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# GUARANTEED A POSTERIORI ERROR ESTIMATES FOR VISCOUS FLOW PROBLEMS

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Abstract. In the talk, we discuss a posteriori error estimates for elliptic and parabolic viscous flow problems and give an overview of the results obtained in this field with the help of a new (functional) approach that was earlier applied to many problems in mathematical physics. This method provides guaranteed and computable error bounds that do not involve mesh dependent constants and are valid for any approximation from the energy space. Functional a posteriori error estimates has been derived on purely functional grounds with the help of the techniques that are close to those used in the analysis of existence and regularity of boundary value problems. They do not attract specific properties of approximations or method used (such as, e.g., Galerkin orthogonality, higher regularity, superconvergence effects). Problems considered in rotating frame (which involve additional terms generated by Coriolis force) are especially interesting in atmosphere and ocean models. We present forms of a posteriori estimates derived for such type (elliptic and parabolic) problems that give guaranteed upper bounds for the energy norm of the error and also provide reliable error indicators. Computational properties of the estimates are demonstrated by a number of numerical examples.

## **1** INTRODUCTION

In the talk we give an overview of new a posteriori error estimation methods that in the last decade were developed for mathematical models in fluid mechanics. These estimates

- contain no mesh-dependent constants;
- valid for any conforming approximation;
- provide *computable* and *guaranteed* bounds of approximation errors.

They provide *fully reliable verification* of approximate solutions obtained by various numerical methods and can be efficiently used in scientific and engineering computations.

First estimates of such a type were obtained by a variational technique<sup>1,2,3</sup>. Later, a different derivation based upon the analysis of integral identities has been developed<sup>4,5</sup>.

### 2 Stokes problem

This classical model in the theory of viscous incompressible is represented by the relations

$$u_t - \nu \Delta u = f - \nabla p \qquad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} u = 0, \tag{2}$$

$$u(x,0) = \hat{u}(x), \tag{3}$$

$$u = u_0 \qquad \text{on } \partial\Omega, \tag{4}$$

where u is the velocity field, p is the pressure function,  $\nu > 0$  is the viscosity parameter, and  $\hat{u}(x)$  and  $u_0$  are solenoidal functions that define the initial and boundary conditions, respectively.

#### 2.1 Stationary Stokes problem

In the stationary case, the problem is to find u(x) and p(x) such that

$$-\nu\Delta u = f - \nabla p \qquad \text{in } \Omega, \tag{5}$$

$$\operatorname{div} u = 0, \tag{6}$$

$$u = u_0$$
 on  $\partial\Omega$ . (7)

Henceforth, we assume that

$$f \in L_2(\Omega, \mathbb{R}^n)$$
 and  $u_0 \in \overset{\circ}{J}(\Omega)$ ,

where  $J(\Omega)$  denotes the closure of smooth solenoidal functions with compact supports in  $\Omega$  with respect to the norm of  $H^1(\Omega, \mathbb{R}^d)$ . We denote  $H^1(\Omega, \mathbb{R}^d)$  by V and define  $V_0$  as the subspace of V containing the functions with zero traces on  $\partial\Omega$  (for problems with

mixed boundary conditions  $V_0$  contains functions vanishing on the Dirichlét part of the boundary). We recall that the (Friedrichs) inequality

$$\|w\| \le C_{F\Omega} \|\nabla w\| \tag{8}$$

holds for  $w \in V_0$ . The set  $\mathring{J}(\Omega) + u_0$  consists of functions  $w + u_0$ , where  $w \in \mathring{J}(\Omega)$ . The space of square summable functions with zero mean is denoted by  $L_0$ .

A generalized solution  $u \in \overset{\circ}{J}(\Omega) + u_0$  of (5)–(7) is defined by the integral relations

$$\int_{\Omega} \nu \nabla u : \nabla w \, dx = \int_{\Omega} f \cdot w \, dx, \qquad \forall w \in \overset{\circ}{J}(\Omega).$$
(9)

or

$$\int_{\Omega} \nu \nabla u : \nabla w \, dx = \int_{\Omega} (f \cdot w + p \operatorname{div} w) \, dx, \quad w \in V_0.$$
(10)

It was shown<sup>4,5</sup> that the above integral relations imply the estimate

$$\nu \|\nabla (u-v)\| \le \|\tau + q\mathbf{I} - \nu \nabla v\| + C_{F\Omega} \|\operatorname{Div} \tau + f\|,$$
(11)

where v is an arbitrary solenoidal function satisfying the boundary conditions. It should be noted that the right hand side does not involve unknown functions and it is directly computable.

If  $q \in H^1(\Omega)$ , then a somewhat different form of the estimate follows by changing  $\tau$  to  $\eta$ , where

$$\tau = \eta - q\mathbf{I},\tag{12}$$

which gives

$$\nu \|\nabla (u-v)\| \le \|\eta - \nu \nabla v\| + C_{F\Omega} \|\operatorname{Div} \eta + f - \nabla q\|.$$
(13)

Estimates (11) and (13) have a clear meaning. Estimate (11) shows that the upper bound of the error can be represented as the sum of two parts related to the decomposition of the Stokes system as

$$\sigma = -p\mathbf{I} + \nu\nabla u,$$
  
-Div $\sigma = f.$ 

Its right-hand side vanishes if and only the above relations are exactly satisfied. Since v is a solenoidal field satisfying the boundary condition, the right-hand side of the majorant is zero if and only if v = u. Similarly, (13) shows that the upper bound of the error can S. Repin

be represented as the sum of two parts related to the decomposition of the Stokes system as

$$\bar{\sigma} = \nu \nabla u, \qquad -\text{Div}\bar{\sigma} = f - \nabla p.$$

If v belongs to a wider class which includes  $H^1$  functions satisfying the boundary conditions, then getting an upper bound of the distance to u exploits the following result.

**Lemma.** Let  $\Omega$  be a bounded domain with Lipschitz continuous boundary. Then, for any function  $f \in L_0$  one can find a function  $w_f \in V_0$  such that  $\operatorname{div} w_f = f$  and

$$\|\nabla w_f\| \le c_\Omega \|f\|,\tag{14}$$

where  $c_{\Omega}$  is a positive constant dependent only on  $\Omega$ .

For n = 2 it was proved by Babuska and A. K. Aziz and in the general case by O. Ladyzhenskaya and V. Solonnikov.

For nonsolenoidal approximations, we have the estimate

$$\nu \|\nabla (u - \hat{v})\| \le \|\tau + q\mathbf{I} - \nu \nabla \hat{v}\| + C_{F\Omega} \|\operatorname{Div} \tau + f\| + 2\nu c_{\Omega} \|\operatorname{div} \hat{v}\|,$$
(15)

where  $\tau \in H(\Omega, \text{Div})$  and  $q \in L_0$  (which means that the function has zero mean).

Estimates of ||p-q|| can also be derived with the help of Lemma. The main idea is as follows. Since  $(p-q) \in L_0$ , we know that

$$\operatorname{div} \widetilde{w} = p - q \quad \text{and} \quad \|\nabla \widetilde{w}\| \le c_{\Omega} \|p - q\|$$

for a certain vector-valued function  $\tilde{w} \in V_0$ . Hence,

$$||p-q||^2 = \int_{\Omega} \operatorname{div} \widetilde{w}(p-q) \, dx$$

Similar estimates can be derived for the generalized Stokes problem<sup>6</sup>, models with nonlinear viscosity<sup>4,7</sup>,

$$\mu u - \nu \Delta u = f - \nabla p \quad \text{in } \Omega, \tag{16}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega. \tag{17}$$

and for the evolutionary problem  $(1)-(4)^8$ .

#### 3 Oseen problem

Oseen problem is considered as a linearization of the Navier–Stokes system

$$-\operatorname{Div}(\nu\nabla u) + \operatorname{Div}(u \otimes u) = f - \nabla p, \qquad (18)$$

$$\operatorname{div} u = 0, \tag{19}$$

$$u = u_0 \qquad \text{on } \partial\Omega. \tag{20}$$

where  $\Omega$  is a connected bounded domain with Lipschitz boundary  $\partial\Omega$ ,  $\nu > 0$  is the viscosity parameter,  $u_0$  is given vector-valued functions such that  $\operatorname{div} u_0 = 0$ ,  $u \otimes w$  is the tensor with components  $\{u \otimes w\}_{ij} := u_i w_j$ . In the Oseen formulation, the convective term is replaced by  $\operatorname{div}(a \otimes u)$ , where a is a given solenoidal vector-valued function.

For this problem, the estimate analogous to (15) has been obtained<sup>4,5</sup>.

$$\nu \|\nabla (u - \hat{v})\| \leq \|\tau + q\mathbf{I} + \mathbf{a} \otimes v - \nu \nabla \hat{v}\| + C_{\nu\Omega} \|f + \operatorname{Div}\tau\| + \overline{c}_{\Omega} \|\operatorname{div}\hat{v}\|, \quad (21)$$

where  $\overline{c}_{\Omega}$  depends on  $c_{\Omega}$ ,  $C_{\nu\Omega}$ , ||a||, and  $\nu$ .

### 4 Viscous flow problems in rotating coordinate system

In certain models, the Navier-Stokes problem is considered in a rotating coordinate system. Then, additional terms arise in the equation of motion, which has the form

$$\partial_t u + \operatorname{div}(u \times u) + 2\omega \times v + \omega \times (\omega \times r) - \operatorname{Div}\sigma = f, \tag{22}$$

$$\sigma = -p\mathbf{I} + \nu\varepsilon(u),\tag{23}$$

$$\operatorname{div} u = 0, \tag{24}$$

$$u = u_0 \qquad \text{on } \partial\Omega.$$
 (25)

In (22), the term  $2\omega \times v$  is due to the Coriolis force and the term  $\omega \times (\omega \times r)$  is related to the centrifugal force (the latter term is usually appended to the source function and disappears from the equation). The vector  $\omega$  is oriented along the axis  $x_3$ , and its value depends on the rotation velocity. Mathematical properties of such models were studied by a number of authors<sup>9</sup>.

A linearized version of (22)–(25) can be viewed as a certain generalization of the Stokes problem. It is defined by the relations

$$-\mathrm{Div}\sigma + \mu u + \omega \times u = f, \tag{26}$$

$$\sigma = -p\mathbf{I} + \nu\varepsilon(u),\tag{27}$$

$$\operatorname{div} u = 0, \tag{28}$$

$$u = u_0 \qquad \text{on } \partial\Omega. \tag{29}$$

This system of equations arises if the problem is solved by semi-discrete approximations (then  $\mu > 0$  comes from an approximation of the term  $\partial_t u$ ).

A generalized solution of the problem (26)–(29) is defined by the integral relation

$$\int_{\Omega} (\nu \nabla u : \nabla w + \mu u \cdot w + (\omega \times u) \cdot w) dx = \int_{\Omega} f \cdot w dx.$$

which holds for any  $w \in \overset{\circ}{J}(\Omega)$ . As in the models considered before, this integral relation generates an estimate of the difference between u and any  $v \in \overset{\circ}{J} + u_0$ .

Let  $\tau \in H_{\text{Div}}, q \in L_0$ , and define

$$\mathbf{r}(v,\tau) := f - \mu v - \omega \times v + \mathrm{Div}\tau,$$
$$\mathbf{d}(v,\tau,q) := \tau + q\mathbf{I} - \nu\nabla v.$$

We obtain the estimate<sup>10</sup> in terms of the weighted norm

$$\|u - v\|_{\nu\mu}^{2} \leq \left( \|\frac{1}{\sqrt{\nu}} \mathrm{d}(v, \tau, q)\| + C_{\nu\Omega} \|(1 - \alpha) \mathrm{r}(v, \tau)\| \right)^{2} + \|\frac{\alpha}{\sqrt{\mu}} \mathrm{r}(v, \tau)\|^{2}, \quad (30)$$

which is valid for any  $\alpha \in [0, 1]$ . Estimates for nonsolenoidal velocity fields and for approximations of the pressure function can be also derived<sup>10</sup>.

### 5 Computational applications

From the computational point of view the estimates has been studied in <sup>11,12,13</sup>. Below we represent one example associated with a flow throughout a rotating cylinder (Fig. 1) On the next picture (Fig. 2), we depict errors computed by comparing with a referenced



Figure 1: Test problem

solution (constructed on a very fine mesh) and errors found by the error majorant. For two different marking criteria the results are quite similar.

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Figure 2: Indication of errors ("bulk" and "max" criteria)

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