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ON THE STABILITY OF LOCALLY ONE-DIMENSIONAL METHOD FOR TWO-DIMENSIONAL PARABOLIC EQUATION WITH NONLOCAL INTEGRAL CONDITIONS

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Abstract. *We construct and analyse a locally one-dimensional method (finite-difference scheme) for a two-dimensional parabolic equation with nonlocal integral conditions. The main attention is paid to the stability of the method. We apply the stability analysis technique which is based on the investigation of the spectral structure of the transition matrix of a finite-difference scheme and demonstrate that depending on the parameters of nonlocal conditions the proposed method can be stable or unstable. The results of numerical experiment with one test problem are also presented and they validate theoretical results.*

1 INTRODUCTION

We consider the two-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t), \quad 0 < x < 1, \quad 0 < y < 1, \quad 0 < t \leqslant T, \tag{1}$$

subject to nonlocal integral conditions

$$u(0, y, t) = \gamma_1 \int_0^1 u(x, y, t)\mathrm{d}x + \mu_1(y, t), \tag{2}$$

$$u(1, y, t) = \gamma_2 \int_0^1 u(x, y, t)\mathrm{d}x + \mu_2(y, t), \quad 0 < y < 1, \quad 0 < t \leqslant T, \tag{3}$$

boundary conditions

$$u(x, 0, t) = \mu_3(x, t), \quad u(x, 1, t) = \mu_4(x, t), \quad 0 < x < 1, \quad 0 < t \leqslant T, \tag{4}$$

and initial condition

$$u(x, y, 0) = \varphi(x, y), \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1, \tag{5}$$

where $f(x, y, t)$, $\mu_1(y, t)$, $\mu_2(y, t)$, $\mu_3(x, t)$, $\mu_4(x, t)$, $\varphi(x, y)$ are given functions, γ_1, γ_2 are given parameters, and function $u(x, y, t)$ is unknown. We assume that for all t, $0 < t \leqslant T$, nonlocal integral conditions (2), (3) and boundary conditions (4) are compatible, i.e., the following compatibility conditions are satisfied:

$$\gamma_1 \int_0^1 \mu_3(x, t)\mathrm{d}x + \mu_1(0, t) = \mu_3(0, t),$$
$$\gamma_1 \int_0^1 \mu_4(x, t)\mathrm{d}x + \mu_1(1, t) = \mu_4(0, t),$$
$$\gamma_2 \int_0^1 \mu_3(x, t)\mathrm{d}x + \mu_2(0, t) = \mu_3(1, t),$$
$$\gamma_2 \int_0^1 \mu_4(x, t)\mathrm{d}x + \mu_2(1, t) = \mu_4(1, t).$$

The present paper is devoted to the numerical solution of the two-dimensional differential problem (1)–(5). We construct the locally one-dimensional (LOD) method and analyse its stability.

The stability of finite-difference schemes for the corresponding one-dimensional parabolic problems with nonlocal integral conditions similar to conditions (2), (3) or with more general integral conditions is investigated by M. Sapagovas[1,2], Ž. Jesevičiūtė and M. Sapagovas[3] and other authors. The alternating direction implicit (ADI) method for the two-dimensional differential problem (1)–(5) has been proposed and the stability of that method has been analysed by S. Sajavičius[4,5]. The LOD technique for two-dimensional parabolic problems with nonlocal integral condition (the specification of mass/energy) has been investigated by M. Dehghan[6]. Paper of M. Sapagovas et. al.[7] deals with the ADI method for the two-dimensional parabolic equation (1) with Bitsadze-Samarskii type nonlocal boundary condition. We use the similar technique and argument in order to construct the LOD method for the differential problem (1)–(5) and to investigate the stability of that method.

The paper is organized as follows. In Section 2, the details of the LOD method are described. Section 3 reviews the well-known stability analysis technique which is based on the spectral structure of the transition matrix of a finite-difference scheme, and discusses possibilities to use such technique in order to analyse the stability of the proposed LOD method. The results of numerical experiment with a particular test problem are presented in Section 4. Some remarks in Section 5 conclude the paper.

2 THE LOCALLY ONE-DIMENSIONAL METHOD

To solve the two-dimensional differential problem (1)–(5) numerically, we apply the finite-difference technique[8]. Let us define discrete grids with uniform steps,

$$\omega_{h_1} = \{x_i = ih_1, i = 1, 2, \dots, N_1 - 1, N_1 h_1 = 1\}, \quad \overline{\omega}_{h_1} = \omega_{h_1} \cup \{x_0 = 0, x_{N_1} = 1\},$$
$$\omega_{h_2} = \{y_j = jh_2, j = 1, 2, \dots, N_2 - 1, N_2 h_2 = 1\}, \quad \overline{\omega}_{h_2} = \omega_{h_2} \cup \{y_0 = 0, y_{N_2} = 1\},$$
$$\omega = \omega_{h_1} \times \omega_{h_2}, \quad \overline{\omega} = \overline{\omega}_{h_1} \times \overline{\omega}_{h_2},$$
$$\omega^\tau = \{t^k = k\tau, k = 1, 2, \dots, M, M\tau = T\}, \quad \overline{\omega}^\tau = \omega^\tau \cup \{t^0 = 0\}.$$

We use the notation $U_{ij}^k = U(x_i, y_j, t^k)$ for functions defined on the grid $\overline{\omega} \times \overline{\omega}^\tau$ or its parts, and the notation $U_{ij}^{k+1/2} = U(x_i, y_j, t^k + 0.5\tau)$ (some of the indices can be omitted). We define one-dimensional discrete operators

$$\Lambda_1 U_{ij} = \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h_1^2}, \quad \Lambda_2 U_{ij} = \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{h_2^2}.$$

Now we explain the main steps of the LOD method for the numerical solution of problem (1)–(5). First of all, we replace the initial condition (5) by equations

$$U_{ij}^0 = \varphi_{ij}, \quad (x_i, y_j) \in \overline{\omega}. \tag{6}$$

Then, for any k, $0 \leqslant k < M - 1$, the transition from the kth layer of time to the $(k+1)$th layer can be carried out by splitting it into two stages and solving one-dimensional finite-difference subproblems in each of them. The both of subproblems are fully-implicit. The first subproblem, i.e., the set of linear algebraic equations systems for all $x_i \in \omega_{h_1}$, is fully-implicit with respect to y:

$$\frac{U_{ij}^{k+1/2} - U_{ij}^{k}}{\tau} = \Lambda_2 U_{ij}^{k+1/2} + f_{ij}^{k+1/2}, \quad y_j \in \omega_{h_2}, \tag{7}$$

$$U_{i0}^{k+1/2} = (\mu_3)_i^{k+1/2}, \tag{8}$$

$$U_{iN_2}^{k+1/2} = (\mu_4)_i^{k+1/2}. \tag{9}$$

In the second subproblem (the set of linear algebraic equations systems for all $y_j \in \omega_{h_2}$), nonlocal integral conditions (2), (3) are approximated by the trapezoidal rule and this subproblem is fully-implicit with respect to x:

$$\frac{U_{ij}^{k+1} - U_{ij}^{k+1/2}}{\tau} = \Lambda_1 U_{ij}^{k+1}, \quad x_i \in \omega_{h_1}, \tag{10}$$

$$U_{0j}^{k+1} = \gamma_1 (1, U)_j^{k+1} + (\mu_1)_j^{k+1}, \tag{11}$$

$$U_{N_1 j}^{k+1} = \gamma_2 (1, U)_j^{k+1} + (\mu_2)_j^{k+1}, \tag{12}$$

where

$$(1, U)_j^{k+1} = h_1 \biggl(\frac{U_{0j}^{k+1} + U_{N_1 j}^{k+1}}{2} + \sum_{i=1}^{N_1 - 1} U_{ij}^{k+1} \biggr).$$

Every transition is finished by calculating

$$U_{i0}^{k+1} = (\mu_3)_i^{k+1}, \quad U_{iN_2}^{k+1} = (\mu_4)_i^{k+1}, \quad x_i \in \overline{\omega}_{h_1}. \tag{13}$$

Thus, the procedure of numerical solution can be stated as follows:

procedure `The LOD Method`
begin
  Calculate U_{ij}^0, $(x_i, y_j) \in \overline{\omega}$, from Eqs. (6);
  for $k = 0, 1, \ldots, M-1$
    for each $x_i \in \omega_{h_1}$
      Solve system (7)–(9) and calculate $U_{ij}^{k+1/2}$, $y_j \in \overline{\omega}_{h_2}$;
    end for
    for each $y_j \in \omega_{h_2}$
      Solve system (10)–(12) and calculate U_{ij}^{k+1}, $x_i \in \overline{\omega}_{h_1}$;
    end for
    Calculate U_{i0}^{k+1} and $U_{iN_2}^{k+1}$, $x_i \in \overline{\omega}_{h_1}$, from Eqs. (13);
  end for
end

It is noteworthy that we can use the well-known Thomas algorithm and efficiently solve systems (7)–(9) because of the tridiagonality of their matrices. In order to solve systems

(10)–(12), the modification of the general algorithm[9] for solving linear equations systems with quasi-tridiagonal matrices can be used.

It is known[8] that the finite-difference scheme (7)–(12) approximates the two-dimensional differential problem (1)–(5) with error $\mathrm{O}(\tau + h_1^2 + h_2^2)$.

Now let us transform the finite-difference scheme (7)–(12) to the matrix form. From Eqs. (11) and (12) we obtain

$$U_{0j}^{k+1} = \overline{\alpha} \sum_{i=1}^{N_1-1} U_{ij}^{k+1} + (\overline{\mu}_1)_j^{k+1},$$
$$U_{N_1 j}^{k+1} = \overline{\beta} \sum_{i=1}^{N_1-1} U_{ij}^{k+1} + (\overline{\mu}_2)_j^{k+1},$$

where

$$\overline{\alpha} = \frac{\gamma_1 h_1}{D}, \quad \overline{\beta} = \frac{\gamma_2 h_1}{D},$$
$$(\overline{\mu}_1)_j^{k+1} = \Big(\frac{1}{D} - \frac{\overline{\beta}}{2}\Big)(\mu_1)_j^{k+1} + \frac{\overline{\alpha}}{2}(\mu_2)_j^{k+1},$$
$$(\overline{\mu}_2)_j^{k+1} = \frac{\overline{\beta}}{2}(\mu_1)_j^{k+1} + \Big(\frac{1}{D} - \frac{\overline{\alpha}}{2}\Big)(\mu_2)_j^{k+1},$$
$$D = 1 - \frac{h_1}{2}(\gamma_1 + \gamma_2).$$

If $M_1 = \max\{|\gamma_1|, |\gamma_2|\} < \infty$ and the grid step $h_1 < 1/M_1$, then $D > 0$.

Let us introduce $(N_1-1)\times(N_1-1)$ and $(N_2-1)\times(N_2-1)$ matrices

$$\widetilde{\Lambda}_1 = h_1^{-2} \begin{pmatrix} -2+\overline{\alpha} & 1+\overline{\alpha} & \overline{\alpha} & \cdots & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ \overline{\beta} & \overline{\beta} & \overline{\beta} & \cdots & \overline{\beta} & 1+\overline{\beta} & -2+\overline{\beta} \end{pmatrix}$$

and

$$\widetilde{\Lambda}_2 = h_2^{-2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

Now we can define matrices of order $(N_1 - 1) \cdot (N_2 - 1)$,

$$A_1 = -E_{N_2-1} \otimes \widetilde{\Lambda}_1, \quad A_2 = -\widetilde{\Lambda}_2 \otimes E_{N_1-1},$$

where E_N is the identity matrix of order N and $A \otimes B$ denotes the Kronecker (tensor) product of matrices A and B. We can directly verify that A_1 and A_2 are commutative matrices, i.e.,

$$A_1 A_2 = A_2 A_1 = \widetilde{\Lambda}_2 \otimes \widetilde{\Lambda}_1.$$

Introducing the matrices A_1 and A_2 allow us to rewrite the finite-difference scheme (7)–(12) in the following form:

$$(E + \tau A_2) U^{k+1/2} = U^k + \tau F^{k+1/2}, \tag{14}$$

$$(E + \tau A_1) U^{k+1} = U^{k+1/2}, \tag{15}$$

where E is the identity matrix of order $(N_1 - 1) \cdot (N_2 - 1)$,

$$U = \left(\widetilde{U}_1, \widetilde{U}_2, \ldots, \widetilde{U}_j, \ldots, \widetilde{U}_{N_2-1}\right)^T, \quad \widetilde{U}_j = \left(U_{1j}, U_{2j}, \ldots, U_{ij}, \ldots, U_{N_1-1,j}\right)^T,$$

and

$$F^{k+1/2} = \left(F_1^{k+1/2}, F_2^{k+1/2}, \ldots, F_j^{k+1/2}, \ldots, F_{N_2-1}^{k+1/2}\right)^T,$$

$$F_1^{k+1/2} = \left(\frac{(\mu_3)_1^{k+1/2}}{h_2^2} + f_{11}^{k+1/2}, \frac{(\mu_3)_2^{k+1/2}}{h_2^2} + f_{21}^{k+1/2}, \ldots, \frac{(\mu_3)_{N_1-1}^{k+1/2}}{h_2^2} + f_{N_1-1,1}^{k+1/2}\right)^T,$$

$$F_j^{k+1/2} = \left(f_{1j}^{k+1/2}, f_{2j}^{k+1/2}, \ldots, f_{ij}^{k+1/2}, \ldots, f_{N_1-1,j}^{k+1/2}\right)^T, \quad j = 2, 3, \ldots, N_2 - 2,$$

$$F_{N_2-1}^{k+1/2} = \left(\frac{(\mu_4)_1^{k+1/2}}{h_2^2} + f_{1,N_2-1}^{k+1/2}, \frac{(\mu_4)_2^{k+1/2}}{h_2^2} + f_{2,N_2-1}^{k+1/2}, \ldots, \frac{(\mu_4)_{N_1-1}^{k+1/2}}{h_2^2} + f_{N_1-1,N_2-1}^{k+1/2}\right)^T.$$

From Eqs. (14) and (15) it follows that

$$U^{k+1} = S U^k + \overline{F}^k, \tag{16}$$

where

$$S = (E + \tau A_1)^{-1} (E + \tau A_2)^{-1},$$

$$\overline{F}^k = \tau (E + \tau A_1)^{-1} (E + \tau A_2)^{-1} F^{k+1/2}.$$

We assume that the existence of the matrices $(E + \tau A_1)^{-1}$ and $(E + \tau A_2)^{-1}$ is ensured by the formulation of the considered two-dimensional differential problem and the proposed finite-difference scheme.

3 ANALYSIS OF THE STABILITY

Let us recall some facts related with the stability of the finite-difference schemes. The finite-difference scheme (16) is called stepwise stable[10] if for all fixed τ and h_1, h_2 there exists a constant $C = C(\tau, h_1, h_2)$ such that $|U_{ij}^k| \leqslant C$, $(x_i, y_j) \in \overline{\omega}$, $k = 0, 1, \ldots$. We know[8] that a sufficient stability condition for the finite-difference scheme (16) can be written in the form

$$\|S\| \leqslant 1 + c_0\tau,$$

where a non-negative constant c_0 is independent on τ and h_1, h_2. The necessary and sufficient condition to define a matrix norm $\|\cdot\|_*$ such that $\|S\|_* < 1$ is the inequality[11]

$$\rho(S) = \max_{\lambda(S)} |\lambda(S)| < 1,$$

where $\lambda(S)$ is the eigenvalues of S and $\rho(S)$ is the spectral radius of S. If S is a simple-structured matrix, i.e., the number of linearly independent eigenvectors is equal to the order of the matrix, then it is possible to define the norm[2]

$$\|S\|_* = \|P^{-1}SP\|_\infty = \rho(S),$$

which is compatible with the vector norm

$$\|U\|_* = \|P^{-1}U\|_\infty,$$

where columns of the matrix P are linearly independent eigenvectors of S, and the norms

$$\|B\|_\infty = \max_{1\leqslant i\leqslant m} \sum_{j=1}^{m} |b_{ij}|, \quad \|V\|_\infty = \max_{1\leqslant i\leqslant m} |v_i|,$$

m is the order of matrix $B = (b_{ij})_{i,j=1}^m$ and vector $V = (v_1, v_2, \ldots, v_m)^T$. Therefore, we will use the stability condition $\rho(S) < 1$ in the analysis of the stability of the finite-difference scheme (16).

The eigenvalue problem for the matrix $(-\widetilde{\Lambda}_1)$ has been investigated by M. Sapagovas[1]. When $\gamma_1 + \gamma_2 \leqslant 2$, then all the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ are non-negative and algebraically simple real numbers: $\lambda_i(-\widetilde{\Lambda}_1) \geqslant 0$, $i = 1, 2, \ldots, N_1 - 1$. If $\gamma_1 + \gamma_2 > 2$, then there exists one and only one negative eigenvalue of the matrix $(-\widetilde{\Lambda}_1)$. It is well-known[8] that all the eigenvalues of the matrix $(-\widetilde{\Lambda}_2)$ are real, positive and algebraically simple:

$$\lambda_j(-\widetilde{\Lambda}_2) = \frac{4}{h_2^2} \sin^2 \frac{j\pi h_2}{2}, \quad j = 1, 2, \ldots, N_2 - 1. \tag{17}$$

Since A_1 and A_2 are simple-structured matrices as Kornecker products of two simple-structured matrices, then S is a simple-structured matrix too, and the eigenvalues of the matrix S can be expressed as follows:

$$\lambda(S) = \frac{1}{\big(1 + \tau\lambda(A_1)\big)\big(1 + \tau\lambda(A_2)\big)}. \tag{18}$$

Moreover, if the matrix A_1 is such that $\lambda(A_1) \geqslant 0$ for all its real eigenvalues and $\mathrm{Re}\lambda(A_1) \geqslant 0$ for all complex eigenvalues, then $|\lambda(S)| < 1$ for all the eigenvalues of the matrix S, i.e., the LOD method is stable. The eigenvalues of the matrix A_1 coincide with the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ and they are multiple. Thus, the LOD method is stable if all the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ are non-negative, i.e., if $\gamma_1 + \gamma_2 \leqslant 2$. The non-negativity of the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ ensures the stability of the finite-difference scheme (16), but it is notable[7] that the scheme can be stable even if there exists a negative eigenvalue of the matrix $(-\widetilde{\Lambda}_1)$.

4 NUMERICAL RESULTS

In order to demonstrate the efficiency of the considered LOD method and practically justify the stability analysis technique, we solved a particular test problem. In our test problem, functions $f(x, y, t)$, $\mu_1(y, t)$, $\mu_2(y, t)$, $\mu_3(x, t)$, $\mu_4(x, t)$ and $\varphi(x, y)$ were chosen so that the function

$$u(x, y, t) = x^3 + y^3 + t^3$$

would be the solution to the differential problem (1)–(5), i.e.,

$$\begin{gathered}
f(x, y, t) = -3(2x + 2y - t^2),\\
\mu_1(y, t) = y^3 + t^3 - \gamma_1(0.25 + y^3 + t^3),\\
\mu_2(y, t) = 1 + y^3 + t^3 - \gamma_2(0.25 + y^3 + t^3),\\
\mu_3(x, t) = x^3 + t^3,\\
\mu_4(x, t) = x^3 + 1 + t^3,\\
\varphi(x, y) = x^3 + y^3.
\end{gathered}$$

The LOD method was implemented in a stand-alone C application. All numerical experiments with $\tau = 10^{-4}$, $h_1 = h_2 = 10^{-2}$, $T = 2.0$ and with different values of γ_1, γ_2 were performed using the technologies of grid computing. To estimate the accuracy of the numerical solution, we calculated the maximum norm of computational error,

$$\|\varepsilon\|_{C_h} = \max_{0\leqslant k\leqslant M} \max_{\substack{0\leqslant i\leqslant N_1\\ 0\leqslant j\leqslant N_2}} |U_{ij}^k - u(x_i, y_j, t^k)|.$$

Note that

$$\min_{0\leqslant t\leqslant T} \min_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant y\leqslant 1}} u(x, y, t) = u(0, 0, 0) = 0, \qquad \max_{0\leqslant t\leqslant T} \max_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant y\leqslant 1}} u(x, y, t) = u(1, 1, T) = 10.$$

Similarly as in paper of S. Sajavičius and M. Sapagovas[12], for the numerical analysis of the spectrum of the matrix S, MATLAB (The MathWorks, Inc.) software package[13] was used. The eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ were calculated numerically. Then all different eigenvalues of the matrix S were calculated using expressions (17) and formula (18).

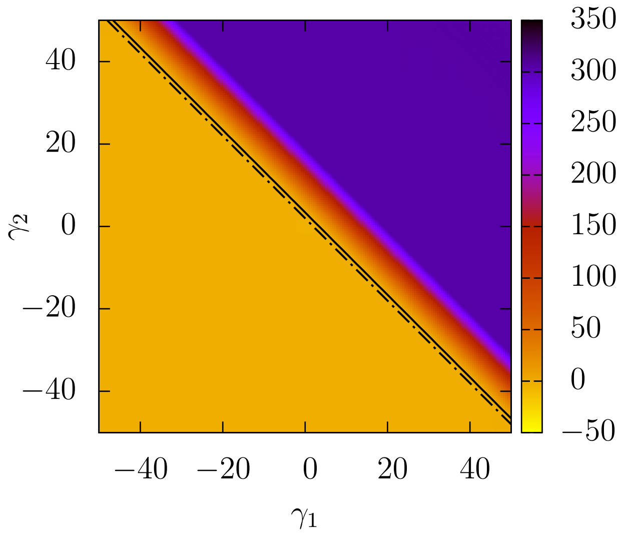

Figure 1: The dependence of $\log_{10}\|\varepsilon\|_{C_h}$ on the values of parameters γ_1 and γ_2. The dash-dot and solid straight lines denote the lines $\gamma_1+\gamma_2=2$ and $\gamma_1+\gamma_2=\gamma^*$, respectively.

The numerical analysis of the spectrum of the matrix S shown that all the eigenvalues of the matrix S hold property $|\lambda(S)|<1$ when $\gamma_1+\gamma_2\leqslant\gamma^*\approx 3.42366$. The dependence of $\log_{10}\|\varepsilon\|_{C_h}$ on the values of parameters γ_1 and γ_2 are presented in Fig. 1. We can see how the values of $\|\varepsilon\|_{C_h}$ grow when $\gamma_1+\gamma_2$ becomes greater than γ^*.

Note that the case $\gamma_1=0$ and $\gamma_2=0$ corresponds to the differential problem with classical boundary conditions and it is known[8] that the LOD method is stable in this case. If, for example, $\gamma_1=0$ and $\gamma_2\neq 0$, then we have the problem with classical boundary conditions (2), (4) and nonlocal integral condition (3). From Fig. 2 we see that in this case the norm $\|\varepsilon\|_{C_h}$ starts to grow when $2<\gamma_2\leqslant\gamma^*$ and the growing becomes extremely fast when $\gamma_2>\gamma^*$.

5 CONCLUDING REMARKS

We developed the LOD method for the two-dimensional parabolic equation with two nonlocal integral conditions. Applying quite a simple technique allow us to investigate the stability of this method. The stability analysis technique is based on the analysis of the spectrum of the transition matrix of a finite-difference scheme. We demonstrate that the proposed LOD method can be stable or unstable depending on the parameters of nonlocal conditions. The results of numerical experiment with a particular test problem justify theoretical results.

The LOD method can be generalized for the corresponding two-dimensional differential

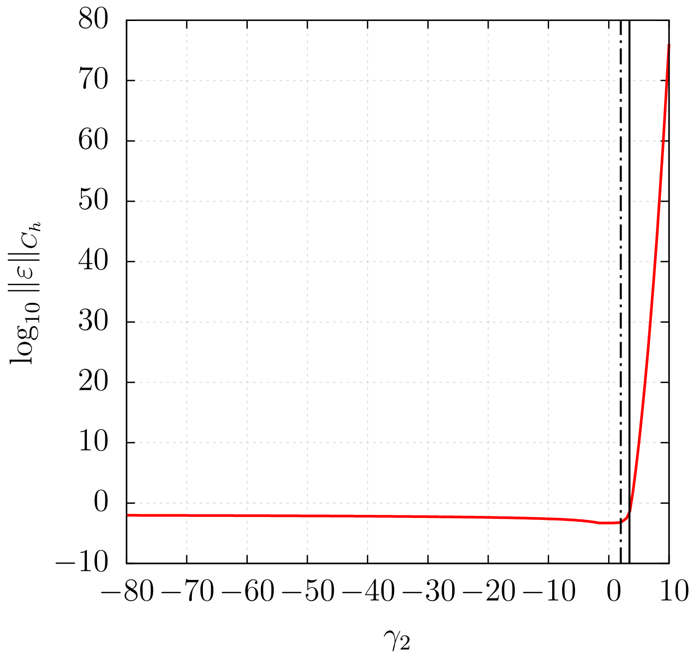

Figure 2: The dependence of $\log_{10}\|\varepsilon\|_{C_h}$ on the values of parameter γ_2 ($\gamma_1 = 0$). The dash-dot and solid vertical straight lines denote the lines $\gamma_2 = 2$ and $\gamma_2 = \gamma^*$, respectively.

problem with nonlocal integral conditions

$$u(0, y, t) = \gamma_1 \int_0^1 \alpha(x)u(x, y, t)\mathrm{d}x + \mu_1(y, t),$$
$$u(1, y, t) = \gamma_2 \int_0^1 \beta(x)u(x, y, t)\mathrm{d}x + \mu_2(y, t), \quad 0 < y < 1, \quad 0 < t \leqslant T,$$

where $\alpha(x)$ and $\beta(x)$ are given functions.

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