# BOUNDARY ALGORITHMS FOR HIGH-ACCURACY AEROACOUSTIC SCHEMES 

Ludwig W. Dorodnicyn<br>Moscow State University,<br>Faculty CMC, Vorobievy gory, 119992 Moscow, Russia<br>e-mail: dorodn@cs.msu.su

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Abstract. For high-accuracy finite-difference approximations to hyperbolic equations, namely, DRP scheme by Tam and Webb and its modifications by Bailly and Bogey, boundary conditions have been constructed. The dispersion relations of discrete equations at near-boundary nodes must be close to the dispersion of a governing scheme. Starting from the $1 D$ transport equation, the technique is extended to the Euler equations both in one and two dimensions.

## 1 INTRODUCTION

Numerical simulation of aeroacoustic problems requires high accuracy of discrete approximations to governing equations. On uniform spatial grids, finite differences preserving dispersion relations are widely used, including DRP scheme by Tam and Webb ${ }^{1}$ and its modifications by Bailly and Bogey ${ }^{2}$.

We consider schemes involving wide grid stencils: 7 -point schemes from ${ }^{1}$ and up to 11point from ${ }^{2}$. This fact complicates the imposition of boundary conditions, because such finite-difference operators are inapplicable not only at boundary nodes, but also within some areas near boundaries.

To overcome this difficulty, non-centered differences were applied in ${ }^{3}$. Meanwhile, such formulations are commonly stable for the outflow boundary and unstable for the inflow boundary. $\mathrm{In}^{2}$, reduced stencils were used near boundaries at the expense of decreasing the scheme accuracy. Mention also papers ${ }^{4,5}$, where some kinds of non-centered differences near the boundaries were implemented.

We continue our previous research (reported at ECCOMAS 2008 Congress) dealt with algorithms for 1D problems - the transport equation and the linearized Euler equations. The approach is based on the technique of 'consistent boundary conditions', that combines the idea of 'discrete nonreflecting boundary conditions' from ${ }^{6,7}$ with dispersion relation optimization. Namely, dispersion relations suggested by the internal scheme are accurately reproduced by some discrete operators in a number of near-boundary nodes. This procedure treats both "physical" modes and spurious sawtooth modes that appear in the scheme chosen.

The time advancing is performed with the use of explicit Runge-Kutta methods by Bailly and Bogey ${ }^{2}$.

At present, we have extended the discrete boundary technique to two-dimensional problems. For the 2D transport equation, the artificial boundary conditions are accurate as well as in the 1D case. For the 2D linearized Euler equations, the asymptotic boundary conditions by Tam and Webb ${ }^{1}$ have been modified, and the discrete consistent approximation has been implemented to them.

The algorithms are validated on numerical simulation of one- and two-dimensional test problems.

## 2 LARGE-STENCIL SCHEMES AND BOUNDARY CONDITIONS

Most of the technique can be explained for the linear one-dimensional transport equation

$$
\begin{equation*}
\partial u / \partial t+\partial u / \partial x=0, \quad 0<x<X \tag{1}
\end{equation*}
$$

Consider a uniform spatial grid

$$
x_{0}=0, \quad x_{j}=x_{0}+j h, j=0, \ldots, N, \quad x_{N}=X
$$

In the class of high-accuracy algorithms studied here, derivatives in space and time are treated separately. Consider a semi-discrete (continuous in time and finite-difference in space) form of the schemes proposed in ${ }^{1,2}$ represented generally as

$$
\begin{equation*}
d u_{j} / d t+\frac{1}{h} \sum_{l=-m}^{m} a_{l} u_{j+l}=0, \quad j=m, \ldots, N-m \tag{2}
\end{equation*}
$$

Particularly, the 7-point DRP scheme ${ }^{1}$ looks like

$$
\begin{equation*}
d u_{j} / d t+\frac{1}{h} \sum_{l=-3}^{3} a_{l} u_{j+l}=0, \quad j=3,4, \ldots, N-3 \tag{3}
\end{equation*}
$$

Describe briefly the method to construction of schemes from class (2). The aim is to replace derivative $d u / d x$ with a highly accurate discrete analogue

$$
D_{x}^{h} u \equiv \frac{1}{h} \sum_{j=-m}^{m} a_{j} u_{j}
$$

Inserting

$$
u(x)=\exp (i k x), \quad \text { where } \quad 0 \leq k \leq \pi / h
$$

we obtain

$$
\begin{equation*}
d u / d x=i k u, \quad D_{x}^{h} u=\frac{1}{h} \sum_{j=-m}^{m} a_{j} e^{i j \varphi} u(x) \equiv i \tilde{k} u \tag{4}
\end{equation*}
$$

where

$$
\varphi=k h
$$

The effective, or modified, wavenumber $\tilde{k}$ has been introduced. Coefficients $a_{j}$ are to be chosen to provide similarity between the modified and the exact wavenumbers $\tilde{k} \approx k$.

The first requirement to operator $D_{x}^{h}$ is its approximation $O\left(h^{4}\right)$. Applying to $k$ and $\tilde{k}$ in (4) the differentiation with respect to $k$ of orders from 0 to 4 at $k=0$, one obtains the conditions of approximation $O\left(h^{4}\right)$ as

$$
\sum_{j=-m}^{m} a_{j}=0, \quad \sum_{j=-m}^{m} j a_{j}=1, \quad \sum_{j=-m}^{m} j^{n} a_{j}=0, \quad n=2,3,4 .
$$

Next, under these restrictions, the dispersion relation optimization, e.g.,

$$
\int_{0}^{L}\left|\sum_{j=-m}^{m} a_{j} e^{i j \varphi}-i \varphi\right|^{2} d \varphi \rightarrow \min _{\left\{a_{j}\right\}}
$$

is performed. In standard DRP scheme (3), $m=3$ and $L=\pi / 2$.

We have stated the general formulation. In fact, the problem is simplified owing to the symmetry of the coefficients

$$
\begin{equation*}
a_{j}=-a_{-j}, \quad j=0, \ldots, m \tag{5}
\end{equation*}
$$

This follows the symmetry of the stencil and can be guessed in advance. Consequently, the conditions of approximation $O\left(h^{4}\right)$ take the form

$$
2 \sum_{j=1}^{m} j a_{j}=1, \quad \sum_{j=1}^{m} j^{3} a_{j}=0
$$

For 7-point DRP scheme from ${ }^{1}$, the optimization problem is

$$
\int_{0}^{\pi / 2}\left(2 \sum_{j=1}^{3} a_{j} \sin (j \varphi)-\varphi\right)^{2} d \varphi \rightarrow \min _{\left\{a_{j}\right\}}
$$

In algorithms ${ }^{2}$, the error estimation is essentially different:

$$
\int_{\pi / 16}^{\pi / 2}\left|2 \sum_{j=1}^{m} a_{j} \sin (j \varphi)-\varphi\right| d(\ln \varphi) \rightarrow \min _{\left\{a_{j}\right\}}
$$

Let us compare the continuous and the discrete problems in their complete setup. For equation (1), an initial-boundary value problem is formulated, e.g.,

$$
\begin{equation*}
\partial u / \partial t+\partial u / \partial x=0, \quad 0<x<X, t>0, \quad u(0, t)=0, \quad u(x, 0)=f(x) \tag{6}
\end{equation*}
$$

Thus, the transport equation should be supplied with a left-hand boundary condition, while any prescriptions on the right-hand boundary are not needed. In what follows, the initial condition will not be taken into account.

Scheme (2) is specified for internal nodes only (we shall call it internal scheme). There exist two sets of nodes near the left and the right boundaries

$$
j=0, \ldots, m-1 \quad \text { and } \quad j=N-m+1, \ldots, N
$$

where the scheme equations should be altered. In this paper, equations specified at such nodes (to be called boundary nodes) will be treated as boundary conditions. In contrast to the unique left boundary condition for the differential equation (6), the large-stencil scheme (2) requires $m$ conditions on both the left-hand and the right-hand boundaries.

## 3 NORMAL MODE ANALYSIS AND NONREFLECTING BOUNDARY CONDITIONS

In order to understand how to specify boundary conditions, consider some properties of internal scheme (2) which rewrite in the form

$$
\begin{equation*}
\partial u / \partial t+D_{x}^{h} u=0 \tag{7}
\end{equation*}
$$

Normal mode

$$
u(x, t)=\exp \{i k x-i \omega t\}, \quad \text { where } 0 \leq k \leq \pi / h, \quad \omega \in \mathrm{R},
$$

obeying (7) satisfies the dispersion relation

$$
\begin{equation*}
\omega-\tilde{k}(k)=0 \tag{8}
\end{equation*}
$$

where $\tilde{k}(k)$ is the modified wavenumber from (4).


Figure 1: Effective wavenumber $\tilde{k} h$ versus $k h$ for schemes DRP (blue), FDo9p (orange), and FDo11p (magenta)

Figure 1 shows the dependences $\tilde{k}(k)$ for the standard 7 -point DRP scheme ${ }^{1}$ and two schemes from ${ }^{2}$ FDo9p and FDo11p, using 9 and 11 points, respectively. For comparison, the exact wavenumber $k$ is also plotted with thin line. We can see that all the schemes reproduce accurately modes in a large interval of wavenumbers. The greater number of points is involved the wider range is achieved. Then, as $k$ increases, modified wavenumber deflects substantially from the straight line and takes zero value at $k=\pi / h$. For the schemes considered, function $\tilde{k}(k)$ is positive in $0<k<\pi / h$.

Each scheme determines the critical wavenumber $k h=\varphi^{*}$ where limiting value $\tilde{k}=$ $\tilde{k}_{\max }$ is attained. In Fig. 1 these values are marked for the DRP scheme. For any frequency $\omega=\tilde{k}(k)$ lower than $\tilde{k}_{\text {max }}$, there exist exactly two values of $k$ responsible for different modes. The first mode $\left(0 \leq k h<\varphi^{*}\right)$ is a "physical" wave which corresponds to the solution of the differential equation (1). The second $\left(\varphi^{*}<k h \leq \pi\right)$ is a "spurious" wave of discrete nature. It is similar to a sawtooth grid-to-grid oscillation.

The waves described by scheme (7) propagate with group velocities ${ }^{8}$. The latter are obtained by differentiating equation (8) with respect to $k$,

$$
c_{g} \equiv d \omega / d k=d \tilde{k} / d k
$$

It is seen from Fig. 1 that physical modes propagate rightward and spurious do leftward.

## 4 CONSISTENT BOUNDARY CONDITIONS

The application of large-stencil schemes near the boundaries finds obstacles due to the fact that some grids involved in approximations are unavailable. Let us, not touching the time derivative, replace the spatial part of the operator by another linear combination of $u$-values. At each near-boundary node, non-symmetric stencils must be used. The technique will be based on an accurate reproduction of dispersion relations of outgoing waves suggested by the internal scheme. Only modes having real wavenumbers will be taken into account. This approach one may consider as an extension of 'discrete nonreflecting boundary conditions' for the compact 3-point scheme $O\left(h^{4}\right)$ stated in ${ }^{6}$.

First consider the procedure of obtaining boundary conditions for the general scheme (2). Then illustrate the results for the 7 -point DRP scheme from ${ }^{1}$.

In centered scheme (2), the discrete physical mode is outgoing for the right-hand boundary, and this kind of waves should be approximated on non-symmetric stencils. On the left-hand boundary, an outgoing mode is the spurious sawtooth wave propagating leftward.

For schemes (2) with stencil half-width $m$, we construct the set of right-hand boundary equations in the form

$$
\begin{equation*}
d u_{N-j} / d t+\frac{1}{h} \sum_{l=0}^{2 m} b_{j l} u_{N-l}=0, \quad j=0, \ldots, m-1 \tag{9}
\end{equation*}
$$

where the same number of nodes $2 m+1$ is involved.
Specify number $j$. For convenience, assume that the time derivative is applied at point $x=0$. However, this will not affect the problem formulation. Given set of coefficients

$$
a_{l}, \quad l=-m, \ldots, m
$$

Find coefficients

$$
b_{j l}, \quad l=0, \ldots, 2 m,
$$

with the purpose to make close the two sums

$$
\begin{equation*}
\sum_{l=-m}^{m} a_{l} u_{l} \sim \sum_{l=0}^{2 m} b_{j l} u_{j-l} \tag{10}
\end{equation*}
$$

The values of $b_{j l}$ must obey the conditions of approximation $O\left(h^{4}\right)$ which read

$$
\begin{equation*}
\sum_{l=0}^{2 m} b_{j l}=\sum_{l=-m}^{m} a_{l}, \quad \sum_{l=0}^{2 m}(j-l)^{n} b_{j l}=\sum_{l=-m}^{m} l^{n} a_{l}, \quad n=1,2,3,4 . \tag{11}
\end{equation*}
$$

and provide the dispersion relation optimization

$$
\begin{equation*}
\boldsymbol{\Phi}_{j}^{2}\left(b_{j l}\right)=\int_{0}^{L}\left|\sum_{l=0}^{2 m} b_{j l} e^{i(j-l) \varphi}-\sum_{l=-m}^{m} a_{l} e^{i l \varphi}\right|^{2} d \varphi \rightarrow \min \tag{12}
\end{equation*}
$$

Here, $L \leq \varphi^{*}$ should be chosen more or less arbitrarily.
The construction of left-hand boundary conditions is slightly more sophisticated. Nonetheless, it employs the same technique.

Let the set of left-hand boundary conditions be

$$
\begin{equation*}
d u_{j} / d t+\frac{1}{h} \sum_{l=0}^{2 m} c_{j l} u_{l}=0, \quad j=0, \ldots, m-1 \tag{13}
\end{equation*}
$$

For given node number $j$ and internal scheme coefficients $a_{l}$, find coefficients

$$
c_{j l}, \quad l=0, \ldots, 2 m
$$

in order to replace

$$
\begin{equation*}
\sum_{l=-m}^{m} a_{l} u_{l} \sim \sum_{l=0}^{2 m} c_{j l} u_{l-j} \tag{14}
\end{equation*}
$$

The left boundary must absorb oscillatory modes propagating leftward. Let us single out the sawtooth function and represent $u_{j}=(-1)^{j} v_{j}$ with a smooth continuous function

$$
v(x)=e^{i k x}, \quad k h \in\left[0, \pi-\varphi^{*}\right] .
$$

Now the previous procedure is applied to coefficients of $v_{j}$. After this, we return to the present notation.

The conditions of approximation $O\left(h^{4}\right)$ take the form

$$
\begin{equation*}
\sum_{l=0}^{2 m}(-1)^{l-j} c_{j l}=\sum_{l=-m}^{m}(-1)^{l} a_{l}, \quad \sum_{l=0}^{2 m}(-1)^{l-j}(l-j)^{n} c_{j l}=\sum_{l=-m}^{m}(-1)^{l} l^{n} a_{l}, \quad n=1,2,3,4 \tag{15}
\end{equation*}
$$

and the dispersion relation optimization results in

$$
\begin{equation*}
\boldsymbol{\Phi}_{j}^{2}\left(c_{j l}\right)=\int_{0}^{L}\left|\sum_{l=0}^{2 m}(-1)^{l-j} c_{j l} e^{i(l-j) \varphi}-\sum_{l=-m}^{m}(-1)^{l} a_{l} e^{i l \varphi}\right|^{2} d \varphi \rightarrow \min \tag{16}
\end{equation*}
$$

Here, $L \leq \pi-\varphi^{*}$.
By analogy with the effective wavenumber (4) defined by the governing equation, introduce modified wavenumbers of boundary equations. Systems (9) and (13) give, respectively,

$$
\begin{align*}
k^{*}=k_{j}^{*}=-i \frac{1}{h} \sum_{l=0}^{2 m} b_{j l} e^{i(j-l) \varphi}, & j=0, \ldots, m-1  \tag{17}\\
k^{*}=k_{j}^{*}=-i \frac{1}{h} \sum_{l=0}^{2 m} c_{j l} e^{i(l-j) \varphi}, & j=0, \ldots, m-1 \tag{18}
\end{align*}
$$

Let us summarize. The construction of high-accuracy schemes combines two kinds of grid approximation. The first is traditional, when at internal nodes the differential operator is replaced with a finite-difference analogue in order to minimize quantity

$$
|\tilde{k}-k|
$$

The second kind is uncommon, since at near-boundary nodes, the finite-difference operator already specified is approximated with another finite-difference operator yielding the minimization of

$$
\left|k^{*}-\tilde{k}\right|
$$

We will call such equations consistent boundary conditions, meaning their consistency with a governing numerical scheme. We stress that the previous boundary treatments from ${ }^{2-5}$ employed quantity

$$
\left|k^{*}-k\right| .
$$

Sets of right-hand boundary conditions have rather expectable and typical form. So far as physical scheme waves are close to their differential prototypes $(\tilde{k} \approx k)$, the coefficients determined by system of (11) and (12) occur close to the values from non-centered differences in ${ }^{3,4}$.

In contrast, left boundary conditions are very peculiar. They are associated with sawtooth modes which, first, have no relation to actual physics, and second, are specific for each scheme. Values of coefficients obtained from (15) and (16) strongly depend on the internal scheme.

Consider in detail standard 7-point DRP scheme (3) and both right-hand (9) and left-hand (13) boundary conditions for it:

$$
\begin{equation*}
d u_{N-j} / d t+\frac{1}{h} \sum_{l=0}^{6} b_{j l} u_{N-l}=0, \quad j=0,1,2 \tag{19}
\end{equation*}
$$



Figure 2: Effective wavenumbers $\tilde{k} h$ (solid line), $\operatorname{Re} k^{*} h$ (dotted line), and ( $-\operatorname{Im} k^{*} h$ ) (dash line) versus $k h$ for DRP scheme at nodes: (a) $N-2$, (b) $N-1$, (c) $N$, (d) 0 , (e) 1 , (f) 2 .

$$
\begin{equation*}
d u_{j} / d t+\frac{1}{h} \sum_{l=0}^{6} c_{j l} u_{l}=0, \quad j=0,1,2 . \tag{20}
\end{equation*}
$$

Examine how good such boundary equations reproduce the dispersion relation determined by the governing equation.

Begin with the right-hand boundary conditions consistent for length $L=\pi / 2$ which is equal to that chosen in ${ }^{1}$ for the internal scheme optimization.

Figure $2(\mathrm{a}-\mathrm{c})$ illustrates the comparison of modified wavenumbers obtained from governing equation (3) by formula (4) and from boundary equations (19) by Eq. (17). So far as function $k^{*}(k)$ in (17) has complex values, both its real and imaginary parts are shown.

All the three equations (19) display a good agreement between $\operatorname{Re} k^{*}(k)$ and $\tilde{k}(k)$ at the reference segment $k h \in[0, \pi / 2]$. Discrepancy increases as the node approaches the boundary.

Now proceed to left-hand boundary conditions. Take length $L=\pi / 4$ which is much less than $\pi-\varphi^{*}$.

Figure 2 (d-f) shows the modified wavenumbers defined by left boundary conditions (20) according to (18) as well as the effective wavenumber of governing equation (3). Oppositely to the right-hand conditions, the pairs of curves $\operatorname{Re} k^{*}(k)$ and $\tilde{k}(k)$ are very close in range $k h \in[3 \pi / 4, \pi]$ referring to sawtooth modes.

## 5 TIME ADVANCING

To the semi-discrete schemes constructed above, time-integration methods are applied. All the schemes, including their internal and boundary equations, are representable in the form

$$
\begin{equation*}
d U / d t=\mathbf{A} U \tag{21}
\end{equation*}
$$

where

$$
U=\left(\begin{array}{llllll}
u_{0} & u_{1} & u_{2} & \cdots & u_{N-1} & u_{N}
\end{array}\right)
$$

and $\mathbf{A}$ is the linear discrete spatial operator used in a scheme. In our computations, the time derivative in (21) is replaced with an explicit 5-stage Runge-Kutta scheme RKo5s from ${ }^{2}$

$$
\begin{equation*}
\frac{U^{(k+1)}-U}{\Delta t}=\alpha_{k} \mathbf{A} U^{(k)}, k=1, \ldots, 4, \quad U^{(0)}=U, \quad \widehat{U}=U^{(5)} \tag{22}
\end{equation*}
$$

where $U$ and $\widehat{U}$ denote values from the previous and the new time levels, respectively. Algorithm (22) may be classified as a generalized Jameson (low-storage) method of 2nd order approximation. Coefficients $\alpha_{k}$ of RKo5s have been chosen to optimize spectral resolution in time.

Recall that the time advancing is carried out uniformly at all the nodes-both internal and boundary. The algorithms proposed do not require any special coordination between the integration of governing equations and the boundary conditions.

We have constructed a family of algorithms for the numerical solution of the onedimensional transport equation. The basic elements of the technique proposed, namely, boundary approximations to normal derivatives and the time advancing, can be incorporated in more general models.

## 6 1D LINEARIZED EULER EQUATIONS

The technique described above is implemented to one-dimensional constant-coefficient hyperbolic systems of equations, in particular, to the Euler equations linearized upon a uniform background flow.

Represent the 1D Euler equations in the vector form

$$
\begin{equation*}
\frac{\partial \mathbf{U}}{\partial t}+C_{x} \frac{\partial \mathbf{U}}{\partial x}=0, \quad 0<x<X \tag{23}
\end{equation*}
$$

where

$$
\mathbf{U}=\left(\rho^{\prime} u^{\prime} v^{\prime} p^{\prime}\right)^{T}
$$

is the vector of primitive-variable perturbations,

$$
C_{x}=\left(\begin{array}{cccc}
u & \rho & 0 & 0  \tag{24}\\
0 & u & 0 & 1 / \rho \\
0 & 0 & u & 0 \\
0 & \rho c^{2} & 0 & u
\end{array}\right)
$$

$\rho$ is density, $u$ and $v$ velocity components, $p=\rho c^{2} / \gamma$ pressure, $c$ sound speed, $\gamma$ ratio of specific heats.

A large-stencil scheme for system (23) is a simple generalization of (2) as

$$
\begin{equation*}
d \mathbf{U}_{j} / d t+C_{x} \frac{1}{h} \sum_{l=-m}^{m} a_{l} \mathbf{U}_{j+l}=0, \quad j=m, \ldots, N-m . \tag{25}
\end{equation*}
$$

Boundary conditions for scheme (25) are based on its characteristic form

$$
\begin{equation*}
d w_{j}^{(s)} / d t+\frac{c^{(s)}}{h} \sum_{l=-m}^{m} a_{l} w_{j+l}^{(s)}=0, \quad j=m, \ldots, N-m, \quad s=1,2,3,4 \tag{26}
\end{equation*}
$$

Here, the characteristic velocities and variables are involved,

$$
\begin{align*}
c^{(1)}=u+c, & w^{(1)}=\rho c u^{\prime}+p^{\prime} \\
c^{(2)}=u-c, & w^{(2)}=\rho c u^{\prime}-p^{\prime} \\
c^{(3)}=u, & w^{(3)}=c^{2} \rho^{\prime}-p^{\prime}  \tag{27}\\
c^{(4)}=u, & w^{(4)}=v^{\prime}
\end{align*}
$$

Eq. (26) is a scalar scheme (2) for quantity $w^{(s)}$. So we apply so-called characteristic boundary conditions in the form of (9) and (13). Depending on the sign of $c^{(s)}$, the sets of left and right boundary conditions either repeat these formulae or exchange them as written below,

$$
\begin{align*}
& d w_{j}^{(s)} / d t+\frac{c^{(s)}}{h} \sum_{l=0}^{n} c_{j l} w_{l}^{(s)}=0, \quad j=0, \ldots, m-1, \\
& d w_{N-j}^{(s)} / d t+\frac{c^{(s)}}{h} \sum_{l=0}^{n} b_{j l} w_{N-l}^{(s)}=0, \quad j=0, \ldots, m-1, \quad \text { if } \quad c^{(s)}>0,  \tag{28}\\
& d w_{j}^{(s)} / d t-\frac{c^{(s)}}{h} \sum_{l=0}^{n} b_{j l} w_{l}^{(s)}=0, \quad j=0, \ldots, m-1, \\
& d w_{N-j}^{(s)} / d t-\frac{c^{(s)}}{h} \sum_{l=0}^{n} c_{j l} w_{N-l}^{(s)}=0, \quad j=0, \ldots, m-1, \quad \text { if } \quad c^{(s)}<0 . \tag{29}
\end{align*}
$$

Consider a subsonic flow directed rightward, when

$$
0<u<c .
$$

This implies $c^{(1)}, c^{(3)}, c^{(4)}>0$ and $c^{(2)}<0$. Consequently, Eqs. (28) are specified for $w^{(1)}, w^{(3)}$, and $w^{(4)}$, whereas Eqs. (29) for $w^{(2)}$.

## 7 2D TRANSPORT EQUATION

In order to show the two-dimensional implementation of consistent boundary conditions, consider the 2D scalar transport equation. In this case, we are able to obtain highly accurate approximations to the differential problem.

The governing equation

$$
\begin{equation*}
\partial u / \partial t+c_{x} \partial u / \partial x+c_{y} \partial u / \partial y=0, \quad 0<x<X, \quad 0<y<Y \tag{30}
\end{equation*}
$$

for $(x, y)$ Cartesian coordinates and $c_{x}, c_{y}>0$ some constants, is replaced by the internal scheme

$$
\begin{array}{r}
d u_{j k} / d t+\frac{c_{x}}{h_{x}} \sum_{l=-m}^{m} a_{l} u_{j+l, k}+\frac{c_{y}}{h_{y}} \sum_{l=-m}^{m} a_{l} u_{j, k+l}=0 \\
j=m, \ldots, N_{x}-m, k=m, \ldots, N_{y}-m \tag{31}
\end{array}
$$

Here, a rectangular grid uniform both in $x$ and $y$ is used,

$$
\begin{array}{lll}
x_{0}=0, & x_{j}=x_{0}+j h_{x}, j=0, \ldots, N_{x}, & x_{N_{x}}=X, \\
y_{0}=0, & y_{k}=y_{0}+k h_{y}, & k=0, \ldots, N_{y},  \tag{32}\\
y_{N_{y}}=Y
\end{array}
$$

Coefficients $a_{l}$ are specified in the same manner as in the 1D case (2).

Equation (30) needs inflow boundary conditions at the left and the lower sides of the rectangle, e.g.,

$$
\left.u\right|_{x=0}=\left.u\right|_{y=0}=0
$$

For the large-stencil scheme (31), the set of boundary nodes comprises four stripes of $m$-point width near all the sides, i.e.,

$$
j=0, \ldots, m-1, j=N_{x}-m+1, \ldots, N_{x}, k=0, \ldots, m-1, k=N_{y}-m+1, \ldots, N_{y} .
$$

The whole mesh is separated into 9 subdomains.
The complete set of boundary conditions can be expressed, together with the governing equation (31), in a uniform way as

$$
\begin{equation*}
d u_{j k} / d t+c_{x} D_{x}^{h} u+c_{y} D_{y}^{h} u=0, \quad j=0, \ldots, N_{x}, k=0, \ldots, N_{y} . \tag{33}
\end{equation*}
$$

where

$$
\left(D_{x}^{h} u\right)_{j k}= \begin{cases}\frac{1}{h_{x}} \sum_{l=-m}^{m} a_{l} u_{j+l, k}, & j=m, \ldots, N_{x}-m, \\ \frac{1}{h_{x}} \sum_{l=0}^{n} b_{N_{x}-j, l} u_{N_{x}-l, k}, & j=N_{x}-m+1, \ldots, N_{x} \\ \frac{1}{h_{x}} \sum_{l=0}^{n} c_{j l} u_{l k}, & j=0, \ldots, m-1,\end{cases}
$$

and

$$
\left(D_{y}^{h} u\right)_{j k}= \begin{cases}\frac{1}{h_{y}} \sum_{l=-m}^{m} a_{l} u_{j, k+l}, & \\ \frac{1}{h_{y}} \sum_{l=0}^{n} b_{N_{y}-k, l} u_{j, N_{y}-l}, & \\ k=N_{y}-m+1, \ldots, N_{y}-m, \\ \frac{1}{h_{y}} \sum_{l=0}^{n} c_{k l} u_{j l}, & k=0, \ldots, m-1 .\end{cases}
$$

Thus, in scheme (31), $x$ - and $y$-differences are, when necessary, replaced with their consistent approximations according to the rules stated above. We illustrate this with conditions on the right and the left sides of the rectangular domain:

$$
\begin{array}{r}
d u_{N_{x}-j, k} / d t+\frac{c_{x}}{h_{x}} \sum_{l=0}^{n} b_{j l} u_{N_{x}-l, k}+\frac{c_{y}}{h_{y}} \sum_{l=-m}^{m} a_{l} u_{N_{x}-j, k+l}=0, \\
j=0, \ldots, m-1, \quad k=m, \ldots, N_{y}-m,
\end{array}
$$

and

$$
\begin{array}{r}
d u_{j k} / d t+\frac{c_{x}}{h_{x}} \sum_{l=0}^{n} c_{j l} u_{l k}+\frac{c_{y}}{h_{y}} \sum_{l=-m}^{m} a_{l} u_{j, k+l}=0 \\
j=0, \ldots, m-1, k=m, \ldots, N_{y}-m
\end{array}
$$

The whole procedure is clear and the resulting boundary equations are accurate as well as in the case of 1 D transport equation.

## 8 2D LINEARIZED EULER EQUATIONS

Unlike the 1D case, the multidimensional Euler equations do not admit an implementation of the scalar consistent boundary conditions in a straightforward way. First of all, even differential equations are difficult for imposition of artificial boundary conditions. There do not exist local nonreflecting boundary conditions for the multidimensional wave equation ${ }^{9}$. All the problems of continuous boundary conditions are inevitably passed to the discrete approximations.

We will demonstrate an improved version of continuous artificial boundary conditions by Tam and Webb ${ }^{1}$ and its discrete approximation involving the consistent boundary conditions.

The two-dimensional Euler equations on a rectangular domain look analogously with (23) as

$$
\begin{equation*}
\frac{\partial \mathbf{U}}{\partial t}+C_{x} \frac{\partial \mathbf{U}}{\partial x}+C_{y} \frac{\partial \mathbf{U}}{\partial y}=0, \quad 0<x<X, \quad 0<y<Y \tag{34}
\end{equation*}
$$

where the matrices are

$$
C_{x}=\left(\begin{array}{cccc}
u & \rho & 0 & 0  \tag{35}\\
0 & u & 0 & 1 / \rho \\
0 & 0 & u & 0 \\
0 & \rho c^{2} & 0 & u
\end{array}\right), \quad C_{y}=\left(\begin{array}{cccc}
v & 0 & \rho & 0 \\
0 & v & 0 & 0 \\
0 & 0 & v & 1 / \rho \\
0 & 0 & \rho c^{2} & v
\end{array}\right) .
$$

The wide-stencil scheme for system (34) is represented as

$$
\begin{array}{r}
d \mathbf{U}_{j k} / d t+C_{x} \frac{1}{h_{x}} \sum_{l=-m}^{m} a_{l} \mathbf{U}_{j+l, k}+C_{y} \frac{1}{h_{y}} \sum_{l=-m}^{m} a_{l} \mathbf{U}_{j, k+l}=0, \\
j=m, \ldots, N_{x}-m, \quad k=m, \ldots, N_{y}-m . \tag{36}
\end{array}
$$

Let the mean flow be subsonic and horizontal, i.e., put $v=0$.
Outflow conditions by Tam and Webb ${ }^{1}$ are specified on the right-hand, lower and upper boundaries:

$$
\begin{array}{r}
\frac{\partial \rho^{\prime}}{\partial t}+u \frac{\partial \rho^{\prime}}{\partial x}=\frac{1}{c^{2}}\left(\frac{\partial p^{\prime}}{\partial t}+u \frac{\partial p^{\prime}}{\partial x}\right) \\
\frac{\partial u^{\prime}}{\partial t}+u \frac{\partial u^{\prime}}{\partial x}+\frac{1}{\rho} \frac{\partial p^{\prime}}{\partial x}=0 \\
\frac{\partial v^{\prime}}{\partial t}+u \frac{\partial v^{\prime}}{\partial x}+\frac{1}{\rho} \frac{\partial p^{\prime}}{\partial y}=0  \tag{37}\\
\frac{\partial p^{\prime}}{\partial t}+V \xi \frac{\partial p^{\prime}}{\partial x}+V \eta \frac{\partial p^{\prime}}{\partial y}+\frac{V}{2 r} p^{\prime}=0
\end{array}
$$

Here, $\left(x_{0}, y_{0}\right)$ is an assumed point of initial acoustic perturbation,

$$
V=\sqrt{c^{2}-u^{2} \eta^{2}}+u \xi, \quad \xi=\frac{x-x_{0}}{r}, \quad \eta=\frac{y-y_{0}}{r}, \quad r=\sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}} .
$$

Discretization of (37) employs centered and consistent outflow differences as well as in the case of 2D transport equation.

Conditions on the left-hand boundary include radiation conditions by Tam and Webb for quantities $u^{\prime}$ and $p^{\prime}$,

$$
\begin{equation*}
\frac{\partial u^{\prime}}{\partial t}+V \xi \frac{\partial u^{\prime}}{\partial x}+V \eta \frac{\partial u^{\prime}}{\partial y}+\frac{V}{2 r} u^{\prime}=0, \quad \frac{\partial p^{\prime}}{\partial t}+V \xi \frac{\partial p^{\prime}}{\partial x}+V \eta \frac{\partial p^{\prime}}{\partial y}+\frac{V}{2 r} p^{\prime}=0 \tag{38}
\end{equation*}
$$

Two remaining equations are built up in the following way.
The first equation in outflow set (37) describes in fact the convection of quantity $w^{(3)}=c^{2} \rho^{\prime}-p^{\prime}$ from (27). The technique of consistent boundary conditions should be applied to its approximation near all the sides of the rectangle. In particular, the left-hand boundary equation becomes

$$
\begin{equation*}
\frac{d \rho_{j k}^{\prime}}{d t}+\frac{u}{h_{x}} \sum_{l=0}^{n} c_{j l} \rho_{l k}^{\prime}=\frac{1}{c^{2}}\left(\frac{d p_{j k}^{\prime}}{d t}+\frac{u}{h_{x}} \sum_{l=0}^{n} c_{j l} p_{l k}^{\prime}\right), \quad j=0, \ldots, m-1, k=0, \ldots, N_{y} \tag{39}
\end{equation*}
$$

Based on the condition of vorticity absence

$$
\partial v^{\prime} / \partial x-\partial u^{\prime} / \partial y=0
$$

we transform the third equation of (37) and get

$$
\begin{equation*}
\frac{\partial v^{\prime}}{\partial t}+u \frac{\partial u^{\prime}}{\partial y}+\frac{1}{\rho} \frac{\partial p^{\prime}}{\partial y}=0, \quad x=0 \tag{40}
\end{equation*}
$$

The $y$-derivatives featuring in Eq. (40) may be replaced with the centered differences of an internal scheme or, in the upper left and the lower left corners, with consistent non-centered differences of outflow type (9).

## 9 NUMERICAL RESULTS

The algorithms proposed are validated on numerical simulation of test problems.
Begin with a 1D example. Consider the motion of a rectangular pulse governed by transport equation (1). The process is modeled by DRP scheme (3) coupled with timeintegration scheme RKo5s from ${ }^{2}$. The mesh is composed of $N=250$ nodes. Two settings of boundary conditions are tested.

Case (i) is internal scheme (3) with imposed zero values at the left boundary and conditions (19) consistent for segment $\varphi \in[0, \pi / 2]$ at the right-hand boundary:

$$
\begin{aligned}
d u_{j} / d t+\frac{1}{h} \sum_{l=-3}^{3} a_{l} u_{j+l} & =0, & j=3,4, \ldots, N-3, \\
u_{0}=u_{1}=u_{2} & =0, & \\
d u_{N-j} / d t+\frac{1}{h} \sum_{l=0}^{6} b_{j l} u_{N-l} & =0, & j=0,1,2 .
\end{aligned}
$$

(a)

(b)


Figure 3: Motion of rectangular pulse; spatial distribution for time $t=20$, (a) case (i), (b) case (ii)

Case (ii) is governing scheme (3), Eq. (19) on the right for segment $[0, \pi / 2]$ (as previously), and Eq. (20) on the left consistent for $[0, \pi / 4]$.

Remark. It is important that high-accuracy schemes considered in the paper are not intended for the description of discontinuous solutions. However, we do not use selective filters after ${ }^{2,10}$ or any other stabilization procedure. Hence such problems serve as very strong tests for revealing undesired features of computational algorithms.

Computations occurred stable for the Courant-Friedrichs-Lewy number

$$
\mathrm{CFL} \equiv \Delta t / h \leq 2.06
$$

in both cases of boundary conditions. This corresponds to the theoretical stability limit. Note that too big time step is undesirable from the accuracy point of view.

Figure 3 shows the numerical solutions computed with CFL $=1$ in cases (i) and (ii) at dimensionless time $t=20$, when the initial pulse (plotted with dashed line) has not yet reached the right-hand boundary.

The two discontinuities generate short waves propagating leftward. Wave packages (better seen in Fig. 3 (b)) are coming to the left boundary and being reflected from it. The reflection is strong in case (i) and small in case (ii) of consistent boundary conditions.

To validate the technique for the two-dimensional transport equation, the motion of a discontinuous pulse with a constant value on a square support has been modeled. In Eq. (30), propagation velocities $c_{x}=1$ and $c_{y}=1$ have been set, and initial function

$$
u(x, y, 0)= \begin{cases}1, & 10<x<20,10<y<20 \\ 0, & \text { otherwise }\end{cases}
$$

The computational domain is $\{0 \leq x \leq 40 \times 0 \leq y \leq 40\}$. The mesh consists of $N_{x} \times N_{y}=200 \times 200$ nodes. As a governing scheme (31) the 2 D form of seven-point DRP scheme by Tam and Webb has been taken. Time-advancing method is RKo5s, with

$$
\mathrm{CFL} \equiv c_{x} \Delta t / h_{x} \equiv c_{y} \Delta t / h_{y}=0.5
$$



Figure 4: Motion of 2D pulse; spatial distribution for time $t=15$.

A stable computation is achieved with the use of consistent boundary conditions (33). The coefficients are taken for integration segments $[0, \pi / 2]$ on the outflow and $[0, \pi / 4]$ on the inflow. Fig. 4 shows the numerical solution at time $t=15$ when the pulse had not


Figure 5: Propagation of 2D Gaussian pulses; spatial distribution of $\rho^{\prime}$ for time $t=25$ : (a) Tam's boundary conditions; (b) modified boundary conditions
yet left the computational domain. The results practically repeat the properties of 1D rectangular pulse discussed above. Effects of two-dimensionality are almost unnoticeable.

The last problem concerns the 2D linearized Euler equations and belongs to benchmark aeroacoustic problems having exact solutions ${ }^{11}$. The propagation of Gaussian pulses of both acoustic and convective types is considered.

The computational domain is square $\{0 \leq x \leq 50 \times 0 \leq y \leq 50\}$. The mesh dimension is $N_{x}=N_{y}=200$ and sizes are $h_{x}=h_{y}=h=0.25$. Internal scheme (36) is based on the 7-point DRP approximation. Time-advancing method is RKo5s.

The background flow is horizontal $(v=0)$; Mach number $\mathrm{M} \equiv u / c=0.2$. The initial perturbation is

$$
\begin{aligned}
& p^{\prime}(x, y, 0)= \rho c^{2} \exp \left[-\ln (2)\left(\frac{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}}{(b h)^{2}}\right)\right] \\
& \begin{aligned}
\rho^{\prime}(x, y, 0)= & \rho \exp \left[-\ln (2)\left(\frac{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}}{(b h)^{2}}\right)\right] \\
& +0.1 \rho \exp \left[-\ln (2)\left(\frac{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}}{(a h)^{2}}\right)\right] \\
u^{\prime}(x, y, 0)= & 0.04 c\left(y-y_{1}\right) \exp \left[-\ln (2)\left(\frac{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}}{(a h)^{2}}\right)\right] \\
v^{\prime}(x, y, 0)= & -0.04 c\left(x-x_{1}\right) \exp \left[-\ln (2)\left(\frac{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}}{(a h)^{2}}\right)\right] .
\end{aligned} .
\end{aligned}
$$

Here, $\left(x_{0}, y_{0}\right)=(15,15)$ and $\left(x_{1}, y_{1}\right)=(35,35)$ are the centers of acoustic and convective


Figure 6: Error $\rho^{\prime}-\rho_{\text {exact }}^{\prime}$ versus $x$ at $y=0$ for time $t=25$ : solid line-Tam's boundary conditions; dash line - modified boundary conditions
pulses, respectively; constants $a=10$ and $b=9$ define the characteristic widths of convective and acoustic pulses.

We compare two versions of boundary conditions-the original formulation by Tam and Webb versus the modified algorithm from Eqs. (37)-(40).

Spatial distributions of density perturbation $\rho^{\prime}$ for dimensionless time $t=25$ are shown in Fig. 5. At this instant, the convective pulse occurs on the wavefront from the initial acoustic peak. In Fig. 5 (a) the violation of symmetry of the acoustic wave is well seen. In the case of modified boundary conditions (b), this effect is present too but displays itself much weaklier.

Figure 6 shows the error of density perturbation $\rho^{\prime}-\rho_{\text {exact }}^{\prime}$ on the lower boundary of the computational domain $(y=0)$ for the time instant chosen above. The modified boundary conditions have about two times increased accuracy compared to the approach of Tam and Webb.

## 10 CONCLUSIONS

- The technique has been shown to be accurate and stable in the case of large-stencil centered schemes for linear 1D hyperbolic systems. No additional stabilizing procedure is required.
- For the 2D transport equation, the large-stencil internal schemes supplied with consistent boundary conditions demonstrate the same accuracy as in the 1D case.
- For 2D gasdynamic problems, we have obtained a tool for approximation of spatial derivatives and inflow fluxes which has been successfully implemented to a version of continuous boundary conditions.


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