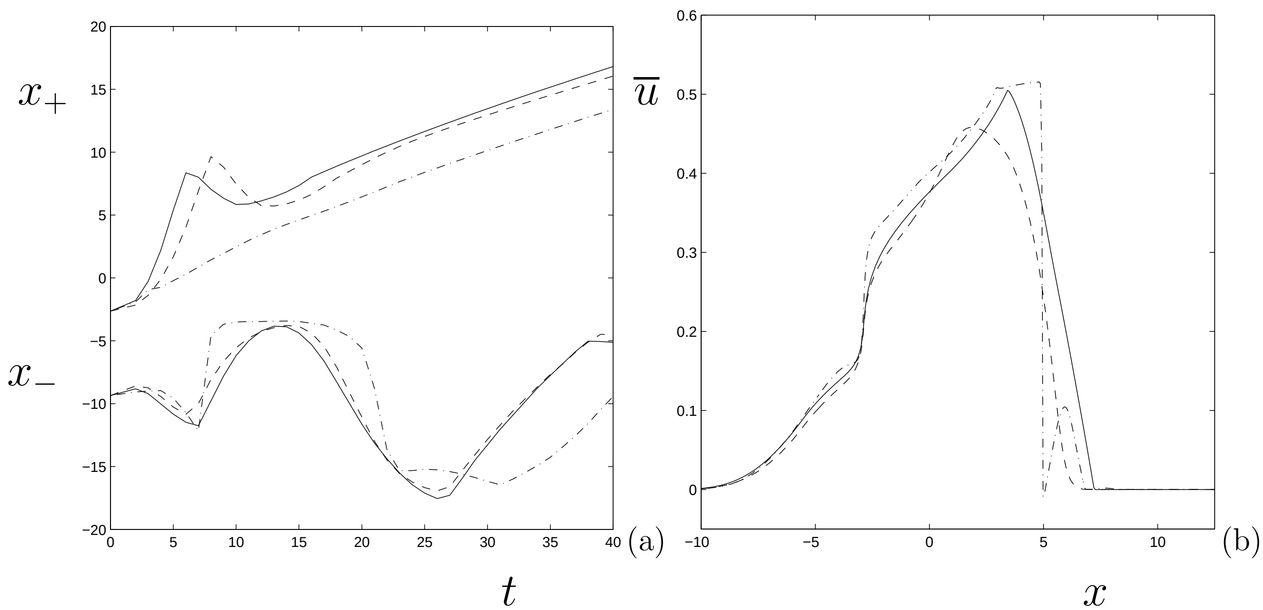

Figure 3: Location of the head x_+ of the cliff (upper curves) and the tail x_- of the ramp (lower curves) in (a) and in (b) snapshot of the filtered solution: filtered Burgers (solid), Leray (dashed) and Bardina (dash-dotted) for $A = 0.85$ and $n_1 = n_2 = 64$ at $N = 2048$.

varying filter-widths in complex geometries, from the point of view of avoiding explicitly modeling of the commutator errors. In view of maintaining appropriate efficiency in large-eddy simulations of turbulent flows in/around complex geometries it may, however, be required to allow for sharp variations in the filter. In such cases the dynamic importance of the commutator errors summons an explicit parameterization of the commutator errors.

At sufficiently large filter non-uniformities explicit modeling will become necessary. An extension of the similarity approach was formulated and compared with the Leray regularization approach. The Leray parameterization captures both the flux due to the turbulent stresses and the commutator errors in one model. Consequently, it combines computational efficiency with high accuracy. This result motivates the use of the Leray model in complex flows and it stimulates the study of more general regularization approaches for the closure of commutator errors in large-eddy simulation. As a first illustration, the prediction of the solution to the non-uniformly filtered Burgers equation was studied and the Leray approach was found to provide higher accuracy than the full similarity modeling. The extension to turbulent flow in three dimensions in a turbulent channel and over a backward facing step are presently being considered and will be presented elsewhere.

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