V European Conference on Computational Fluid Dynamics
ECCOMAS CFD 2010
J. C. F. Pereira and A. Sequeira (Eds)
Lisbon, Portugal,14-17 June 2010

A FULLY EULERIAN FORMULATION FOR FLUID-STRUCTURE-INTERACTION PROBLEMS WITH LARGE DEFORMATIONS AND FREE STRUCTURE MOVEMENT

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Key words: Fluid Structure Interaction, Monolithic Formulation, Eulerian Coordinates, Finite Elements, Adaptivity

Abstract. *In this work we present a monolithic model for fluid-structure-interaction problems formulated in Eulerian coordinates. This approach differs from the classical Arbitrary Lagrangien Eulerian (ALE) coordinates, where the flow domain is implicitely transformed to a reference domain.*

We formulate both problem, Structure and Fluid in an Eulerian framework. Thus, while the flow problem is given in natural coordinates the structure problem needs to be transformed. This transformation however is what we call a natural transformation, since it is given by the deformation of the structure itself. A degeneration of the flow-domain mapping is usually the cause for a breakdown of discretization schemes and solution methods of fluid-structure-interaction in ALE coordinates. The natural structure transformation is always well defined.

Our approach is able to handle very large deformation, free movement of the structure in the fluid and also contact of the structure with the boundary of the domain or the structure itself. This comes at the price of higher computational effort and additional discretization errors since the interface is only tracked implicitely by the solution itself.

1 Introduction

We consider fluid-structure-interaction problems of the following type: let $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ be an open domain which is partitioned into $\Omega = \hat{\Omega}_f \cup \hat{\Omega}_s$. Here, by $\hat{\Omega}_f$ we denote the flow-domain and by $\hat{\Omega}_s$ the solid-domain, where an elastic structure is given. The problem is driven either by volume forces on fluid or solid or by boundary conditions (e.g. an inflow-condition for the fluid). The dynamics of the coupled fluid-structure-interaction problem is governed by an equilibrium of forces on the interface $\hat{\Gamma}_i := \hat{\Omega}_f \cap \hat{\Omega}_s$. A typical example can be the flow around an elastic obstacle as shown in Figure 1. Forces of the evolving flow lead to a deformation of the structure $\hat{\Omega}_s \mapsto \Omega_s(t)$ and thus also change

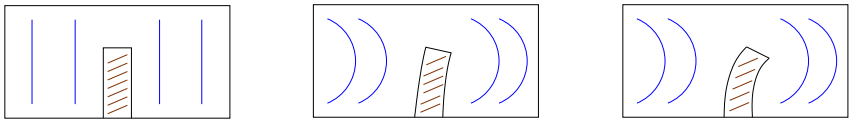

Figure 1: Flow around an elastic obstacle.

the flow domain $\hat{\Omega}_f \mapsto \Omega_f(t)$. Here, we only consider problems, where the common fluid-structure domain does not change: $\Omega = \Omega_f(t) \cup \Omega_s(t)$ for all $t \geq 0$.

The problem of modeling fluid-structure-interaction problems is due to the two different coordinate systems usually considered for describing flow and structure problems. For flow problems, the Eulerian coordinates with the focus of interest on a fixed point $x \in \Omega_f$ in space are natural. In this point $x \in \Omega_f$, the flow is modeled by describing the velocity $v(t,x)$ and pressure $p(t,x)$. We do not follow the trajectories $\{x(t)\} \subset \Omega_f(t)$ of particles over time t. This would be natural for describing structure problems: for every particle $\hat{x} \in \hat{\Omega}_s$, we model the deformation $\hat{u}(t,\hat{x})$ to get the deformed position $x(t) = \hat{x} + \hat{u}(t,\hat{x})$. The speed of deformation $\hat{v} = d_t\hat{u}$ is of lesser interest. These different viewpoints stem from the observation that we call a flow-problem stationary if the velocity in a certain point does not change with time $\partial_t v = 0$, while we call a structure-problem stationary if the velocity is zero $\hat{v} = d_t\hat{u} = 0$ for all particles $\hat{x} \in \hat{\Omega}_s$.

So called *partitioned approaches* for fluid-structure-interaction treat both problems decoupled in their own natural coordinate system. We refer to Bungartz and Schäfer[4] for various examples on these approaches. One of the big drawbacks of partitioned approaches is their demand for very small time-steps due to the explicit character. Partitioned approaches do not include the interface as part of the problem description. There exists no closed variational formulation for the coupled fluid-structure-interaction problem. This however is inevitable for rigorous a posteriori error estimation as well as for gradient-based optimization schemes and for implicit solution schemes allowing for large time-steps.

To allow for a *monolithic* formulation of fluid-structure-interaction problems, both systems need to be given in the same coordinate system. In the *Arbitrary Lagrangian Eulerian* formulation (ALE) both problems are described in the reference domains $\hat{\Omega}_s$ and $\hat{\Omega}_f$. Therefor, the flow-domain needs to be *mapped* onto the current domain via the ALE-transformation $\hat{T}_f : \hat{\Omega}_f \mapsto \Omega_f(t)$. This arbitrary mapping (arbitrary, since there is no natural deformation given inside the flow-domain) can be understood as a transformation of the computational mesh, see Figure 2. Details on the ALE-formulation are given in Bungartz and Schäfer[4].

In this paper we present a different monolithic formulation, called the *Fully Eulerian Coordinates*, where both sub-problems are given in Eulerian coordinates, on the domains $\Omega_f(t)$ and $\Omega_s(t)$. Now, the structure-problems needs to be described via a transformation $\hat{T}_s : \hat{\Omega}_s \mapsto \Omega_s(t)$. This transformation however is *natural* in the sense, that it is given by the physical variable $\hat{u}$, the deformation itself.

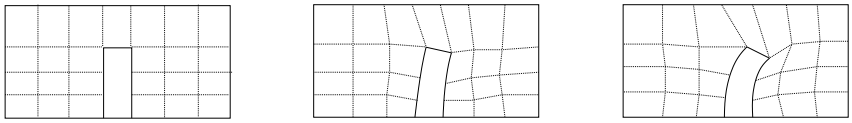

Figure 2: Implicit mesh transformation using ALE formulation of fsi-problems.

2 Eulerian Formulation for Fluid-Structure-Interaction Problems

In this section we present the Eulerian monolithic formulation for fluid-structure-interaction problems. This Eulerian formulation has first been proposed by Dunne[5] and has then been simplified by Richter and Wick[13]. For a complete discussion and derivation of the governing equations see Rannacher and Richter[11].

Here, we present the fluid-structure interaction problem in a form that allows for very large deformation of the structure and for free movement of the structure in the flow domain. Especially, the solid domain does not need to have contact to the domain's boundary $\partial\Omega$.

2.1 Notation

We begin with introducing some notation which will be used throughout this paper. By $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$), we denote the domain of the FSI problem. The domain Ω is supposed to be *time independent* but to consist of two possibly time-dependent subdomains, the fluid domain $\Omega_f(t)$ and the structure domain $\Omega_s(t)$. Unless needed, the explicit time dependency will be skipped in this notation. The boundaries of Ω, Ω_f, and Ω_s are denote by $\partial\Omega$, $\partial\Omega_f$, and $\partial\Omega_s$, respectively. The common interface between Ω_f and Ω_s is $\Gamma_i(t)$, or simply Γ_i. At time $t = 0$ we denote the *reference domains* by $\hat{\Omega}_s := \Omega_s(0)$ and $\hat{\Omega}_f := \Omega_f(0)$. Likewise, when a problem is formulated in Lagrangian coordinates on the reference partitioning we indicate this (for domains, coordinates, values and operators) by a "hat".

Partial derivatives of a function f with respect to the i-th coordinate are denoted by $\partial_i f$, and the total time-derivative by $d_t f$. The divergence of a vector and tensor is written as $\operatorname{div} f = \sum_i \partial_i f_i$ and $(\operatorname{div} F)_i = \sum_j \partial_j F_{ij}$. The gradient of a vector valued function v is the tensor with components $(\nabla v)_{ij} = \partial_j v_i$.

For a Lebesgue measurable set X, we denote by $L^2(X)$ the Lebesgue space of square-integrable functions on X equipped with the usual inner product and norm

$$(f,g)_X := \int_X fg\, dx, \quad \|f\|_X^2 = (f,f)_X,$$

respectively, and correspondingly for vector- and matrix-valued functions. Mostly the domain X will be Ω, in which case we will skip the domain index in products and norms. For Ω_f and Ω_s, we similarly indicate the associated spaces, products, and norms by a corresponding index "f" or "s".

We will generally use Roman letters, V, for denoting spaces of functions depending only on spatial variables and calligraphic letters, $\mathcal{V}$, for spaces of functions depending additionally on time. Let $L_X := L^2(X)$ and $L^0_X := L^2(X)/\mathbb{R}$. The functions in L_X (with $X = \Omega$, $X = \Omega_f(t)$, or $X = \Omega_s(t)$) with first-order distributional derivatives in L_X make up the Sobolev space $H^1(X)$. Further, $H^1_0(X) = \{v \in H^1(X) : v_{|\partial X_D} = 0\}$, where ∂X_D is that part of the boundary ∂X at which Dirichlet boundary conditions are imposed. Further, we will use the function spaces $V_X := H^1(X)^d$, $V^0_X := H^1_0(X)^d$, and for time-dependent functions

$$\begin{aligned} \mathcal{L}_X &:= \mathcal{L}^2[0,T;L_X], \quad \mathcal{V}_X := \mathcal{L}^2[0,T;V_X] \cap \mathcal{H}^1[0,T;V^*_X], \\ \mathcal{L}^0_X &:= \mathcal{L}^2[0,T;L^0_X], \quad \mathcal{V}^0_X := \mathcal{L}^2[0,T;V^0_X] \cap \mathcal{H}^1[0,T;V^*_X], \end{aligned}$$

where V^*_X is the dual of V^0_X, and $\mathcal{L}^2$ and $\mathcal{H}^1$ indicate the corresponding properties in time. Again, the X-index will be skipped in the case of $X = \Omega$, and for $X = \Omega_f$ and $X = \Omega_s$ a corresponding index "f" or "s" will be used.

2.2 Fluid

For the liquid part, we assume Newtonian incompressible flow governed by the usual Navier-Stokes equations, i.e., the equations describing conservation of mass and momentum. The (constant) density and kinematic viscosity of the fluid are ρ_f and ν_f, respectively.

The equations are written in an Eulerian framework in the time-dependent domain $\Omega_f(t)$. The physical unknowns are the scalar pressure field $p_f \in \mathcal{L}_f$ and the vector velocity field $v_f \in v^D_f + \mathcal{V}^0_f$. Here, v^D_f is a suitable extension of the prescribed Dirichlet data on the boundaries (both moving or stationary) of Ω_f, and g_f is a suitable extension to all of $\partial\Omega_f$ of the Neumann data for $\sigma_f \cdot n$ on the boundaries. We have "hidden" the fluid-structure interface conditions of continuity of velocity and normal stress in parts of the boundary data v^D_f and g_f.

The variational form of the Navier-Stokes equations in an Eulerian framework is obtained by multiplying them with suitable test functions from the test space V^0_f for the momentum equations and L_f for the mass conservation equation.

Problem 1 (Fluid model in Eulerian formulation)**.** *Find* $\{v_f, p_f\} \in \{v^D_f + \mathcal{V}^0_f\} \times \mathcal{L}_f$*, such that* $v_f(0) = v^0_f$*, and*

$$\begin{aligned} (\rho_f(\partial_t + v_f \cdot \nabla)v_f, \psi^v)_f + (\sigma_f, \psi^v)_f &= (g_f, \psi^v)_{\partial\Omega_f} + (f, \psi^v)_f, \\ (div\, v_f, \psi^p)_f &= 0, \end{aligned} \tag{1}$$

for all $\{\psi^v, \psi^p\} \in V^0_f \times L_f$*, where*

$$\sigma_f := -p_f I + 2\rho_f\nu_f(\nabla v_f + \nabla v_f^T)$$

2.3 Structure

For ease of notation we here only consider the compressible *St. Venant-Kirchhoff* (STVK) model. The STVK model is detailed by a set of two parameters, the Poisson ratio ν_s and the Young modulus E_s, or alternatively, the Lamé coefficients λ_s and μ_s. These parameters satisfy the following relations:

$$\nu_s = \frac{\lambda_s}{2(\lambda_s+\mu_s)}, \quad E_s = \mu_s\frac{3\lambda_s+2\mu_s}{\lambda_s+\mu_s},$$
$$\mu_s = \frac{E_s}{2(1+\nu_s)}, \quad \lambda_s = \frac{\nu_s E_s}{(1+\nu_s)(1-2\nu_s)},$$

where $\nu_s = \frac{1}{2}$ for incompressible and $\nu_s < \frac{1}{2}$ for compressible material. The physical unknowns are the vector velocity $\hat{v}_s \in \mathcal{V}_s^0$ and the vector deformation $\hat{u}_s \in \hat{u}_s^D + \mathcal{V}_s^0$. $\hat{u}_s^D \in \mathcal{V}_s$ is a suitable extension of the prescribed Dirichlet data on the boundary of Ω_s. For the structure's velocity we usually only consider homogenous Dirichlet conditions. By $\hat{g}_s$ we denote the Neumann boundary data (and the fluid-structure interface condition) on parts of $\partial\Omega_s$. We first state the structure equations in the natural Lagrangian coordinates:

Problem 2 (STVK structure model in Lagrangian formulation)**.** *Find* $\{\hat{u}_s, \hat{v}_s\} \in \{\hat{u}^D + \hat{\mathcal{V}}_s^0\} \times \hat{\mathcal{V}}_s^0$*, such that* $\hat{u}_s(0) = \hat{u}_s^0$*,* $\hat{v}_s(0) = \hat{v}_s^0$*, and*

$$\begin{aligned}(\rho_s d_t\hat{v}_s, \hat{\psi}^u)_{\hat{s}} + (\hat{J}\ \hat{\sigma}_s\ \hat{F}^{-T}, \hat{\nabla}\hat{\psi}^u)_{\hat{s}} &= (\hat{g}_s, \hat{\psi}^u)_{\partial\hat{\Omega}_s} + (\hat{f}_s, \hat{\psi}^u)_{\hat{s}},\\ (d_t\hat{u}_s - \hat{v}_s, \hat{\psi}^v)_{\hat{s}} &= 0,\end{aligned} \tag{2}$$

for all $\{\hat{\psi}^u, \hat{\psi}^v\} \in \hat{V}_s^0 \times \hat{V}_s^0$*, where*

$$\hat{F}_s = I + \hat{\nabla}\hat{u}_s \qquad \hat{J}_s = det\,(I + \hat{\nabla}\hat{u}_s) \tag{3}$$

$$\hat{E} = \frac{1}{2}(\hat{F}_s^T\hat{F}_s - I) \quad \hat{\sigma}_s = \hat{J}_s^{-1}\hat{F}_s(\lambda_s(tr\,\hat{E})I + 2\mu_s\hat{E})\hat{F}_s^T \tag{4}$$

To rewrite the above conservation equations in an Eulerian frame, we need the pressure $\hat{p}_s$, the displacement $\hat{u}_s$ and its gradient $\hat{\nabla}\hat{u}_s$ in the Eulerian sense, which are denoted by p_s, u_s and ∇u_s, respectively. The transformation of the structure domain is given by

$$\hat{T}_s(t) : \hat{\Omega}_s \to \Omega_s(t), \quad \hat{T}_s(t,\hat{x}) = \hat{x} + \hat{u}(t,\hat{x}), \quad \hat{F}_s = \hat{\nabla}\hat{T}_s, \quad \hat{J}_s = \det\,(F_s).$$

For an incompressible material it holds $\hat{J}_s = 1$ and in the general case for physical reasons $\hat{J}_s > 0$. Thus, the gradient and its inverse of $\hat{T}_s$ are well defined. By $T_s(t) : \Omega_s(t) \to \hat{\Omega}_s$ we denote the inverse mapping back to the reference domain. Then, with $\hat{x} = T_s(t,x)$ the physical variables in the Eulerian coordinate system are given as:

$$p_s(t,x) := \hat{p}_s(t,\hat{x}), \quad u_s(t,x) := \hat{u}_s(t,\hat{x}), \quad v_s(t,x) := \hat{v}_s(t,\hat{x}).$$

The inverse mapping T_s can be written using the Eulerian deformation field u_s:

$$T_s(t,x) = x - u_s(t,x), \quad F_s := \nabla T_s = I - \nabla u_s, \quad J_s := \det(F_s)$$

To access the gradient of the deformation in Eulerian coordinates, we use the identity $T_s(\hat{T}_s(\hat{x})) = \hat{x}$. Differentiating this yields:

$$(I - \nabla u)(I + \hat{\nabla}\hat{u}) = I \quad \Leftrightarrow \quad \hat{\nabla}\hat{u} = (I - \nabla u)^{-1} - I = F_s^{-1} - I$$

Thus, the Cauchy stress tensor $\hat{\sigma}_s$ for STVK materials (3) and (4) in an Eulerian framework is as follows:

$$\begin{aligned} F_s = I - \nabla u_s = \hat{F}_s^{-1} \quad & J_s = \det(I - \nabla u_s) = \hat{J}_s^{-1} \\ E = \frac{1}{2}(F_s^{-T}F_s^{-1} - I) \quad & \sigma_s = J_s F_s^{-1}(\lambda_s(\operatorname{tr} E)I + 2\mu_s E)F_s^{-T} \end{aligned}$$

Finally we can state the structure equations in Eulerian framework for STVK materials:

Problem 3 (STVK structure model in Eulerian formulation)**.** *Find* $\{u_s, v_s\} \in \{u_s^D + \mathcal{V}_s^0\} \times \{v_s^D + \mathcal{V}_s^0\}$*, such that* $u_s(0) = u_s^0$*,* $v_s(0) = v_s^0$*, and*

$$\begin{aligned} (\hat{\rho}_s J_s \partial_t v_s, \psi^v)_s + (\hat{\rho}_s J_s v_s \cdot \nabla v_s, \psi^v)_s + (\sigma_s, \nabla\psi^v)_s, & \\ = (g_s, \psi^v)_{\Gamma_{sN}} + (\sigma_s n_s, \psi^v)_{\Gamma_i} + (\hat{\rho}_s J_s f_s, \psi^v)_s, & \\ (\partial_t u_s + v_s \cdot \nabla u_s - v_s, \psi^u)_s = 0, & \\ (1 - \det F_s, \psi^p) = 0, & \end{aligned} \tag{5}$$

for all $\{\psi^u, \psi^v\} \in V_s^0 \times V_s^0$*, where*

$$F_s = I - \nabla u_s \qquad J_s = det(I - \nabla u_s) \tag{6}$$

$$E = \frac{1}{2}(F_s^{-T}F_s^{-1} - I) \quad \sigma_s = J_s F_s^{-1}(\lambda_s(tr\, E)I + 2\mu_s E)F_s^{-T} \tag{7}$$

2.4 The FSI problem in Eulerian formulation

For the coupled FSI problem Problems 1 and 3 can be combined into a complete variational formulation in Eulerian coordinates. On the flow domain $\Omega_f(t)$ the system is governed by Problem 1, while in the structure domain $\Omega_s(t)$ Problem 3 is valid. On the interface $\Gamma_i(t) = \Omega_f(t) \cap \Omega_s(t)$ the dynamics are driven by requiring continuity of the velocity and by imposing a balance of forces

$$\sigma_f \cdot n = \sigma_s \cdot n \text{ on } \Gamma_i(t).$$

The continuity is enforced strongly by requiring one common continuous field for the velocity $v \in \mathcal{V}$ in Ω. Then, by multiplying the momentum equations of the flow and

structure problem with the continuous (across the interface) test-functions ψ^v, the balance of normal stresses is given due to integration by parts:

$$\int_{\Gamma_i} (\sigma_f - \sigma_s) \cdot n \psi^v do = 0$$

The compressible STVK material does not have a pressure variable p_s. For the formulation of the coupled problem we use one pressure field $p \in Ł$ in Ω and we harmonically extend the fluid's pressure to the structure domain $\Omega_s(t)$.

The deformation u_s is defined on all of Ω by a continuous extension to the flow domain. We denote the deformation by $u \in \mathcal{V}$. We save the details on this extension for a later discussion. Opposed to ALE formulations, this extended deformation field does not generate the ALE mapping and it does not need to provide a mapping from $\hat{\Omega}_f$ to $\Omega_f(t)$ at all. Instead we have more freedom designing the extension. This is crucial for the proposed fully Eulerian formulation.

The Dirichlet boundary data v_f^D and v_s^D on parts of $\partial\Omega$ are merged into a suitable velocity field $v^D \in \mathcal{V}$. By $u^D \in \mathcal{V}$ we denote the extension of u_s^D to all of Ω.

For ease of notation, we introduce the characteristic functions χ_f of Ω_f and χ_s of Ω_s respectively. We write the Cauchy stress tensor and the density in the whole domain as follows:

$$\sigma := \chi_f \sigma_f + \chi_s \sigma_s, \quad \rho := \chi_f \hat{\rho}_f + \chi_s \hat{\rho}_s. \tag{8}$$

The characteristic functions depend on the time t as well as on the shape of the domain partitioning and thus on the deformation u.

Extension of the deformation and the Eulerian domain partitioning The layout of the transformed domains is controlled by the transformation $\hat{T}_s : \hat{\Omega}_s \mapsto \Omega_s(t)$ and thus by the solution $u_s(t,x) = \hat{u}_s(t,\hat{x})$ itself. Hence, also the characteristic functions χ_f and χ_s are only defined implicitly by the solution itself. In the solid part of the domain, the deformation u_s can be used as mapping between the deformed $\Omega_s(t)$ and the undeformed $\widehat{\Omega}_s$ domain, via

$$x \in \Omega_s(t): \quad \widehat{\Omega}_s \ni \hat{x} = x - u_s(t,x).$$

Thus, the characteristic function of the solid domain is described by

$$\chi_s(t,x) = \hat{\chi}_s(x-u).$$

In the flow domain, there is no natural deformation given. By $u \in \mathcal{V}$ we denote a continuous extension of u_s to the whole domain Ω. Then, we can use this artificial deformation field u on all Ω to define:

$$x \in \Omega \;\Rightarrow\; \begin{cases} x - u(t,x) \in \hat{\Omega}_s & \Rightarrow\; x \in \Omega_s(t) \\ x - u(t,x) \notin \hat{\Omega}_s & \Rightarrow\; x \in \Omega_f(t). \end{cases}, \qquad \begin{aligned} \chi_s(t,x) &:= \hat{\chi}_s(x-u), \\ \chi_f(t,x) &:= \hat{\chi}_f(x-u). \end{aligned} \tag{9}$$

The extension of u to the flow domain does not necessarily define a mapping $T_f(t) : \Omega_f(t) \to \hat{\Omega}_f$ from the moving flow domain to the reference domain, only the information $T_f(t,x) \notin \hat{\Omega}_s$ is used. In particular, we do not need to specify homogenous Dirichlet conditions for u on the outer boundary of the flow domain $\partial\Omega \cap \partial\Omega_f(t)$. Hence, we also have the freedom to use different equations but the harmonic or bi-harmonic equation. Here, for the "fluid's deformation" we use the same governing equation as for the "solid's deformation" in Problem 3:

$$\partial_t u + v \cdot \nabla u = v \text{ in } \Omega.$$

In ALE context the use of this equation is prohibited in two ways: first, particles transported by the fluid's velocity in the general do not stay inside the flow-domain. Thus, the deformation u defined above does not create a mapping of $\hat{\Omega}$ onto $\Omega_f(t)$. Second, using the fluid-velocity for transporting the deformation field leads to very entangled meshes if used as ALE mapping. Here however we only use the fact that a fluid-particle will not be transported into the structure domain.

By this definition of the deformation extension $u \in \mathcal{V}$, arbitrary large deformation as well as free movement of the structure within the flow-domain is possible. Further, we can model contact of the structure with the boundary of the domain as well as self-contact of the structure.

Finally, we can combine formulas (1) and (5) to obtain the complete variational formulation of the FSI problem in Eulerian coordinates.

Problem 4 (FSI Problem in Eulerian formulation, STVK material)**.** *Find* $\{u, v, p\} \in \{u^D + \mathcal{V}^0\} \times \{v^D + \mathcal{V}^0 \times \mathcal{L}\}$*, such that* $u(0) = u^0$*,* $v(0) = v^0$*, and*

$$\begin{aligned} (\rho(\partial_t v + v \cdot \nabla v), \psi^v) + (\sigma, \psi^v) &= (g, \psi^v)_{\partial\Omega} + (f, \psi^v), \\ (\chi_f \, div\, v, \psi^p) + \alpha_p\{(\chi_s \nabla p, \nabla \psi^p) - (\partial_n p, \psi^p)_{\Gamma_i}\} &= 0, \\ (\partial_t u + v \cdot \nabla u - v, \psi^u) &= 0, \end{aligned} \tag{10}$$

for all $\{\psi^u, \psi^v\} \in V^0 \times V^0$ *and* $\psi^p \in L$ *where* ρ*,* F*, and* σ *are defined as above and* $\alpha_p > 0$ *is a small constant.*

In these variational formulations the location of the interface Γ_i is given implicitly by the deformation:

$$\Gamma_i(t) := \{x \in \Omega,\ x - u(t,x) \in \hat{\Gamma}_i\}.$$

The resulting system is nonlinear even if linear models are used for the two subproblems, e.g. a Stokes fluid and a linear elastic structure.

In some situations the solution of an FSI problem may tend to a "steady state" as $t \to \infty$. The equations for stationary fluid structure interaction in Eulerian formulation are given in Richter & Wick[13].

2.5 Comparison to ALE formulations

In the classical ALE formulation for FSI problems the structure problem is formulated on the static reference domain $\hat{\Omega}_s$. The flow problem is transformed back to the reference flow domain $\hat{\Omega}_f$ by an artificial transformation $\hat{T}_f : \hat{\Omega}_f \to \Omega$ usually derived by the continuation of the deformation to the flow domain. The situation in Eulerian coordinates is similar: here, the flow problem resides in the natural coordinate system while the structure problem needs to be transformed. This transformation $T_s : \Omega_s \to \hat{\Omega}_s$ however is a "natural transformation" since it is given by the deformation itself. If the structure solver is capable of handling a certain deformation, the transformation is well behaved and cannot deteriorate in the context of FSI. The deformation field is extended to the flow domain in both formulations. But while the extended (and artificial) field is used to transform the flow variables in the ALE formulation, it is only necessary to lookup the domain of influence in the Eulerian formulation. Here, no gradient evaluation is required and the regularity is of lesser importance. In the fully Eulerian formulation, the extension of the deformation to the flow domain does not have to define a mapping between the reference and the deformed fluid domain. Hence, we have more freedom in modeling this extension.

Both formulations contain equations for velocity, deformation and pressure, a total of five solution variables in two spatial dimensions. The Eulerian formulation is strongly nonlinear due to the implicit dependence of the domain on the deformation. In the ALE formulation the transformation of the flow domain imposes strong nonlinearities which can prohibit large deformation. The Eulerian formulation tends to be slightly more costly due to the implicit definition of the moving fluid-structure interface $\Gamma_i(t)$.

3 Discretization and Solution Methods

For ease of presentation, we introduce a short notation for equation (10). Find $U = \{v, u, p\} \in \{v^D + \mathcal{V}^0, \mathcal{V}^0, \mathcal{L}\}$ such that

$$(\rho\partial_t v, \psi^v) + (\partial_t u, \psi^u) + a(U)(\Psi) + b(U)(\Psi) = F(\Psi) \quad \forall \Phi := \{\psi^v, \psi^u, \psi^p\} \in V^0 \times V^0 \times L,$$

where the semi-linear forms $a(\cdot)(\cdot)$ and $b(\cdot)(\cdot)$ and the right hand side $F(\Psi)$ are given by

$$\begin{aligned} a(U)(\Psi) &= (\rho v \cdot \nabla v, \psi^v) + (\sigma, \nabla\psi^v) + (v \cdot \nabla u - v, \psi^u) \\ b(U)(\Psi) &= (\chi_f \operatorname{div} v, \psi^p) + \alpha_p\{(\chi_s \nabla p, \nabla\psi^p) - (\partial_n p, \psi^p)_{\Gamma_i}\}, \\ F(\Psi) &= (g, \psi^v)_{\partial\Omega} + (f, \psi^v). \end{aligned}$$

Discretization in space and time is then accomplished by adaptive Galerkin methods.

3.1 Temporal discretization

We usually consider Galerkin methods for time-discretization to make a posteriori error estimation and space-time adaption accessible, see Schmich and Vexler[14] or Rannacher

and Richter[11]. The resulting methods are however up to integration error similar to standard time stepping scheme. Thus we here only present the one-step θ-scheme. The time interval $I = [0, T]$ is split into subsets

$$0 = t_0 < t_1 < \cdots < t_M = T,$$

with $I_m := (t_{m-1}, t_m]$ and $k_m := t_m - t_{m-1}$. By $U_k^m \in V \times V \times L$ we denote the time-discrete solution at time $t = t_m$. Then, with $v_k^0 := v^0$ and $u_k^0 := u^0$ we find - with U_k^{m-1} given - the next time-step for $\theta \in [0, 1]$ by

$$\begin{aligned}(\rho v^m, \psi^v)+(u^m, \psi^u)+k_m\theta a(U^m)(\Psi)+k_m b(U^M)(\Psi) = k_m\theta F(t_m, \Psi)+k_m(1-\theta)F(t_{m-1}, \Psi)\\ + (\rho v^{m-1}, \psi^v) + (u^{m-1}, \psi^u) + k_m(\theta - 1)a(U^{m-1})(\Psi) \qquad \forall \Psi \in V^0 \times V^0 \times L. \quad (11)\end{aligned}$$

For $\theta = 0$ this is the explicit forward Euler and for $\theta = 1$ the fully implicit backward Euler-scheme. For $\theta = \frac{1}{2}$ we get the second order Crank-Nicolson scheme. This scheme lacks stability and does not smooth out numerical errors. The optimal scheme for fluid-structure-interaction problems is the fractional-step-θ-scheme, consisting of three sub-steps of the θ-scheme with different values of θ, see Dunne, Rannacher and Richter[6]. As an alternative we sometimes use the easier (and globally A-stable, however not strongly A-stable) implicitly shifted Crank-Nicolson scheme with $\theta|_{I_m} = \theta_m := \frac{1}{2} + \frac{1}{2}k_m$, see Heywood and Rannacher[7]. All θ-schemes for $\theta \in (0, 1]$ can be written as Galerkin discretizations and thus a closed variational formulation for the time-discrete problem exists.

3.2 Spatial discretization

For discretization in space continuous equal-order finite elements for pressure, velocity and deformation are utilized. By Ω_h we denote a triangulation of the domain Ω into open quadrilaterals in two and hexahedrals in three dimensions. The triangulation has to fulfill the standard shape-regularity conditions (a maximum angle condition and no large anisotropies). To allow for local refinement, we use the concept of hanging nodes, that is of degrees of freedom, which lie in the middle of edges (or faces) of adjacent elements. These degrees of freedom are replaced by interpolations of neighboring nodes. We allow at most one such hanging nodes per edge (or face). On the *reference element* $\hat{K} = (0, 1)^d$ we define the space Q^p by (in two dimensions)

$$Q^p = \text{span}\{x^\alpha y^\beta,\ 0 \le \alpha, \beta \le p\}.$$

On the mesh Ω_h the piece-wise p-th order parametric space is given by

$$V_h^p = \{\phi \in C(\Omega) :\ \phi\big|_K \circ T_K^{-1} \in Q^p\}, \quad (12)$$

where $T_K \in Q^p$ is the mapping $T_K : \hat{K} \to K$.

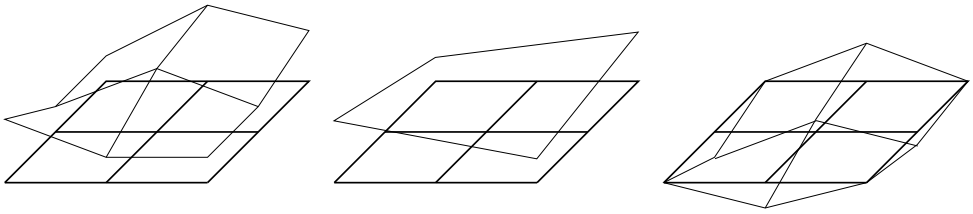

Figure 3: The fluctuation operator $\pi_h u_h := (id - i_{2h})u_h$ used for the Local Projection Stabilization method. Left: a piece-wise linear u_h, middle: the interpolation $i_{2h}u_h$, right: the fluctuation $\pi_h u_h$.

We consider every time-step of (11) as a stationary pde. The discrete solution $U_h^m = \{v_h^m, u_h^m, p_h^m\}$ is then sought in the space

$$U_h^m \in [V_h^p]^d \times [V_h^p]^d \times V_h.$$

We usually neglect the superscript "m". Dirichlet conditions are built into these discrete spaces by skipping the finite element basis functions in nodes x_i on the boundary of Ω_h. We usually consider the space of piece-wise quadratic functions $p = 2$. Since these equal-order spaces are not *inf-sup stable* we add further stabilization terms. We use the local projection stabilization as described by Becker and Braack[1] both for stabilizing the *inf-sup condition* and dominant convection. To the semi-linear form (11) we add the stabilization term

$$S_{\text{lps}}(U_h)(\Psi_h) = \sum_{K\in\Omega_h} \Big\{ \delta_K^v (v_h \cdot \nabla \pi_h v_h, v \cdot \nabla \pi_h \phi_h)_K + \delta_K^p (\nabla \pi_h p_h, \nabla \pi_h \xi_h)_K \Big\},$$

where $\pi : V_h \to V_h$ is the local *fluctuation operator* which filters all fine mesh frequencies:

$$\pi_h v_h := (id - i_{2h})v_h,$$

with the interpolation $i_{2h} : V_h \to V_{2h}$ onto the mesh with double mesh spacing. In Figure 3 we illustrate this fluctuation operator for biquadratic elements. The stabilization parameters are locally chosen as

$$\delta_K^v = \delta_K^p = \delta_0 \left(\frac{\chi_f \nu_f + \chi_s \mu_s}{h_K^2} + \frac{\|v_h\|_{K,\infty}}{h_K} \right)^{-1}.$$

The advantage of the LPS-method compared to classical residual based stabilization schemes as PSPG or SUPG, see Brooks and Hughes[3] and Hughes, Franca and Balestra[8] is that no additional coupling between different solution variables (that is pressure, velocity and deformation) are introduced. Further, for time-dependent problems, the residual based stabilization schemes introduce couplings between the solution at the new and old time-step.

The local projection scheme is however not suited for pure transport problems without physical diffusion. Thus, for stabilizing the deformation equation we use the SUPG method and add the stabilization form

$$S_{\text{supg}}(U_h,\Phi_h) = \sum_{K\in\Omega_h} \frac{1}{k_m\theta}(U_h^m - U_h^{m-1}, \delta_K^u v_h^m\cdot\nabla\psi_h) + (v_h^m\cdot\nabla u_h^m - v_h^m, \delta_K^u v_h^m\cdot\nabla\psi_h).$$

In every time slot $I_m = (t_{m-1},t_m]$ the discrete problem is finally given by

$$\begin{aligned}(\rho v_h^m,\psi_h^v) + (u_h^m,\psi_h^u) + k_m\theta a(U_h^m)(\Psi_h) + k_m b(U_h^m)(\Psi_h)\\ + k_m\theta S_{\text{lps}}(U_h^m)(\Psi_h) + k_m\theta S_{\text{supg}}(U_h^m)(\Psi_h) = k_m\theta F(t_m,\Psi_h) + k_m(1-\theta)F(t_{m-1},\Psi_h)\\ + (\rho v_h^{m-1},\psi_h^v) + (u_h^{m-1},\psi_h^u) + k_m(\theta-1)a(U_h^{m-1})(\Psi_h)\\ \forall\Psi_h\in V_h^0\times V_h^0\times L_h. \quad (13)\end{aligned}$$

3.3 Solution Scheme

The nonlinear discrete problem (13) is solved by a Newton's method yielding iterates $U_h^{m,(i)}$ for $i\geq 0$. In every iteration linear problems need to be solved for the update $U_h^{m,(i+1)} := U_h^{m,(i)} + W_h^{m,(i)}$:

$$A'(U_h^{m,(i)})(W_h^{m,(i)},\Psi_h) = G(U_h^{m-1},U^{m,(i-1)}).$$

Here, $A'(\cdot)(\cdot,\cdot)$ is the Jacobian, that is the directional derivatives of all operators on the left hand of (13) evaluated in $U_h^{m,(i)}$ in direction of $W_h^{m,(i)}$ and tested with Ψ_h. The right hand side $G(\cdot,\cdot)$ is the residual of equation (13), depending on the solution at the last time-step U_h^{m-1} and the last Newton-iterative $U_h^{m,(i-1)}$. We compute the derivatives of the semilinear form $A(\cdot)(\cdot)$ either analytically or with automatic differentiation, see Dunne[5] for details on the automatic differentiation as well as for most analytic directional derivatives.

For solving the linear problems, a GMRES iteration is preconditioned with a geometric multigrid method. Here, the critical part is the smoothing operation, since the condition number of the matrices can get very large.

With equal-order finite elements, the degrees of freedom for different solution components (pressure, velocity and deformation) are situated in the same mesh-nodes and we can apply a special blocking. Let $\{\phi_h^i, i=1,\dots,N\}$ be the nodal basis of the scalar finite element space V_h^p in (12). Then (in two spatial dimensions), we have the representation

$$v_h^d = \sum_{i=1}^N \boldsymbol{v}_i^d\phi_h^i,\ u_h^d = \sum_{i=1}^N \boldsymbol{u}_i^d\phi_h^i,\ p_h = \sum_{i=1}^N \boldsymbol{p}_i\phi_h^i,\quad \boldsymbol{v}^d,\boldsymbol{u}^d,\boldsymbol{p}\in\mathbb{R}^N,\quad d=1,2,$$

with the same basis functions ϕ_h^i. Thus for the component variable U_h, we can write

$$U_h = \sum_{i=1}^{N} \boldsymbol{U}_i \phi_h^i, \quad \boldsymbol{U}_i = \begin{pmatrix} \boldsymbol{v}_i^1 \\ \boldsymbol{v}_i^2 \\ \boldsymbol{u}_i^1 \\ \boldsymbol{u}_i^2 \\ \boldsymbol{p}_i \end{pmatrix} \in \mathbb{R}^5.$$

This local blocking allows us to hide the number of equation components (here 5) in the linear algebra, by considering all vectors as block-vectors $\boldsymbol{U} \in \mathbb{R}^{N\times 5}$ and the Jacobian as a block-matrix $J_h \in \mathbb{R}^{(N\times N)\times(5\times 5)}$. The entries of the Jacobian are local block matrices:

$$J_h = \left(J_h^{ij}\right)_{i,j=1}^{N}, \quad J_h^{ij} = A'(U_h)\big((1\,1\,1\,1\,1)^T\phi_h^j, (1\,1\,1\,1\,1)^T\phi_h^i\big) \in \mathbb{R}^{5\times 5}.$$

The solver now works independent on the number of equations on the entries J_h^{ij} and U_h^i. As smoother we use some steps of an block-incomplete LU decomposition of J_h, where we consider the entries $J_h^{ij} \in \mathbb{R}^{5\times 5}$ as closed units which are inverted exactly. Locally in every block, the saddle-point character of the Navier-Stokes equations and on the interface the fsi-coupling is taken into account. If we write the Jacobian in an additive decomposition $J_h = J_f + J_s$ for the fluid- and structure-part, it locally behaves like (neglecting the stabilization terms)

$$J_s = \begin{bmatrix} \frac{1}{k}\mathrm{id} & \sigma_s & 0 \\ \frac{1}{k}\mathrm{id} & \mathrm{id}+\nabla & 0 \\ 0 & 0 & \Delta \end{bmatrix}, \quad J_f = \begin{bmatrix} \frac{1}{k}\mathrm{id}+\sigma_f & 0 & \nabla \\ \frac{1}{k}\mathrm{id} & \mathrm{id}+\nabla & 0 \\ \mathrm{div} & 0 & 0 \end{bmatrix}.$$

This sparsity pattern is used by not storing the zero-entries. More important, the effort for matrix-vector products and for inverting the blocks J_h^{ij} is significantly reduced.

As multigrid-smoother we use a block-ILU-decomposition of the Jacobian J_h, where we invert the local blocks J_h^{ij} exactly. This smoother has shown to be very robust and efficient for various complex flow problems, see Braack and Richter[2] for reactive flow problems or Lin and Richter[10] for liquid crystal flows. However, for fluid-structure interaction problems, we do not get mesh-independent convergence rates. This is due to the very bad conditioning of the Jacobians on finer mesh levels. Numerical truncation errors in the smoothing process lead to a slight impairment of the convergence rate on finer meshes. The development of a solution scheme with optimal complexity is still upcoming work for fluid-structure-interaction problems.

3.4 Parallelization

The multigrid solver is parallelized for distributed memory architectures. For details on the parallelization of the multigrid solver we refer to Richter[12] or Kimmritz and Richter[9].

The parallel solver differs from the standard solver by using an additive Schwarz iteration as an outer smoothing loop. On every subdomain, an ILU-preconditioned Richardson iteration is used to approximate the solution. Thus if $\Omega_h^p,\ p = 1, \ldots, N_p$ is a partitioning of the mesh to the domains (without overlap elements), the smoothing iterations is given by

$$W_h^{(t+1)} = W_h^{(t)} + \sum_{p=1}^{N_p} \mathcal{P}^p \mathcal{ILU}(J_h\big|_{\Omega_p})^{-1} \mathcal{R}^p (b - J_h W_h^{(t)}),$$

with prolongation $\mathcal{P}^p$ and restriction $\mathcal{R}^p$ operators between Ω_h and the subdomain Ω_h^p. This minimal-overlap partitioning (only degrees of freedom on the interface between two subdomains are duplicated) does not yield a solver, it however serves very well as smoothing operation for the high frequencies, see[12] for details on the parallel analysis and implementation.

3.5 Approximation of the Interface

Since the interface $\Gamma_i(t)$ is moving through the domain and crossing mesh elements, one cannot align the mesh nodes with the interface. This is the one severe drawback compared to ALE formulations. Numerical integration thus has to carefully consider the interface regions where the dynamics of the coupled problem is dominated. We now look at one mesh element K at the interface $\Gamma_i \cup K \neq \emptyset$:

$$\int_K F(x)\mathrm{d}x = \int_K (\chi_f F_f(x) + \chi_s F_s(x))\mathrm{d}x.$$

The function F is usually neither continuous nor differentiable across the interface. In Dunne[5] it was proposed to used summed integration formulas to evaluate these functionals. For an accurate integration a very large number of integration points is necessary. This method, not taking the specific layout of the interface into account in not efficient. Here, we approximate the interface $\Gamma_K := \Gamma_i \cup K$ by a linear or quadratic function and split the quad into four (curved) triangles. In Figure 4 we show examples for the splitting of an element into triangles. Next we use a seven-point Gauss formula to evaluate the

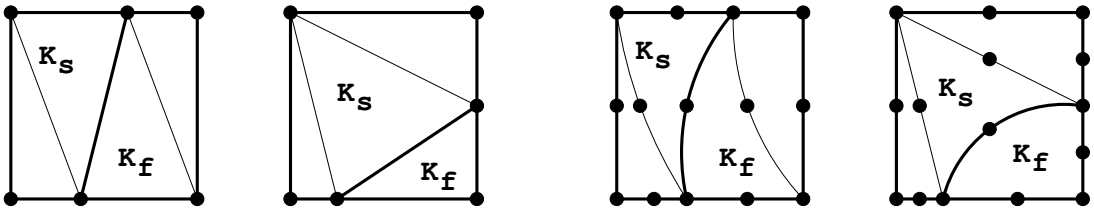

Figure 4: Splitting of an interface element K into triangles for integration. Left: linear approximation of the interface, right: quadratic approximation.

integrals in the triangles:

$$\int_K F(x)\mathrm{d}x = \int_{K_f} F_f \mathrm{d}x + \int_{K_s} F_s \mathrm{d}x \approx I_h(K_f)(F_f) + I_h(K_s)(F_s).$$

The control points for specifying the triangles are calculated in advance for all elements touching the interface. Since the location of the interface depends on the deformation u, the splitting into triangles needs to be recalculated for every residual evaluation. This calculation can be done very efficient and fast and does not require a significant share of the overall computing time.

3.6 Evaluating the structure derivatives

For assembling the Jacobian J_h of the semi-linear form in equation (13) derivatives with respect to the interface appear since it's location depends on the deformation $\Gamma_i(t) = \Gamma_i(u)$. On elements touching the interface $K \cap \Gamma_i(u)$ we need to express integrals like

$$I_K(u) = \frac{\partial}{\partial s} \int_K \chi_f(x, u + s\Phi)F_f(x) + \chi_s(x, u + s\Phi)F_s(x)\,\mathrm{d}x\Big|_{s=0}.$$

By the Hadamard structure theorem, see e.g. Zolesio[16], these integrals turn to lower-dimensional integrals over the interface:

$$I_K(U) = -\int_{\Gamma_i(U)} F_f(x)n_f \cdot \Phi + F_s(x)n_s \cdot \Phi\,\mathrm{d}s.$$

Including these integrals in the Jacobian is important for obtaining good Newton convergence. We can evaluate the interface-integrals by simple quadrature rules using the splitting of the elements into triangle as explained above and shown in Figure 4. In the numerical examples it was sufficient to approximate the normal vector by the normal of the approximated interface line splitting the two subdomains in Figure 4 and by applying the mid-point rule along the interface line.

4 Numerical Examples

First examples to verify the new Eulerian formulation for fluid-structure-interaction problems have been demonstrated in Dunne[5] as well as in Rannacher and Richter[11]. In Richter and Wick[13] more detailed studies for stationary test-cases have shown an agreement in the approximation order between ALE formulation and the fully Eulerian formulation.

Schäfer and Turek[15] have proposed a numerical benchmark problem for fluid-structure interaction problems. This, however very difficult benchmark problem, has been studied by Dunne[5] and Rannacher and Richter[11] for both, ALE and Eulerian formulation. In the quantities of interest, agreement up to 5% has been reached.

Here, we focus on the special extension of the deformation field allowing for large deformation and movement of the structure within the flow domain.

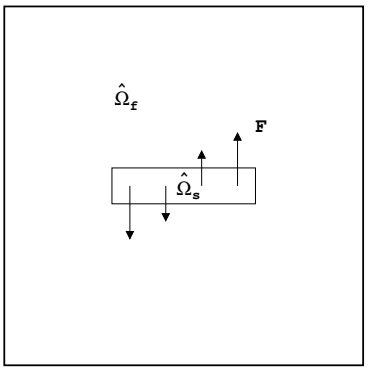

Figure 5: Rectangular elastic structure in unit square.

4.1 First example: rotating obstacle

Limiting factor for large deformation using the ALE framework is the implicit transformation of the fluid mesh to artificial coordinates, usually defined via $\hat{T}_f = id + \hat{u}_f$, where $\hat{u}_f$ is an extension of the deformation to the flow domain $\hat{\Omega}_f$. In the Eulerian formulation, the flow problem is not transformed but instead given in natural coordinates. Here, the artificial extension of the deformation field u to Ω_f is not used as transformation, but just needed to trace the domain of influence in the definition of the characteristic functions (9).

In the middle of the unit square $\Omega = (0,1)^2$, we prescribe an elastic object $\hat{\Omega}_s = \{(x,y) : \ x \in (0.3, 0.7), \ y \in (0.45, 0.55)\}$ (See Figure 5). On the boundary $\partial\Omega$, the *do-nothing* condition is given for the velocity and a homogenous Neumann condition for the deformation:

$$-\nu_f \partial_n v + p \cdot n = 0 \text{ on } \partial\Omega, \quad \partial_n u = 0 \text{ on } \partial\Omega.$$

The dynamics of the system is driven by a force acting on the elastic structure:

$$(\rho_s(\partial_t v + v \cdot \nabla v), \phi)_{\Omega_s(t)} + (\sigma_s, \nabla\phi)_{\Omega_s(t)} = (\rho_s J_s F, \phi)_{\Omega_s(t)},$$

with the volume force F given by

$$F = \begin{pmatrix} -y + \frac{1}{2} \\ x - \frac{1}{2} \end{pmatrix}.$$

This force results in a rotation of the rectangle and a rotational flow evolving in the domain Ω. By the standard ALE approach (without remeshing and reinitialization) this problem is not computable, since the extension of the deformation u to the flow domain would lead to an increasingly large distortion of the meshes. In Figure 6 we show snapshots of the solution for different time points.

For comparison we show corresponding results obtained using an ALE formulation. For the construction of the mapping $\hat{T}_f : \hat{\Omega}_f \to \Omega_f(t)$ a bi-harmonic extension of the

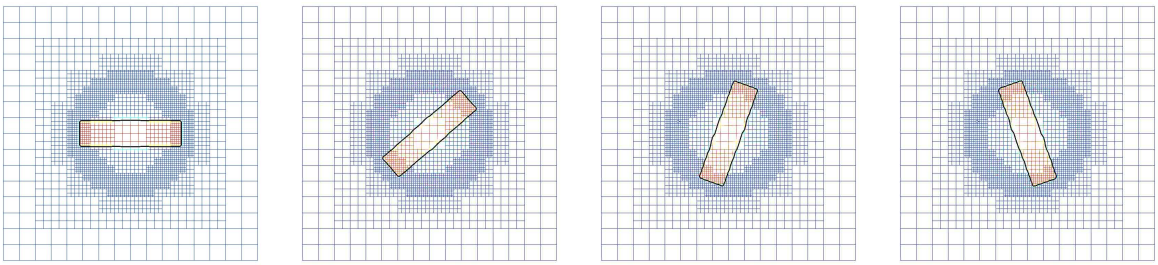

Figure 6: Rotation of the elastic rectangle at different time-steps using the Eulerian formulation.

deformation u to the flow domain is used. This usually yields the best results and allows for the largest possible deformation, see Dunne[5] for details. In Figure 7 we show the deformation and the resulting distorted meshes for different time-steps. The last snapshot indicates the largest possible deformation of the flow domain before breakdown of the solution scheme in the next time-step. For the ALE comparison a very coarse mesh is used for a better visualization of the effects.

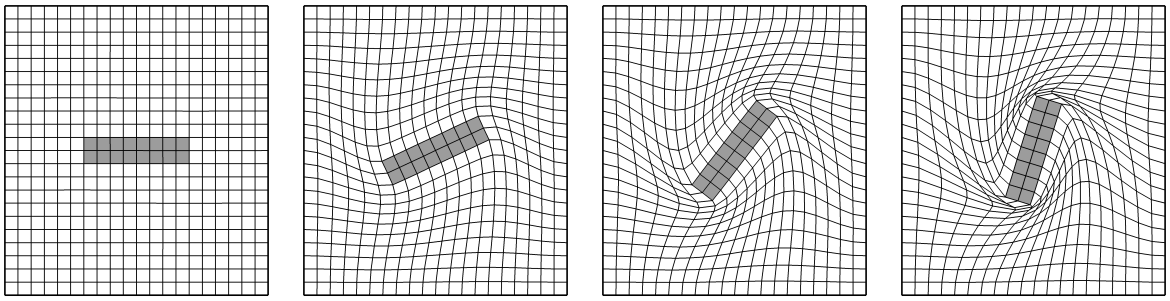

Figure 7: Implicit mesh deformation using ALE coordinates for the rotating rectangular structure.

4.2 Second example: contact

The ALE formulation has problems when large deformation appears close to the boundary of the domain. In particular the contact of the elastic structure with the boundary is not possible within a monolithic formulation using simple ALE coordinates without remeshing techniques. The transformation $\hat{T}_f(t) : \hat{\Omega}_f \to \Omega_f(t)$ of the flow domain would degenerate. Here, we model a pressure induced flow in a narrowing channel, blocked by an elastic structure, see Figure 8.

At the inflow and outflow boundary the *do-nothing* condition is prescribed with a pressure drop causing a flow to the right:

$$-\nu_f \partial_n v + p \cdot n = 0 \text{ on } \Gamma_{\text{in}}, \quad -\nu_f \partial_n v + p \cdot n = -1 \text{ on } \Gamma_{\text{out}}.$$

Prescribing a pressure drop instead of a Dirichlet inflow condition, the problem stays

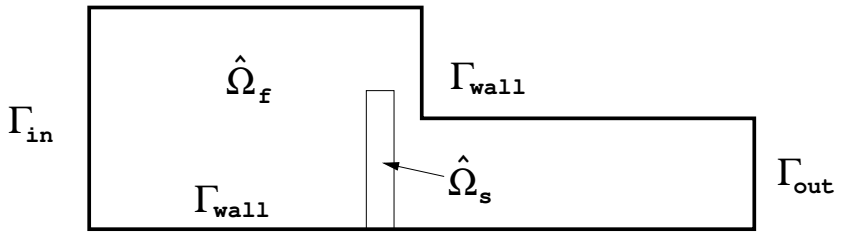

Figure 8: Narrowing channel with elastic obstacle. Flow driven by pressure drop: $p = 0$ on Γ_{in} and $p = -1$ on Γ_{out}.

well-posed even if the channel is closed. Along the remaining boundary Γ_{wall} homogenous Dirichlet condition $v = 0$ is prescribed. For the extension of the displacement u to the flow domain a slip-like boundary condition is enforced on the outer boundary,

$$u \cdot n = 0 \text{ on } \Gamma_{\text{wall}}.$$

In Figure 9, we show snapshots of the deformed structure $\Omega_s(t)$ for different time points. First, by the evolving flow the structure is bended to the right. "Contact" of the structure with the wall is realized up to one layer of mesh elements. After mesh refinement, the structure gets closer to the boundary of the domain.

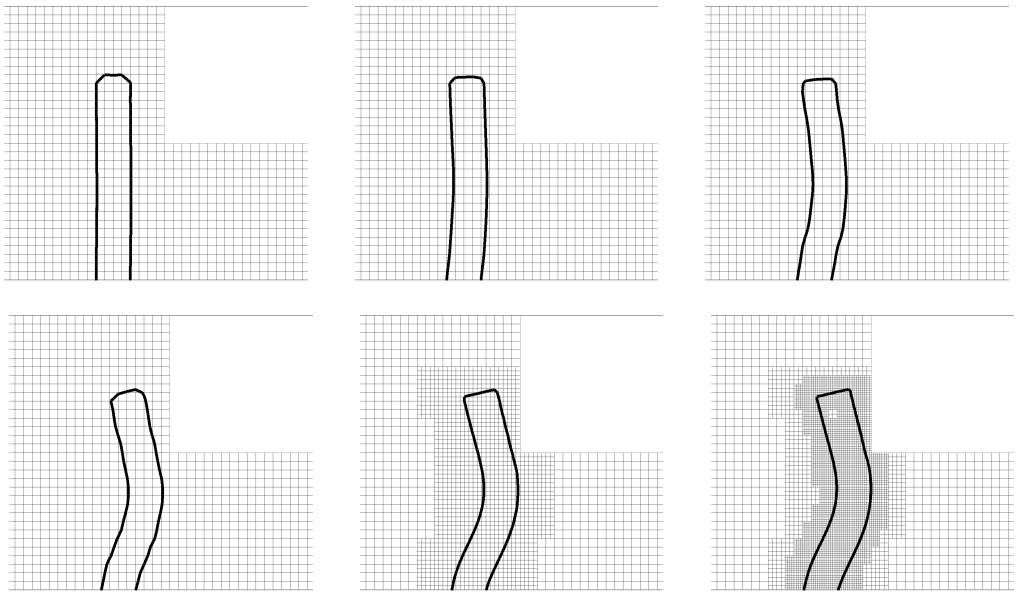

Figure 9: Near-contact of the elastic structure with the wall. "Contact" is possible up to one layer of mesh elements.

5 Summary

In this paper we further investigate a new, fully Eulerian variational formulation for fluid-structure interaction problems. This approach has already proven to yield the correct results for easy benchmark-problems. The fully Eulerian formulation allows us to treat FSI problems with free bodies and large deformations. This is the main advantage of this method compared to interface tracking methods such as the arbitrary Lagrangian-Eulerian (ALE) method.

The method based on the Eulerian approach is inherently more expensive than the ALE method, by about a factor of two, but it allows to treat also large deformations and boundary contact of the structure. This potential has been investigated for several model configurations. Special attention is given to a robust and efficient parallel multigrid-solver for the coupled fluid-structure interaction problems.

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