V European Conference on Computational Fluid Dynamics
ECCOMAS CFD 2010
J. C. F. Pereira and A. Sequeira (Eds)
Lisbon, Portugal,14-17 June 2010

DISCRETE KINETIC FINITE VOLUME SCHEMES FOR MULTIDIMENSIONAL SYSTEMS OF CONSERVATION LAWS ON UNSTRUCTURED GRID

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Key words: Conservation Laws, BGK Models, Kinetic Schemes, Finite Volume Schemes, Unstructured Meshes

Abstract. *We construct a family of Riemann solver free finite volume schemes for systems of conservation laws. The method is based on a relaxation approximation which takes the form of a BGK system. We prove convergence for unstructured grids in the scalar case. We present 2D computations for Euler equations.*

1 INTRODUCTION

In this paper we construct some finite volume discretisations for hyperbolic systems of conservation laws in two space dimensions, in an open subset $\Omega \subset \mathbb{R}^2$ with C^2 boundary $\partial\Omega$:

$$\partial_t U + \partial_{x_1} A_1(U) + \partial_{x_2} A_2(U) = 0 \quad \text{in} \quad]0, +\infty[\times\Omega, \tag{1}$$

where $U(x, y, t) \in \mathbb{R}^p$ and A_1, A_2 are smooth functions with values in $\mathbb{R}^p$. We denote u_0 an initial value for this system:

$$U(x_1, x_2, 0) = U_0(x_1, x_2). \tag{2}$$

D. Aregba-Driollet and R. Natalini[2] designed a class of numerical schemes based on a discrete kinetic approximation of the Cauchy problem for (1). These schemes were extended to the initial-boundary value problem by D. Aregba-Driollet and V. Milišić[1]. All these schemes were designed on cartesian grids, here we extend these works to unstructured grids.

To approximate the solutions of (1), we choose a semilinear hyperbolic system of BGK type :

$$\begin{cases} \partial_t f_l^\epsilon + \lambda_{l1}\partial_{x_1} f_l^\epsilon + \lambda_{l2}\partial_{x_2} f_l^\epsilon = \dfrac{1}{\epsilon}(M_l(U^\epsilon) - f_l^\epsilon) \qquad l \in \{1, \cdots, N\} \\ f_l^\epsilon(0, x) = M_l(U_0(x)) \end{cases} \tag{3}$$

where (λ_{ld}), $1 \leq l \leq N$, $d = 1, 2$ are fixed real numbers, U^ϵ is defined by $U^\epsilon = \sum_{l=1}^N f_l^\epsilon$, and ϵ is a positive parameter. The link between (1) and (3) is done *via* the following compatibility conditions:

Definition 1.1. *Let $I \subseteq \mathbb{R}^p$ be a fixed domain. A Lipschitz continuous function $M = (M_l)_{1\leq l\leq N} : I \to (\mathbb{R}^p)^N$ is a (local) Maxwellian Function for (1) and with respect to I if the following (compatibility) conditions are verified for all $U \in I$:*

$$\sum_{l=1}^N M_l(U) = U \tag{4}$$

$$\sum_{l=1}^N \lambda_{ld} M_l(U) = A_d(U) \quad d = 1, 2. \tag{5}$$

It is easy to see formally that if conditions (4)(5) are satisfied and if U^ϵ tends to a limit function U as $\epsilon \to 0$ then U is a solution of (1). In the following we shall assume that the function M is a Maxwellian Function.

Actually one needs for some stability conditions to ensure this convergence, namely:

Definition 1.2. *The functions $M = (M_l)_{1\leq l\leq N}$ is Monotone Non Decreasing (MND) on I if all the eigenvalues of $M_l'(U)$ are real and non negative for all $U \in I$ and all $l = 1, \ldots, N$. In particular in the scalar case $p = 1$ it means that $M_l'(U) \geq 0$ on I for all l.*

It can be shown that the MND property is closely related to Liu's subcharacteristic condition interlacing the eigenvalues of system (1) and the characteristic velocities λ_{ld}, see Aregba-Driollet and Milišić[1]. In the scalar case, suppose that the maxwellian function is MND on $[-\|U_0\|_\infty, \|U_0\|_\infty]$. Then $U^\epsilon \to U$, the unique entropy solution of the Cauchy problem (1-2), see Natalini[8]. This result has been extended to the initial-boundary problem in Milišić[7]. In the case of systems with a strictly convex entropy, see Bouchut[3] for the relationship between the MND condition and the entropy properties of the BGK approximation.

In this paper, for the numerical experiments we focus our attention on the following 4 orthogonal velocities model because the needed information for applying the MND property is minimal: one has just to know the spectral ray of the jacobian of the fluxes. The velocities vectors are

$$\begin{aligned}
\overrightarrow{\lambda_1} &= \lambda_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\overrightarrow{\lambda_2} &= \lambda_y \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\
\overrightarrow{\lambda_3} &= \lambda_x \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
\overrightarrow{\lambda_4} &= \lambda_y \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{aligned}$$

and the Maxwellian Functions are

$$\begin{aligned}
M_1(U) &= \frac{1}{4}[U + \frac{2A_1(U)}{\lambda_x}] \\
M_2(U) &= \frac{1}{4}[U - \frac{2A_2(U)}{\lambda_y}] \\
M_3(U) &= \frac{1}{4}[U - \frac{2A_1(U)}{\lambda_x}] \\
M_4(U) &= \frac{1}{4}[U + \frac{2A_2(U)}{\lambda_y}]
\end{aligned}$$

Here the stability condition reads

$$\begin{aligned}
\lambda_x &> 2 \sup_{U\in I} |\sigma(A_1'(U))| \\
\lambda_y &> 2 \sup_{U\in I} |\sigma(A_2'(U))|,
\end{aligned}$$

where $\sigma(\cdot)$ is the spectrum of the jacobian matrices.

The plan of the paper is the following: in section 2 we construct the schemes and give a convergence result for the scalar case. In the last section numerical experiments are performed on Euler equations of gas dynamics.

2 KINETIC FINITE VOLUME SCHEMES

In this part we design a numerical scheme for the relaxing semilinear system (3) associated with (1), (2). We present here a cell-center type finite volume method on unstructured grid. Our aim is to construct a scheme of the following form :

$$\begin{cases} U_i^{n+1} = U_i^n - \dfrac{\Delta t}{|C_i|} \displaystyle\sum_{e=C_i\cap C_j} |e| g_{ij}(U_i^n, U_j^n, n_e) \\ \\ U_i^0 = \dfrac{1}{|C_i|} \displaystyle\int_{C_i} U_0(x_1, x_2)\, dx_1 dx_2. \end{cases} \tag{6}$$

2.1 Geometry and notations

We shall collect here the assumptions concerning geometry and the notations used in this work.

Definition 2.1. *Let a k-polygon be a closed, convex polygon with k vertices. The set* $\mathcal{T}_h := \{C_i / C_i$ *is a k-polygon for* $i \in \mathcal{I} \subseteq \mathbb{N}\}$ *(where* $\mathcal{I} \subseteq \mathbb{N}$ *is an index set) is called an unstructured grid of* $\Omega \subset \mathbb{R}^n$ *if the two following properties are satisfied :*

1. $\Omega = \bigcup_{i\in\mathcal{I}} C_i$
2. *For two different* C_i *and* C_j *, we have* $C_i \cap C_j = \emptyset$*, or* $C_i \cap C_j =$ *a common vertex of* C_i*,* C_j *or* $C_i \cap C_j =$ *a common edge of* C_i*,* C_j*.*

Definition 2.2. *Let* $(C_i)_{i\in\mathcal{I}}$ *denote an unstructured grid of* $\mathbb{R}^2$*. We shall use the following notation :*

C_i *: The cell number i of the grid.*
$|C_i|$ *: Area of the cell* C_i *(2-dimensional Lebesgue measure).*
e_{ij}*: Edge* $C_i \cap C_j$ *of the cell* C_i
$\mathcal{E}_\mathcal{T}$*: Set of all the edges of the mesh.*
$|e|$ *: Length of the edge e (1-dimensional Lebesgue measure).*
Γ_i *: Boundary of* C_i*.*
$P_i = \sum_{e\in\Gamma_i} |e|$ *: Perimeter of* C_i*.*
Γ_{ij} *: Common edge of the boundary between* C_i *and* C_j*.*
$\mathcal{V}_i$ *: The set of cells* C_j *which have a common edge with* C_i*.*
$\mathcal{N}_i$ *: The set of the indices j of the cells* C_j *which have a common edge with* C_i*.*
$n_e =$ *: Outer unit normal from edge e.*

$\lambda_l(n_e) = \lambda_l \cdot n_e$.
$h = \sup_{i\in I} h_i$, *where* h_i *is the exterior diameter of* C_i.
U_i^n *: approximation of the exact solution* U *on* C_i *at time* $n\Delta t$.

Let us make the following assumptions on the geometry :

Assumption 2.1. *We assume that there are two constants* c_1 *et* c_2 *such that*

$$0 < c_1 \leq \frac{\Delta t}{h} \leq c_2 \tag{7}$$

if Δt, $h \to 0$. *Moreover, we assume that there exists a constant* $c_V > 0$ *such that*

$$\sup_i \frac{h^2}{|C_i|} \leq c_V \tag{8}$$

2.2 Design of the schemes

The timestep is denoted by Δt and discrete times are $t_0 = 0$ and $t_{n+1} = t_n + \Delta t$. Each U_i^n is an approximation of the mean value of $U(\cdot, t_n)$ on the cell C_i. At time $t_0 = 0$, if U_0 is the initial value, we take

$$U_i^0 = \frac{1}{|C_i|} \int_{C_i} U_0(x_1, x_2) dx_1 dx_2.$$

For the microscopic (kinetic) variables, we use as initial conditions :

$$f_{l,i}^0 = M_l(U_i^0).$$

We use a fractional step method : first we solve the homogeneous hyperbolic problem on $[t_n, t_{n+1}]$:

$$\begin{cases} \partial_t f_l^\epsilon + \lambda_{l1} \partial_{x_1} f_l^\epsilon + \lambda_{l2} \partial_{x_2} f_l^\epsilon = 0 \qquad l \in \{1, \cdots, N\} \\ f_l^\epsilon(x_1, x_2, t_n) = f^{\epsilon,n}(x_1, x_2) \end{cases} \tag{9}$$

As the system is diagonal, we may consider each equation separately. That is, we choose a conservative scheme of flux φ_l for the l^{th}-equation. Then we obtain a scheme under conservative form :

$$f_{l,i}^{\epsilon,n+1/2} = f_{l,i}^{\epsilon,n} - \frac{\Delta t}{|C_i|} \sum_{e = C_i \cap C_j} |e| \varphi_l(f_{l,i}^{\epsilon,n}, f_{l,j}^{\epsilon,n}, n_e).$$

Next, taking $f_{l,i}^{\epsilon,n+1/2}$ as initial condition at time t_n, we solve the ordinary differential system :

$$\begin{cases} \partial_t f_{l,i}^\epsilon = \frac{1}{\epsilon}(M_l(U_i^{\epsilon,n}) - f_{l,i}^\epsilon) \\ f_{l,i}^\epsilon(t_n) = f_{l,i}^{\epsilon,n+1/2} \end{cases}$$

Thanks to the compatibility condition (4), we may solve explicitly this system. Then we write the solution :

$$f_{l,i}^{\epsilon,n+1} = (1 - e^{-\frac{\Delta t}{\epsilon}}) M_l(U_i^{\epsilon,n+1/2}) + e^{-\frac{\Delta t}{\epsilon}} f_{l,i}^{\epsilon,n+1/2},$$

where

$$U_i^{\epsilon,n+1/2} = \sum_{l=1}^{N} f_{l,i}^{\epsilon,n+1/2}.$$

Note that

$$U^{\epsilon,n+1} = U^{\epsilon,n+1/2}.$$

We have constructed a family of numerical scheme for the semilinear system (3), that differ by the choice of the homogeneous scheme.

When $\epsilon \to 0$, we obtain the relaxed limit of the scheme :

$$\begin{cases} f_{l,i}^n = & M_l(U_i^n) \\ f_i^{n+1/2} = & H_\Delta(\Delta t) f_i^n \\ U_i^{n+1} = & \displaystyle\sum_{l=1}^{N} f_{l,i}^{n+1/2} \end{cases}$$

where $H_\Delta(\Delta t)$ represents the numerical scheme applied to the kinetic equations (9).

In order to obtain a finite volume formulation (6), we have to choose for each kinetic equation a numerical scheme, which can be defined by its numerical flux, called φ_l for the l^{th} equation. According to the form of the relaxed scheme, the numerical scheme is the following :

$$\begin{cases} U_i^{n+1} = U_i^n - \dfrac{\Delta t}{|C_i|} \displaystyle\sum_{e=C_i \cap C_j} \sum_{l=1}^{N} |e| \varphi_l(M_l(U_i^n), M_l(U_j^n), n_e) \\ U_i^0 = \dfrac{1}{|C_i|} \displaystyle\int_{C_i} U_0(x_1, x_2) dx_1 dx_2. \end{cases} \tag{KFVS}$$

This scheme is of the form (6) with

$$g_{ij}(U_i, U_j, n_e) = \sum_{l=1}^{N} \varphi_l(M_l(U_i), M_l(U_j), n_e). \tag{10}$$

3 Convergence in the scalar case

In the scalar case $p = 1$ we are able to prove convergence of schemes (KFVS) following the lines of Kröner and Rokyta's work[6]. A preliminary result is the existence of kinetic entropies:

Theorem 3.1 (Bouchut[3]). *Let η a strictly C^1-convex entropy, the condition*

$$M_l'(u) \geq 0, \quad \forall u \in I,\ \forall l \in \{1, \dots, N\}$$

is equivalent to existence of kinetic entropies $S_{l,\eta}$ defined in the following way :

$$S_{l,\eta}(f) = \int_{\mathbb{R}} \frac{1}{2}(|f - M_l(k)| - |M_l(k)|)\eta''(k)dk + \frac{1}{2}f(\eta'(-\infty) + \eta'(\infty)).$$

They own the following properties :

$$\text{Functions } S_{l,\eta} \text{ are } C^2 - \text{convex in } [M_l(-\|u_0\|_\infty), M_l(\|u_0\|_\infty)]. \tag{11}$$

$$\forall w \in I, S_{l,\eta}'(M_l(w)) = \eta'(w), \tag{12}$$

$$\sum_{l=1}^{N} \lambda_{ld} S_{l,\eta}(M_l(w)) = A_{\eta,d}(w). \tag{13}$$

$$\sum_{l=1}^{N} S_{l,\eta}(M_l(w)) = \eta(w) - \eta(0). \tag{14}$$

The identity (12) holds in the sense of subdifferentials :
for all $f \in [M_l(-\|u_0\|_\infty), M_l(\|u_0\|_\infty)]$, and for all $w \in I$,

$$S_{l,\eta}(M_l(w)) - S_{l,\eta}(f) - \eta'(w)(M_l(w) - f) \leq 0. \tag{15}$$

It ensures, for each kinetic equation, the existence of an entropy pair $(S_{l,\eta}, F_l)$, where $F_{ld}(s) = \lambda_{ld} S_{l,\eta}(s)$.

We make the following assumptions on the kinetic numerical fluxes:

Assumption 3.1. *For all $l \in \{1, \dots, N\}$, let φ_l be a numerical flux of linear three-points scheme in conservation form, consistent with $\lambda_l f_l \cdot n_e$ i.e.*

$$\varphi_l(u, u, n_e) = \lambda_l u \cdot n_e \tag{16}$$

We assume that φ_l is Lipschitz-continuous. In particular, suppose that for all $M > 0$ there is a constant $C_{\varphi_l}(M)$ such that for all $u, u', v, v' \in [-M, M]$

$$|\varphi_l(u, v) - \varphi_l(u', v')| \leq C_{\varphi_l}(M)(|u - u'| + |v - v'|) \tag{17}$$

and that φ_l is conservative, i.e.

$$\varphi_l(u, v, n_e) = -\varphi_l(v, u, -n_e) \tag{18}$$

Moreover, we assume that φ_l is monotone :

$$\frac{\partial}{\partial u}\varphi_l(u, v) \geq 0 \geq \frac{\partial}{\partial v}\varphi_l(u, v) \tag{19}$$

Assumption 3.2. *Let $\eta : \mathbb{R} \to \mathbb{R}$ be Lipschitz-continuous and convex, and let $(S_{l,\eta}, F_l)$ be an entropy pair for (9). We assume that for all $l \in [1, N]$ there exists a numerical entropy flux G_l which is :*

(i) consistent with $F_l(u) \cdot n_e = \lambda_l \cdot n_e S_{l,\eta}(u)$ i.e

$$G_l(u, u, n_e) = \lambda_l \cdot n_e S_{l,\eta}(u) \tag{20}$$

(ii) Lipschitz-continuous, such that for all $M > 0$ there is a constant $C_{G_l}(M)$ such that for all $u, u', v, v' \in [-M, M] \subset \mathbb{R}$

$$|G_l(u, v) - G_l(u', v')| \leq C_{G_l}(M)(|u - u'| + |v - v'|) \tag{21}$$

(iii) conservative, i.e. for all $j \in \mathcal{N}_i$

$$G_l(u, v, n_e) = -G_l(v, u, -n_e) \tag{22}$$

and

(iv) satisfies the kinetic compatibility condition

$$\frac{\partial G_l}{\partial v}(p, q, n_e) = S'_{l,\eta}(q) \frac{\partial \varphi_l}{\partial q}(p, q, n_e) \tag{23}$$

Let us note that if we choose the (linear) Godunov numerical flux for each φ_l then assumptions 3.1 and 3.2 are verified. On another hand this choice is the optimal one for first order, as it is the less diffusive one.

Lemma 3.1. *For all $i \in \mathcal{I}$, we have :*

$$\sum_{\substack{e \subset C_i \\ e = C_i \cap C_j}} |e| \, n_e = 0$$

A key result to prove convergence is the L^∞ stability of the scheme, which implies monotony and entropy dissipation.

Proposition 3.1 (L^∞ stability)**.** *Let $u_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Let u_i^n defined by the kfv scheme (KFVS) such that the fluxes φ_l satisfy the assumptions 3.1. We denote $M = \|u_0\|_{L^\infty(\mathbb{R}^2)}$ and*

$$C(M) = \sum_l C_l(M_l(\|u_0\|_\infty)).$$

We assume that the CFL-type condition

$$\frac{\Delta t}{h} \leq \frac{1}{k \cdot C(M) c_V}, \tag{24}$$

where c_V is defined as in (8), is fullfilled. We assume that the maxwellian functions M_l are MND on $[-M, M]$. Then we have, for all $n \in \mathbb{N}$,

$$\|u^n\|_{L^\infty(\mathbb{R}^2)} \leq \|u^0\|_{L^\infty(\mathbb{R}^2)}. \tag{25}$$

Proof. The KFVS method is given by :

$$\begin{aligned} u_i^{n+1} &= u_i^n - \frac{\Delta t}{|C_i|} \sum_{\substack{e \subset C_i \\ e = \Gamma_{ij}}} |e|\, g_{ij}(u_i^n, u_j^n, n_e) \\ &= u_i^n - \frac{\Delta t}{|C_i|} \sum_{\substack{e \subset C_i \\ e = \Gamma_{ij}}} |e|\, (g_{ij}(u_i^n, u_j^n, n_e) - g_{ij}(u_i^n, u_i^n, n_e)), \end{aligned}$$

by lemma 3.1. Then

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|C_i|} \sum_{\substack{e \subset C_i \\ e = \Gamma_{ij}}} D_{ij}^n (u_i^n - u_j^n).$$

where

$$D_{ij}^n = |e| \frac{g_{ij}(u_i^n, u_j^n, n_e) - g_{ij}(u_i^n, u_i^n, n_e)}{u_i^n - u_j^n} \quad \text{for} \quad u_j^n \neq u_i^n \tag{26}$$

and

$$D_{ij}^n = 0 \quad \text{for} \quad u_j^n = u_i^n. \tag{27}$$

Next we can write

$$u_i^{n+1} = u_i^n \left(1 - \frac{\Delta t}{|C_i|} \sum_{\substack{e \subset C_i \\ e = \Gamma_{ij}}} D_{ij}^n \right) + \frac{\Delta t}{|C_i|} \sum_{\substack{e \subset C_i \\ e = \Gamma_{ij}}} D_{ij}^n u_j^n.$$

Let us assume that for a fixed $n \geq 0$, $\|u^n\|_{L^\infty(\mathbb{R}^2)} \leq M$. By assumption 3.1 and monotony of the maxwellian functions

$$0 \leq D_{ij}^n \leq C(M)h,$$

and using the CFL-type condition (24) and assumption 2.1

$$\left| \frac{\Delta t}{|C_i|} \sum_{\substack{e \subset C_i \\ e = \Gamma_{ij}}} D_{ij}^n \right| \leq k \frac{\Delta_t}{\min_{i \in \mathcal{I}} |C_i|} C(M)h \leq 1.$$

Finally we have, due to the convex combination

$$
\begin{aligned}
|u_i^{n+1}| &\leq |u_i^n| \left(1 - \frac{\Delta t}{|C_i|} \sum_{\substack{e \subset C_i \\ e = \Gamma_{ij}}} D_{ij}^n \right) + \frac{\Delta t}{|C_i|} \sum_{\substack{e \subset C_i \\ e = \Gamma_{ij}}} D_{ij}^n |u_j^n| \\
&\leq \|u^n\|_{L^\infty(\mathbb{R}^2)} \left(1 - \frac{\Delta t}{|C_i|} \sum_{\substack{e \subset C_i \\ e = \Gamma_{ij}}} D_{ij}^n \right) + \frac{\Delta t}{|C_i|} \sum_{\substack{e \subset C_i \\ e = \Gamma_{ij}}} D_{ij}^n \|u^n\|_{L^\infty(\mathbb{R}^2)} \\
&\leq \|u^n\|_{L^\infty(\mathbb{R}^2)} \leq M.
\end{aligned}
$$

Then we get $\|u^{n+1}\|_{L^\infty(\mathbb{R}^2)} \leq M$ and we can complete the proof by induction. □

Kröner and Rokyta[6] proved convergence for upwind finite volume schemes. The above stability result allows us to prove monotony and to define an entropy numerical flux, which satisfies the compatiblity condition (23). We are then able to prove the convergence theorem:

Theorem 3.2. *We consider the Cauchy problem (1)-(2) and the approximate solution $\{u_h\}$ defined by the kinetic finite volume (first-order) method (KFVS). We suppose that the maxwellian functions are MND on the interval $[-\|u_0\|_\infty, \|u_0\|_\infty]$ and we take $M > \|u_0\|_\infty$. We suppose that the conditions (7), (8) for the triangulation hold, and that the CFL condition (24)*

$$
\frac{\Delta t}{h} \leq \frac{1}{k \cdot C(M) C_V}, \tag{28}
$$

is satisfied. Then the method (KFVS) converges weak-$\star$ to the unique entropy solution in the sense of Kružkhov of the Cauchy problem (1)-(2).

$$
u_h \to u \quad weak - \star, \qquad u \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^+).
$$

3.1 Extension to second order in space

We focus now on the extension to second order of these family of kinetic schemes. The semi-discrete version of these scheme is :

$$
|C_i| \frac{dU_i}{dt} + \sum_{e = C_i \cap C_j} |e| g(U_i, U_j, n_e) = 0, \quad C_i \in \mathcal{T}_h. \tag{29}
$$

For internal edges, the fonction $g(.,.,n_e)$ is equal to the kinetic flux as shown above. The extension to second order accury consists in replacing the states U_i and U_j in relation (29) by nonlinear extrapolations $U_{i,e}$ and $U_{j,e}$ on each side of the boundary, constructed from the previous states as explained in the sequel. We follow here the method given by

F. Dubois and O. Michaux[4], which is a generalization of Van Leer's[10] MUSCL scheme for unstructured meshes.

Let us introduce the set V_i of neighbouring cells of a given cell C_i :

$$V_i = \{C_j \in \mathcal{T}_h, \quad e \in \mathcal{E}_\mathcal{T}, \quad e \subset C_i \cap C_j\}.$$

We also introduce the point $y_{i,e}$ on the interface e that links the barycenters x_i and $x_{i,j(e)}$:

$$\begin{cases} y_{i,e} = (1-\theta_{i,e})x_i + \theta_{i,e}x_{i,j(e)}, \quad y_{i,e} \in e, \\ \qquad e \subset \partial C_i, \quad C_i \in \mathcal{T}_h. \end{cases} \tag{30}$$

Next, for z equal to one scalar variable of the family :

$$z \in \{\rho, \rho u, \rho v, p\} \tag{31}$$

we evaluate a mean value $\overline{z_{i,e}}$ on the interface e :

$$\overline{z_{i,e}} = (1-\theta_{i,e})z_i + \theta_{i,e}z_{i,j(e)}, \tag{32}$$

and the gradient $\nabla z|_i$ of field $z(\cdot)$ in volume C_i with a Green formula :

$$\nabla z|_i = \frac{1}{|C_i|}\int_{\partial C_i} \overline{z}\, n\, d\gamma = \frac{1}{|C_i|}\sum_{e\subset\partial C_i} |e|\, \overline{z_{i,e}}\, n_e, \quad C_i \in \mathcal{T}_h. \tag{33}$$

An extrapolation of field $z(\cdot)$ is given by

$$z_{i,e} = z_i + \nabla z|_i \cdot (y_{i,e} - x_i),$$

but the variation $\nabla z|_i \cdot (y_{i,e} - x_i)$ has to be limited. To this end we define the minimum $m_i(z)$ and the maximum $M_i(z)$ of field z in the neighbouring cells :

$$m_i(z) = \min\{z_j, \quad C_j \in \mathcal{V}_i\} \tag{34}$$

$$M_i(z) = \max\{z_j, \quad C_j \in \mathcal{V}_i\}. \tag{35}$$

If the value z_i is extremum among the neighbouring ones, we impose that the extrapolated value $z_{i,e}$ is equal to the cell value z_i :

$$z_{i,e} = z_i \quad \text{if } z_i \le m_i(z) \text{ or } z_i \ge M_i(z),\ e \subset \partial C_i. \tag{36}$$

We introduce a nonlinear extrapolation of the field $z(\cdot)$ between center x_i and boundary face $y_{i,e}$ ($e \subset \partial C_i$) :

$$z_{i,e} = z_i + \alpha_i(z)\nabla z|_i \cdot (y_{i,e} - x_i), \quad e \subset \partial C_i \tag{37}$$

where $\alpha_i(z)$ is a limiting coefficient satisfying the following conditions :

$$\begin{cases} 0 \leq \alpha_i(z) \leq 1, \quad z \in \{\rho, \rho u, \rho v, p\}, \quad C_i \in \mathcal{T}_h \\ k(z_i - m_i(z)) \leq \alpha_i(z) \nabla z|_i \cdot (y_{i,e} - x_i) \leq k(M_i(z) - z_i), \quad \forall e \subset \partial C_i,\ C_i \in \mathcal{T}_h. \end{cases} \tag{38}$$

Then $\alpha_i(z)$ is chosen as large as possible and less than or equal to 1 in order to satisfy the constraints :

$$\alpha_i(z) = \min \left[1, \frac{\min(M_i(z) - z_i, z_i - m_i(z))}{\max\{|\nabla z|_i \cdot (y_{i,e} - x_i)|,\ e \subset \partial C_i\}} \right]. \tag{39}$$

For $k = 1$, we recover the initial limiter proposed by Van Leer[9]. When $k = 1/2$, we obtain the minmod limiter proposed by Harten[5]. The intermediate value $k = 3/4$, which was named STS by Dubois and Michaux is a good choice between the two others.

3.2 Second order in time

When all values $z_{i,e}$ are known for all control voumes, all faces and all fields, extrapolated states $U_{i,e}$ are naturally defined by going back to the conservatives variables. Then we introduce these states as arguments of the flux function g.. and obtain by this way a new system of ordinary differential equations :

$$|C_i| \frac{dU_i}{dt} + \sum_{e \subset \partial C_i} |e| g(U_{i,e}, U_{i,j(e)}, n_e) = 0, \quad C_i \in \mathcal{T}_h. \tag{40}$$

To have a second order accuracy in time, we use a two-steps Runge-Kutta scheme :

$$\frac{|C_i|}{\Delta t}(U_i^{(1)} - U_i^n) + \sum_{e \subset \partial C_i} |e| g(U_{i,e}^n, U_{i,j(e)}^n, n_e) = 0, \quad C_i \in \mathcal{T}_h, \tag{41}$$

$$\frac{|C_i|}{\Delta t}(U_i^{(2)} - U_i^{(1)}) + \sum_{e \subset \partial C_i} |e| g(U_{i,e}^{(1)}, U_{i,j(e)}^{(1)}, n_e) = 0, \quad C_i \in \mathcal{T}_h, \tag{42}$$

$$U_i^{n+1} = \frac{1}{2}(U_i^{(2)} + U_i^n), \quad C_i \in \mathcal{T}_h. \tag{43}$$

4 NUMERICAL EXPERIMENTS

We approximate a weak solution U of Euler's system of gas dynamics in two dimensions

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad A_1(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{pmatrix}, \quad A_2(U) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}, \tag{44}$$

where we have to add the equation of state for a γ-law gas

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u^2 + v^2). \tag{45}$$

The jacobian matrix $A_1'(U)$ has the four eigenvalues

$$a^1 = u - c, \quad a^2 = a^3 = u, \quad a^4 = u + c.$$

As in one dimension, the sound speed is $c = \sqrt{\gamma p/\rho}$. The eigenvalues in the y-direction are similar, with the roles of u and v reversed.

4.1 A mach 3 wind tunnel with a step

This example is a classical test example, which has been proven to be useful for a large number of numerical schemes. A very complete presentation can be found in the paper of Woodward and Colella[11].

The test case describes a Mach 3 flow in a wind tunnel. The tunnel is 1 length unit high and 3 length units long. The step is 0.2 length units high and is located 0.6 length units from the left-hand end of the tunnel. The walls are reflective.

At the left, we impose a supersonic inflow boundary condition, while at the right side we impose an outgoing condition. Initially the wind tunnel is filled with a gamma-law gas, with $\gamma = 1.4$, which everywhere has density 1.4, pressure 1.0, x-velocity 3.0 and y-velocity 0.0.

On the boundary we impose reflecting boundary condition that is

$$\overrightarrow{u} \cdot \overrightarrow{n} = 0.$$

We use four different schemes, all constructed on the four velocities model and the upwind (Godunov) flux. Every solution is displayed with 30 isolines at time t=3.5.

The first-order scheme (figure 1) gives naturally the least accuracy, but the Mach stem is present.

With the the second-order scheme (figure 2), the shocks are thinner but a numerical instability is evident near the bottom wall, and behind the Mach stem.

In order to improve the previous scheme, where the velocities are the same in each cell, we fixed these velocities by cell. The result is clearly better (figure 3), This scheme gives a more accurate reprensation of the general shape and position of the shocks .

The last scheme (figure 4) is the same as the previous combined with a second-order in time Runge -Kutta scheme. The results are quite similar.

4.2 Double mach reflection of a strong shock

This test describes the reflection of a planar Mach shock in air from a wedge. The setup is of a Mach 10 shock which initially makes a 60 degrees angle with a reflecting wall. The undisturbed air ahead of the shock has a density of 1.4 and pressure of 1.0. The computational domain is $[0, 4] \times [0, 1]$ and the reflecting wall lies along the bottom of the domain, beginning at $x = 1/6$. See Woodward and Colella[11] for more precision about this case.

For this test-case we use three different schemes constructed with the four velocities and the upwind flux.

First, the first-order scheme (figure 5) gives the least accurate results. The jet formed by the double Mach reflection is unresolved. The strong shocks are too large.

The second scheme is the second-order scheme with velocities fixed by cells (figure 6). The results are superior to those of first-order. The jet is resolved and the weak shock is better decribed. However there are some unphysical structures.

The third scheme is the same as the previous combined with a second-order in time Runge-Kutta scheme (figure 7). The results are better, the shocks are thinner and the jet is quite well described. However the weak shock generated at the kink in the main reflected shock is quite broad.

5 CONCLUSIONS

The kinetic models (3) offer an alternative to the Godunov approach for the construction of numerical approximations of hyperbolic systems of conservation laws. The higher order extension in space and in time is easy and this approach is very flexible with regard to the choice of the underlying kinetic model as well as to the one of the discretisation parameters.

In the scalar case, the convergence is proved. We have performed numerical experiments on 2D Euler system with unstructured grids. The chosen kinetic model is as simple as possible, and the implementation needs for minimal information on the spectral properties of the jacobian of the fluxes, namely their spectral ray. We observe nevertheless that the numerical results are qualitatively very satisfying: all the essential structures of the solutions are present.

Other choices of the maxwellian function may need for more precise information and lead also to more accurate results but in all cases one has just to solve linear transport equations.

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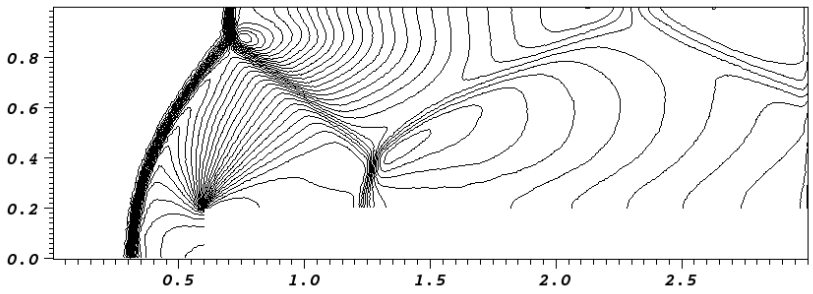

Figure 1: Density order 1 t=3.5

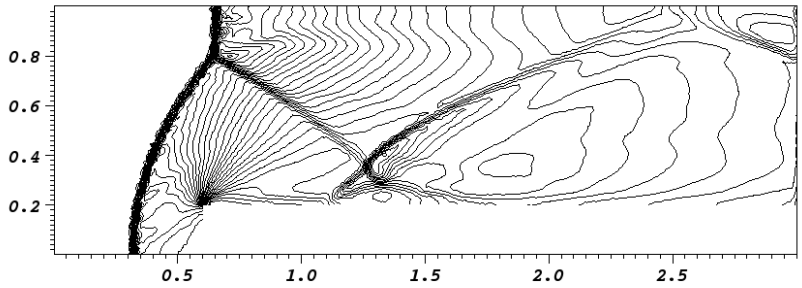

Figure 2: Density order 2 t=3.5

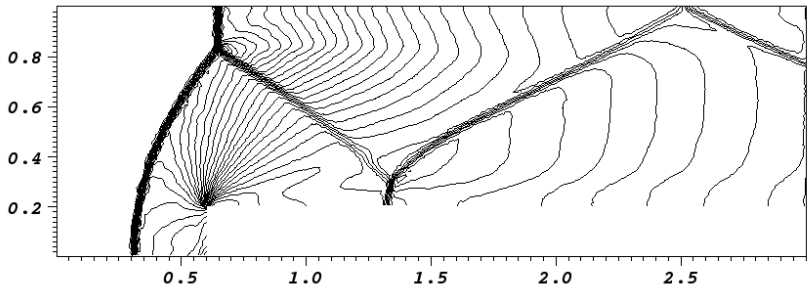

Figure 3: Density order 2 local t=3.5

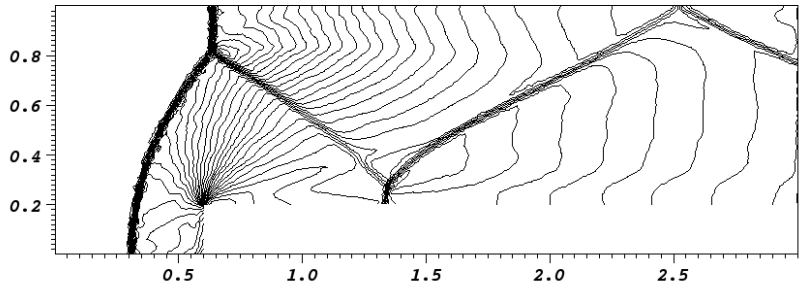

Figure 4: Density order 2 rk2 local t=3.5

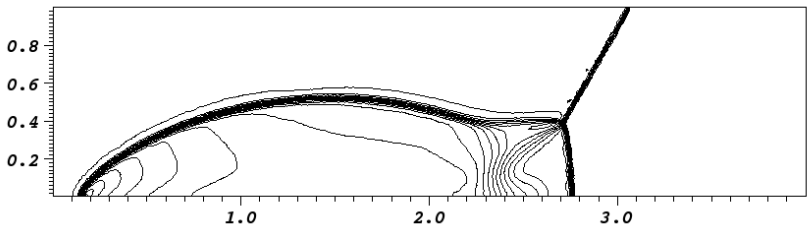

Figure 5: Density order 1 T=0.2s

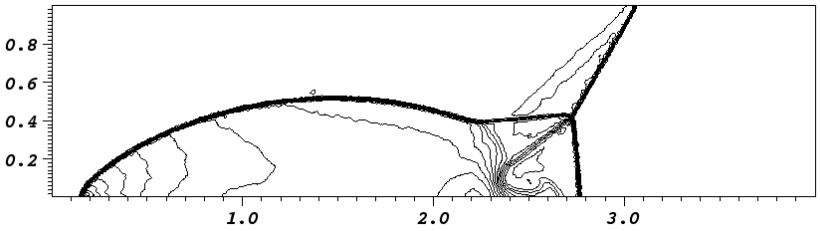

Figure 6: Density order 2 local T=0.2s

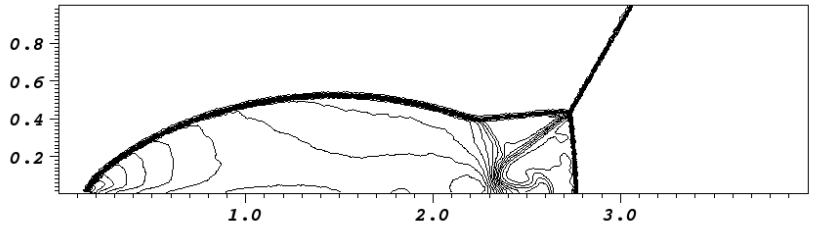

Figure 7: Density order 2 rk2 local T=0.2s