A POSTERIORI ERROR ESTIMATES FOR APPROXIMATE SOLUTIONS OF BARENBLATT-BIOT POROELASTIC MODEL

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Key words: a posteriori error estimates, poroelastic media

Abstract. We are concerned with the Barenblatt-Biott model in the theory of poroelasticity. We derive a guaranteed estimate of the difference between exact and approximate solutions expressed in a combined norm that encompasses errors for the pressure fields computed from the diffusion part of the model and errors related to stresses (strains) of the elastic part. Estimates do not contain generic (mesh-dependent) constants and are valid for any conforming approximation of pressure and stress fields. This is a shortened version of the joint paper [1] with J. M. Nordbotten (Bergen), T. Rahman (Bergen) and S. I. Repin (St. Petersburg).

1 INTRODUCTION

A combination of the Barenblatt's double-diffusion approach and Biot's diffusiondeformation theory leads to what we call the Barenblatt-Biot poroelastic model representing double diffusion in elastic porous media. It takes the form

$$-\nabla \cdot (\mathbb{L} \varepsilon(\mathbf{u})) + \alpha_1 \nabla p_1 + \alpha_2 \nabla p_2 = \mathbf{f}(x, t),$$

$$c_1 \dot{p}_1 - \nabla \cdot (k_1 \nabla p_1) + \alpha_1 \nabla \cdot \dot{\mathbf{u}} + \kappa (p_1 - p_2) = h_1(x, t),$$

$$c_2 \dot{p}_2 - \nabla \cdot (k_2 \nabla p_2) + \alpha_2 \nabla \cdot \dot{\mathbf{u}} + \kappa (p_2 - p_1) = h_2(x, t),$$

(1.1)

u is the displacement of the solid skeleton and p_1 and p_2 are the fluid potentials in the respective components. With the vector gradient operator ∇ , the linear Green strain tensor $\varepsilon(\cdot)$ writes

$$\varepsilon(u) := \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right).$$
(1.2)

The fourth-order elastic stiffness tensor \mathbb{L} defines a stress tensor σ using the Hook's law

$$\sigma := \mathbb{L} \varepsilon(\mathbf{u}).$$

In general, the permeabilities k_1 and k_2 may be heterogeneous and anisotropic tensors, which may be functions of the deformation. Herein, we will neglect this dependence and Jan Valdman

only consider constant, scalar and homogeneous permeabilities. Constants α_1 and α_2 measure changes of porosities due to an applied volumetric strain. Mathematical analysis of this model based on the theory of implicit evolution equations in Hilbert spaces is elaborated in [4].

Our focus in this paper is to derive guaranteed and computable bounds of approximation errors the static Barenblatt-Biot system

$$-\nabla \cdot (\mathbb{L} \varepsilon(\mathbf{u})) + \alpha_1 \nabla p_1 + \alpha_2 \nabla p_2 = \mathbf{f}(x), -\nabla \cdot (k_1 \nabla p_1) + \kappa (p_1 - p_2) = h_1(x), -\nabla \cdot (k_2 \nabla p_2) + \kappa (p_2 - p_1) = h_2(x),$$
(1.3)

which is considered in bounded connected domain $\Omega \subset \mathcal{R}^d$ with Lipschitz continuous boundary Γ .

2 VARIATIONAL FORMULATION OF THE DOUBLE DIFFUSION SYS-TEM

Since the displacement \mathbf{u} is only involved in the first equation of system (1.3), a doublediffusion problem

$$-\nabla \cdot (k_1 \nabla p_1) + \kappa (p_1 - p_2) = h_1(x), \qquad (2.1)$$

$$-\nabla \cdot (k_2 \nabla p_2) + \kappa (p_2 - p_1) = h_2(x)$$
(2.2)

is studied separately. It describes the steady flow of slightly compressible fluid in a general heterogeneous medium consisting of two components. Henceforth, we consider this problem with the Dirichlet boundary conditions $p_1 = p_2 = p_{\Gamma}$ on Γ . Let \bar{p} be a function with square summable coefficients that satisfies this boundary condition. It is convenient to rewrite the problem in terms of new functions

$$p_1 := p_1 - \bar{p}, \quad p_2 := p_2 - \bar{p}$$

Then, a weak formulation of (2.1)-(2.2) leads to

Problem 1. Assume that $(h_1, h_2) \in L^2(\Omega, \mathcal{R}^2)$. Find $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2) \in H^1_0(\Omega, \mathcal{R}^2)$, satisfying the system of variational equalities

$$\int_{\Omega} k_1 \nabla \mathbf{p}_1 \cdot \nabla \mathbf{q}_1 + \int_{\Omega} \kappa(\mathbf{p}_1 - \mathbf{p}_2) \mathbf{q}_1 \, dx = \int_{\Omega} (h_1(x)\mathbf{q}_1 - k_1 \nabla \bar{\mathbf{p}} \cdot \nabla \mathbf{q}_1) \, dx$$

$$\int_{\Omega} k_2 \nabla \mathbf{p}_2 \cdot \nabla \mathbf{q}_2 + \int_{\Omega} \kappa(\mathbf{p}_2 - \mathbf{p}_1) \mathbf{q}_2 \, dx = \int_{\Omega} (h_2(x)\mathbf{q}_2 - k_2 \nabla \bar{\mathbf{p}} \cdot \nabla \mathbf{q}_2) \, dx$$
(2.3)

for all testing functions $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2) \in H_0^1(\Omega, \mathcal{R}^2)$.

This problem can be represented in a general form (which also encompasses other, more complicated models of porous media). For this purpose, we introduce the spaces

$$Q := H_0^1(\Omega, \mathcal{R}^2), \quad Y := L^2(\Omega, \mathcal{R}^{2d}), \tag{2.4}$$

and the corresponding dual spaces

$$Q^* := H^{-1}(\Omega, \mathcal{R}^2), \quad Y^* := L^2(\Omega, \mathcal{R}^{2d}).$$
(2.5)

Hereafter L_2 norms of all functions in Ω are denoted by $\|\cdot\|_{\Omega}$. Duality pairings of (Q, Q^*) and (Y, Y^*) are denoted by $\langle \cdot, \cdot \rangle$ and $\langle \langle \cdot, \cdot \rangle \rangle$, respectively. Also, we introduce a bounded linear operator $\Lambda \in \mathcal{L}(Q, Y)$ and its adjoint operator $\Lambda^* \in \mathcal{L}(Y^*, Q^*)$ by the relations

$$\Lambda \mathbf{q} := (\nabla \mathbf{q}_1, \nabla \mathbf{q}_2), \quad \Lambda^* \mathbb{Y}^* = (-\operatorname{div} y_1^*, -\operatorname{div} y_2^*)^T.$$
(2.6)

The operators Λ and Λ^* satisfy the relation representing integration by parts

$$\langle \langle \mathbb{Y}^*, \Lambda \mathbf{q} \rangle \rangle = \langle \Lambda^* \mathbb{Y}^*, \mathbf{q} \rangle \quad \text{for all } \mathbb{Y}^* \in Y^*, \mathbf{q} \in Q$$

which can be written componentwise as

$$\int_{\Omega} \left(\mathbf{Y}_1^* \cdot \nabla \mathbf{q}_1 + \mathbf{Y}_2^* \cdot \nabla \mathbf{q}_2 \right) \, dx = -\int_{\Omega} \left(\mathbf{q}_1 \operatorname{div} \mathbf{Y}_1^* + \mathbf{q}_2 \operatorname{div} \mathbf{Y}_2^* \right) \, dx, \tag{2.7}$$

where $\mathbf{q} = (\mathbf{q}_1, q_2)$ and $\mathbb{Y}^* = (\mathbf{Y}_1^*, \mathbf{Y}_2^*)$. Now Problem 1 can be represented in the form: Find $\mathbf{p} \in Q$ such that the equality

$$a(\mathbf{p}, \mathbf{q}) = l(\mathbf{q}) \tag{2.8}$$

holds for all $\mathbf{q} \in Q$. The bilinear form $a(\cdot, \cdot)$ and the linear form $l(\cdot)$ are defined as

$$\begin{aligned} a(\mathbf{p}, \mathbf{q}) &:= \int_{\Omega} \left(\Lambda \mathbf{p} : (\mathbb{A}\Lambda \mathbf{q}) + \mathbf{p} \cdot \mathbb{B}\mathbf{q} \right) \, dx, \\ l(\mathbf{q}) &:= \int_{\Omega} \left(\mathbf{h} \cdot \mathbf{q} - \mathbb{C}\Lambda \mathbf{q} \right) dx, \end{aligned}$$

A, B and C are matrices formed by material dependent constants k_1, k_2, k_3 ,

$$\mathbb{A} := \begin{pmatrix} k_1 & 0\\ 0 & k_2 \end{pmatrix}, \quad \mathbb{B} := \begin{pmatrix} \kappa & -\kappa\\ -\kappa & \kappa \end{pmatrix}, \quad \mathbb{C} := \begin{pmatrix} k_1 \nabla \bar{p} & 0\\ 0 & k_2 \nabla \bar{p} \end{pmatrix}$$

and \mathbf{h} is the right hand side vector

$$\mathbf{h} := \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

Remark 1. If \bar{p} is sufficiently regular (so that $\Lambda^*\mathbb{C}$ belongs to Y^*), then

$$l(\mathbf{q}) := \int_{\Omega} (\mathbf{h} \cdot \mathbf{q} - \Lambda^* \mathbb{C} \mathbf{q}) \, dx = \int_{\Omega} \widehat{\mathbf{h}} \cdot \mathbf{q} \, dx,$$

where

$$\widehat{\mathbf{h}} := \begin{pmatrix} h_1 - \operatorname{div} k_1 \nabla \bar{p} \\ h_2 - \operatorname{div} k_2 \nabla \bar{p} \end{pmatrix}.$$

It is easy to verify that (2.8) is the necessary condition for the minimizer of the following convex variational problem.

Problem 2. Find $\mathbf{p} \in Q$ satisfying

$$F(\mathbf{p}) + G(\Lambda \mathbf{p}) = \inf_{\mathbf{q} \in Q} \{F(\mathbf{q}) + G(\Lambda \mathbf{q})\},$$
(2.9)

where

$$F: Q \to \mathcal{R}, \qquad F(\mathbf{q}) := \frac{1}{2} \int_{\Omega} \mathbf{q} \cdot \mathbb{B} \mathbf{q} \, dx - l(\mathbf{q}),$$
 (2.10)

and

$$G: Y \to \mathcal{R}, \qquad G(\Lambda \mathbf{q}) := \frac{1}{2} \int_{\Omega} \Lambda \mathbf{q} : (\mathbb{A}\Lambda \mathbf{q}) \, dx.$$
 (2.11)

Theorem 1 (existence of unique solution). Assume that $k_1, k_2 > 0$ and $\kappa \ge 0$. Then, there exists a unique solution $\mathbf{p} \in Q$ of Problem 2, which also represents the solution of Problem 1.

Proof. This proof and all other proofs can be found in [1].

3 A POSTERIORI ERROR ESTIMATE OF THE DOUBLE DIFFUSION SYSTEM

In this section, we derive guaranteed and directly computable bounds of the difference between exact and approximate solutions. Our analysis is based upon a posteriori error estimation methods suggested in [3, 5]. Following the chapters 6 and 7 in [3], first we need to find explicit forms of dual functionals

$$F^*: Q^* \to \mathcal{R}, \qquad F^*(\Lambda^* \mathbb{Y}^*) := \sup_{\mathbf{q} \in Q} \{ \langle \Lambda^* \mathbb{Y}^*, \mathbf{q} \rangle - F(\mathbf{q}) \}, G^*: Y^* \to \mathcal{R}, \qquad G^*(\mathbb{Y}^*) := \sup_{\Lambda \mathbf{q} \in Y} \{ \langle \langle \mathbb{Y}^*, \Lambda \mathbf{q} \rangle \rangle - G(\Lambda \mathbf{q}) \}.$$
(3.1)

and the corresponding compound functionals

$$D_F: Q \times Q^* \to \mathcal{R}, \qquad D_F(\mathbf{q}, \Lambda^* \mathbb{Y}^*) := F(\mathbf{q}) + F^*(\Lambda^* \mathbb{Y}^*) - \langle \Lambda^* \mathbb{Y}^*, \mathbf{q} \rangle, D_G: Y \times Y^* \to \mathcal{R}, \qquad D_G(\Lambda \mathbf{q}, \mathbb{Y}^*) := G(\Lambda \mathbf{q}) + G^*(\mathbb{Y}^*) - \langle \langle \mathbb{Y}^*, \Lambda \mathbf{q} \rangle \rangle.$$
(3.2)

By the sum of D_F and D_G , we obtain the functional error majorant

$$M(\mathbf{q}, \mathbb{Y}^*) := D_F(\mathbf{q}, \Lambda^* \mathbb{Y}^*) + D_G(\Lambda \mathbf{q}, \mathbb{Y}^*), \qquad (3.3)$$

which provides a guaranteed upper bound of the error:

$$\frac{1}{2}a(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q}) \le M(\mathbf{q}, \mathbb{Y}^*) \quad \text{for all } \mathbb{Y}^* \in Y^*.$$
(3.4)

The majorant is fully computable and depends only on the approximation $\mathbf{q} \in Q$ and arbitrary variable $\mathbb{Y}^* \in Y^*$.

Lemma 1 (dual functionals). For $k_1, k_2 > 0$ and $\kappa > 0$, it holds

$$G^*(\mathbb{Y}^*) = \frac{1}{2} \int_{\Omega} \mathbb{A}^{-1} \mathbb{Y}^* : \mathbb{Y}^* \, dx, \qquad (3.5)$$

$$F^*(\Lambda^*\mathbb{Y}^*) = \begin{cases} \frac{1}{4\kappa} \int (\Lambda^*\mathbb{Y}^* + \mathbf{h})^2 \, dx & \text{if } \Lambda^* y_1^* + h_1 + \Lambda^* y_2^* + h_2 = 0, \\ +\infty & \text{otherwise,} \end{cases}$$
(3.6)

where

$$Y_h^* := \{ (y_1^*, y_2^*) \in Y^* : \Lambda^* \mathbf{Y}_1^* + h_1 + \Lambda^* \mathbf{Y}_2^* + h_2 = 0 \ a.e. \ in \ \Omega \}.$$
(3.7)

After the substitution of (3.5) and (3.6) in the definition (3.2), we obtain explicit expressions for the compound functionals.

$$D_G(\Lambda \mathbf{q}, \mathbb{Y}^*) = \frac{1}{2} \int_{\Omega} \mathbb{A}(\Lambda \mathbf{q} - \mathbb{A}^{-1} \mathbb{Y}^*) : (\Lambda \mathbf{q} - \mathbb{A}^{-1} \mathbb{Y}^*) \, dx, \qquad (3.8)$$

$$D_{F}(\mathbf{q}, \Lambda^{*} \mathbb{Y}^{*}) = \begin{cases} \frac{1}{2} \int_{\Omega} \mathbb{B}\mathbf{q} \cdot \mathbf{q} \, dx + \frac{1}{4\kappa} \int_{\Omega} (\Lambda^{*} \mathbb{Y}^{*} + \mathbf{h})^{2} \, dx \\ & \text{if } \Lambda^{*} \mathbf{Y}_{1}^{*} + h_{1} + \Lambda^{*} \mathbf{Y}_{2}^{*} + h_{2} = 0, \\ +\infty & \text{otherwise.} \end{cases}$$
(3.9)

According to (3.22), the sharpest bound of $a(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q})$ is provided by the estimate

$$\frac{1}{2}a(\mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}) \le \inf_{\mathbb{Y}^* \in Y^*} M(\mathbf{q},\mathbb{Y}^*).$$
(3.10)

Since $M(\mathbf{q}, \mathbb{Y}^*) = +\infty$ if $\mathbb{Y}^* \notin Y_h^*$, we must restrict ourselves to arguments $\mathbb{Y}^* \in Y_h^*$. To construct an element of Y_h^* , an exact equilibration procedure is required. Below, we show a way to avoid the constrain (3.7) by a special penalty term added to the functional majorant. We define

$$Y_{div}^* := \{ (\mathbf{Y}_1^*, \mathbf{Y}_2^*) \in Y^* : \Lambda^* \mathbf{Y}_1^* + \Lambda^* \mathbf{Y}_2^* \in L^2(\Omega) \}$$
(3.11)

and note that $Y_h^* \subset Y_{div}^*$ (since $h_1, h_2 \in L^2(\Omega)$). Further we decompose

$$\mathbb{Y}^* = \hat{\mathbb{Y}}^* + (\mathbb{Y}^* - \hat{\mathbb{Y}}^*)$$

with $\hat{\mathbb{Y}}^* \in Y^*_{div}$ and we extend the dual functionals D_G and D_F by the new variable $\hat{\mathbb{Y}}^*$. We rewrite (3.8) as

$$D_G(\Lambda \mathbf{q}, \mathbb{Y}^*) = \frac{1}{2} \int_{\Omega} \mathbb{A}(\Lambda \mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) : (\Lambda \mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) \, dx + \int_{\Omega} (\Lambda \mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*) \, dx + \frac{1}{2} \int_{\Omega} \mathbb{A}^{-1}(\mathbb{Y}^* - \hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*) \, dx$$

and use the inequality $2\mathbb{M}_1 : \mathbb{M}_2 \leq \beta_1 M_1 : M_1 + \frac{1}{\beta_1} \mathbb{M}_2 : \mathbb{M}_2$ valid for all matrices $\mathbb{M}_1, \mathbb{M}_2$ and for all $\beta_1 > 0$ to bound the middle term as

$$(\Lambda \mathbf{q} - \mathbb{A}^{-1} \hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*) = \mathbb{A}^{1/2} (\Lambda \mathbf{q} - \mathbb{A}^{-1} \hat{\mathbb{Y}}^*) : \mathbb{A}^{-1/2} (\mathbb{Y}^* - \hat{\mathbb{Y}}^*)$$
$$\leq \frac{\beta_1}{2} \mathbb{A} (\Lambda \mathbf{q} - \mathbb{A}^{-1} \hat{\mathbb{Y}}^*) : (\Lambda \mathbf{q} - \mathbb{A}^{-1} \hat{\mathbb{Y}}^*) + \frac{1}{2\beta_1} \mathbb{A}^{-1} (\mathbb{Y}^* - \hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*). \quad (3.12)$$

Obviously, the middle terms adds to the left and the right terms in $D_G(\Lambda \mathbf{q}, \mathbb{Y}^*)$ above and the modified compound functional reads

$$D_{G}(\Lambda \mathbf{q}, \mathbb{Y}^{*}, \hat{\mathbb{Y}}^{*}) := \frac{1+\beta_{1}}{2} \int_{\Omega} \mathbb{A}(\Lambda \mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^{*}) : (\Lambda \mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^{*}) dx + (\frac{1}{2} + \frac{1}{2\beta_{1}}) \int_{\Omega} \mathbb{A}^{-1}(\mathbb{Y}^{*} - \hat{\mathbb{Y}}^{*}) : (\mathbb{Y}^{*} - \hat{\mathbb{Y}}^{*}) dx.$$

$$(3.13)$$

It also contains a scalar factor $\beta_1 > 0$ that value can be chosen arbitrarily. Similar technique is used to modify the compound functional $D_F(\mathbf{q}, \Lambda^* \mathbb{Y}^*)$. For the second integral in (3.9), we have

$$\int_{\Omega} (\Lambda^* \mathbb{Y}^* + \mathbf{h})^2 \, dx \le (1 + \beta_2) \int_{\Omega} (\Lambda^* \hat{\mathbb{Y}}^* + \mathbf{h})^2 \, dx + (1 + \frac{1}{\beta_2}) \int_{\Omega} (\Lambda^* (\mathbb{Y}^* - \hat{\mathbb{Y}}^*))^2 \, dx,$$

where $\beta_2 > 0$. Therefore, a modified dual functional reads

$$D_F(\mathbf{q}, \Lambda^* \mathbb{Y}^*, \Lambda^* \hat{\mathbb{Y}}^*) := \frac{1}{2} \int_{\Omega} \mathbb{B} \mathbf{q} \cdot \mathbf{q} \, dx + \frac{1}{4\kappa} (1 + \beta_2) \int_{\Omega} (\Lambda^* \hat{\mathbb{Y}}^* + \mathbf{h})^2 \, dx + \frac{1}{4\kappa} (1 + \frac{1}{\beta_2}) \int_{\Omega} (\Lambda^* (\mathbb{Y}^* - \hat{\mathbb{Y}}^*))^2 \, dx.$$
(3.14)

By adding (3.13) and (3.14), we extend the functional majorant (3.3) to

$$M(\mathbf{q}, \mathbb{Y}^*, \mathbb{\hat{Y}}^*) := D_F(\mathbf{q}, \Lambda^* \mathbb{Y}^*, \Lambda^* \mathbb{\hat{Y}}^*) + D_G(\Lambda \mathbf{q}, \mathbb{Y}^*, \mathbb{\hat{Y}}^*), \qquad (3.15)$$

in which arbitrary variables satisfy the constrain

$$(\mathbb{Y}^*, \mathbb{\hat{Y}}^*) \in Y_h^* \times Y_{div}^*$$

Clearly, the original and extended majorants satisfy the inequality

$$\frac{1}{2}a(\mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}) \le M(\mathbf{q},\mathbb{Y}^*) \le M(\mathbf{q},\mathbb{Y}^*,\hat{\mathbb{Y}}^*)$$
(3.16)

for all $\hat{\mathbb{Y}}^* \in Y^*_{div}, \beta_1 > 0, \beta_2 > 0$. This estimate is sharp in the sense that there are no irremovable gaps in the inequalities. Indeed, if we set $\mathbb{Y}^* = \hat{\mathbb{Y}}^* = \Lambda \mathbf{p}$ and tend β_1 and β_2 to zero, then $M(\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*)$ tends to $M(\mathbf{q}, \mathbb{Y}^*)$ (and even to the exact error $\frac{1}{2}a(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q})$, cf. (3.10)).

3.1 AN UPPER ESTIMATE OF $M(\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*)$

Let us denote $\mathbb{Y}^* = (\mathbf{Y}_1^*, \mathbf{Y}_2^*)$ and $\hat{\mathbb{Y}}^* = (\hat{\mathbf{Y}}_1^*, \hat{\mathbf{Y}}_2^*)$ and consider a particular subspace

$$(\mathbb{Y}^*, \hat{\mathbb{Y}}^*) \in \{Y_h^* \times Y_{div}^* : \Lambda^* \mathbf{Y}_1^* + h_1 = 0, \mathbf{Y}_2^* = \hat{\mathbf{Y}}_2^* \text{ a.e. in } \Omega\}.$$
(3.17)

In this subspace, it holds (cf. (2.6))

$$\int_{\Omega} (\Lambda^* (\mathbb{Y}^* - \hat{\mathbb{Y}}^*))^2 \, dx = \int_{\Omega} (\operatorname{div}(\hat{\mathbf{Y}}_1^* - \mathbf{Y}_1^*))^2 \, dx = \int_{\Omega} (\operatorname{div}\hat{\mathbf{Y}}_1^* - h_1)^2 \, dx.$$

Therefore, $D_F(\mathbf{q}, \Lambda^* \mathbb{Y}^*, \Lambda^* \hat{\mathbb{Y}}^*)$ defined in (3.14) simplifies as \mathbb{Y}^* -independent

$$D_F(\mathbf{q}, \Lambda^* \hat{\mathbb{Y}}^*) := \frac{1}{2} \int_{\Omega} \mathbb{B} \mathbf{q} \cdot \mathbf{q} \, dx + \frac{1}{4\kappa} (1+\beta_2) \int_{\Omega} (\Lambda^* \hat{\mathbb{Y}}^* + \mathbf{h})^2 \, dx \qquad (3.18)$$
$$+ \frac{1}{4\kappa} (1+\frac{1}{\beta_2}) \int_{\Omega} (\operatorname{div} \hat{\mathbf{Y}}_1^* - h_1)^2 \, dx$$

and only \mathbb{Y}^* -dependent functional in $D_G(\Lambda \mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*)$ defined in (3.13) writes

$$\int_{\Omega} \mathbb{A}^{-1}(\mathbb{Y}^* - \hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*) \, dx = \int_{\Omega} k_1^{-1}(\mathbf{Y}_1^* - \hat{\mathbf{Y}}_1^*) \cdot (\mathbf{Y}_1^* - \hat{\mathbf{Y}}_1^*) \, dx.$$
(3.19)

Lemma 2. Let us define a space

$$Y_{h_1} := \{ \mathbf{Y}_1^* \in L^2(\Omega)^d : \Lambda^* \mathbf{Y}_1^* + h_1 = 0 \ a.e. \ in \ \Omega \}.$$

Then, for all $\hat{\mathbf{Y}}_1^* \in H(\operatorname{div}; \Omega)$, it holds

$$\inf_{\mathbf{Y}_1^* \in Y_{h_1}} \int_{\Omega} \left\| \mathbf{Y}_1^* - \hat{\mathbf{Y}}_1^* \right\|^2 dx \le C^2 \left\| \operatorname{div} \hat{\mathbf{Y}}_1^* + h_1 \right\|^2$$

where C > 0 satisfies Friedrichs' inequality $||w||_{L^2(\Omega)} \leq C ||\nabla w||_{L^2(\Omega)}$ valid for all $w \in H^1_0(\Omega)$.

Application of Lemma 2 to (3.19) and the back substitution to (3.13) defines a \mathbb{Y}^* -independent dual functional

$$D_G(\Lambda \mathbf{q}, \hat{\mathbb{Y}}^*) := \frac{1+\beta_1}{2} \int_{\Omega} \mathbb{A}(\Lambda \mathbf{q} - \mathbb{A}^{-1} \hat{\mathbb{Y}}^*) : (\Lambda \mathbf{q} - \mathbb{A}^{-1} \hat{\mathbb{Y}}^*) \, dx$$
$$+ k_1^{-1} (\frac{1}{2} + \frac{1}{2\beta_1}) C^2 \left\| \operatorname{div} \hat{\mathbf{Y}}_1^* + h_1 \right\|^2. \quad (3.20)$$

which provides an upper estimate of the quantity

$$\inf_{\mathbb{Y}^* \in Y_h^*} D_G(\Lambda \mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*).$$

Therefore, the sum of (3.18) and (3.20) defines a \mathbb{Y}^* -independent functional

$$M_{\beta_1,\beta_2}(\mathbf{q},\hat{\mathbb{Y}}^*) := D_F(\mathbf{q},\Lambda^*\hat{\mathbb{Y}}^*) + D_G(\Lambda\mathbf{q},\hat{\mathbb{Y}}^*)$$
(3.21)

that serves as an upper bound of $M(\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*)$ and provides a computable estimate

$$\frac{1}{2}a(\mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}) \le M_{\beta_1,\beta_2}(\mathbf{q},\hat{\mathbb{Y}}^*) \quad \text{for all } \hat{\mathbb{Y}}^* \in Y^*_{div}.$$
(3.22)

4 A POSTERIORI ERROR ESTIMATE FOR APPROXIMATIONS OF THE COUPLED SYSTEM (1.1)

Assume that the fluid pressures p_1 and p_2 are resolved exactly and substituted to the elasticity equation (cf. (1.1))

$$-\nabla \cdot (\mathbb{L} \varepsilon(u)) = \mathbf{f}(x, t) + \alpha_1 \nabla \mathbf{p}_1 + \alpha_2 \nabla \mathbf{p}_2.$$

Let \mathbf{v} be an approximation of \mathbf{u} (this problem is considered in the same domain Ω as the problem (2.1)-(2.2)). We define the Dirichlet boundary condition by a function $\mathbf{u}_0 \in$ $H^1(\Omega; \mathcal{R}^d)$ and assume

$$\mathbf{v} \in \mathbf{u}_0 + H_0^1(\Omega; \mathcal{R}^d).$$

Lemma 3. For every function $\tau \in Q := \{\sigma \in L^2(\Omega; \mathbb{R}^{d \times d}_{sym}) : \operatorname{div} \sigma \in L^2(\Omega; \mathcal{R}^d)\}$ it holds

$$\|\varepsilon(\mathbf{u}-\mathbf{v})\|_{\mathbb{L};\Omega} \leq \|\varepsilon(\mathbf{v})-\mathbb{L}^{-1}\tau\|_{\mathbb{L};\Omega} + C \|\operatorname{div}\tau+\mathbf{f}-\alpha_1\nabla \mathsf{p}_1-\alpha_2\nabla \mathsf{p}_2\|_{\Omega}, \quad (4.1)$$

where the constant C > 0 satisfies an inequality

$$\|\mathbf{w}\|_{\Omega} \le C \|\varepsilon(\mathbf{w})\|_{\mathbb{L};\Omega} \quad \text{for all } \mathbf{w} \in H^1_0(\Omega; \mathcal{R}^d).$$

$$(4.2)$$

and the norm $\|\cdot\|$ is defined as $\|\varepsilon\|^2_{\mathbb{L};\Omega} := \int_{\Omega} \mathbb{L} \varepsilon : \varepsilon \, dx.$

Let q_1 and q_2 be approximation of exact pressure fields p_1 and p_2 respectively. By triangle inequalities, we obtain

$$\begin{aligned} \|\operatorname{div} \tau + \mathbf{f} - \alpha_1 \nabla \mathbf{p}_1 - \alpha_2 \nabla \mathbf{p}_2\|_{\Omega} &\leq \|\operatorname{div} \tau + \mathbf{f} - \alpha_1 \nabla \mathbf{q}_1 - \alpha_2 \nabla \mathbf{q}_2\|_{\Omega} \\ &+ \|\nabla(\mathbf{p}_1 - \mathbf{q}_1)\|_{\Omega} + \|\nabla(\mathbf{p}_2 - \mathbf{q}_2)\|_{\Omega}. \end{aligned}$$
(4.3)

Use (4.3) and square both parts of (4.1) to obtain

$$\begin{aligned} \|\varepsilon(\mathbf{u} - \mathbf{v})\|_{\mathbb{L};\Omega}^{2} &\leq \left(\left\|\varepsilon(\mathbf{v}) - \mathbb{L}^{-1}\tau\right\|_{\mathbb{L};\Omega} + C \left\|\operatorname{div}\tau + \mathbf{f} - \alpha_{1}\nabla \mathbf{q}_{1} - \alpha_{2}\nabla \mathbf{q}_{2}\right\|_{\Omega} + C \left\|\nabla(\mathbf{p}_{1} - \mathbf{q}_{1})\right\|_{\Omega} + C \left\|\nabla(\mathbf{p}_{2} - \mathbf{q}_{2})\right\|_{\Omega}\right)^{2}. \end{aligned}$$

$$(4.4)$$

By the algebraic inequality

$$(a+b+c)^2 \le (1+\beta_4+\beta_5) a^2 + (1+\frac{1}{\beta_4}+\beta_6) b^2 + (1+\frac{1}{\beta_5}+\frac{1}{\beta_6}) c^2$$

valid for all scalars a, b, c and for all $\beta_4, \beta_5, \beta_6 > 0$, inequality (4.4) and the following inequality (β_3 is an arbitrary positive constant)

$$(\|\nabla(\mathbf{p}_{1} - \mathbf{q}_{1})\|_{\Omega} + \|\nabla(\mathbf{p}_{2} - \mathbf{q}_{2})\|_{\Omega})^{2} \leq (1 + \beta_{3}) \|\nabla(\mathbf{p}_{1} - \mathbf{q}_{1})\|_{\Omega}^{2} + (1 + \frac{1}{\beta_{3}}) \|\nabla(\mathbf{p}_{2} - \mathbf{q}_{2})\|_{\Omega}^{2} \leq \max\{\frac{1 + \beta_{3}}{k_{1}}, \frac{1 + \beta_{3}}{k_{2}\beta_{3}}\} \ a(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q}) \leq 2\max\{\frac{1 + \beta_{3}}{k_{1}}, \frac{1 + \beta_{3}}{k_{2}\beta_{3}}\} \ M_{\beta_{1},\beta_{2}}(\mathbf{q}, \hat{\mathbb{Y}}^{*}).$$
(4.5)

Now we obtain the final estimate in terms of the coupled error norm

$$a(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q}) + \|\varepsilon(\mathbf{u} - \mathbf{v})\|_{\mathbb{L};\Omega}^{2} \leq (1 + \beta_{4} + \beta_{5}) \|\varepsilon(\mathbf{v}) - \mathbb{L}^{-1}\tau\|_{\mathbb{L};\Omega}^{2} + \left(1 + \frac{1}{\beta_{4}} + \beta_{6}\right) C^{2} \|\operatorname{div}\tau + \mathbf{f} - \alpha_{1}\nabla \mathbf{q}_{1} - \alpha_{2}\nabla \mathbf{q}_{2}\|_{\Omega}^{2} + 2\widehat{C} \ M_{\beta_{1},\beta_{2}}(\mathbf{q},\hat{\mathbb{Y}}^{*}), \quad (4.6)$$

where

$$\widehat{C} = 1 + C^2 \left(1 + \frac{1}{\beta_5} + \frac{1}{\beta_6} \right) \max\left\{ \frac{1 + \beta_3}{k_1}, \frac{1 + \beta_3}{k_2\beta_3} \right\}$$

This estimate holds for all $\tau \in Q$, $\hat{\mathbb{Y}}^* \in Y^*_{div}$ and all $\beta_1, \ldots, \beta_6 > 0$.

Remark 2. Finally, we comment on that how this estimate can be used in practical computations. Assume that numerical solutions of the Barenblatt-Biot system (1.3) are obtained on certain finite dimensional subspace generated by the mesh \mathcal{T}_h . We denote them \mathbf{q}_h and \mathbf{v}_h . In the simplest case, we need to post-process the functions $\mathbf{q}_h := \nabla \mathbf{q}_h$ and $\tau_h := \mathbb{L}\varepsilon(\mathbf{v}_h)$ in such a way that their post-processed images $\tilde{\mathbf{q}}_h$ and $\tilde{\tau}_h$ belong to Q and Y_{div}^* , respectively. Then a guaranteed upper bound follows from (4.6) by direct substitution and optimization with respect to the parameters $\beta_1, \ldots, \beta_6 > 0$. A sharper estimate can be obtained if the majorant is further minimized with respect to \mathbf{q}_h and τ_h with the help of some direct minimization procedure (e.g., gradient descent). Another way may be efficient if the problem is solved on a sequence of consequently refined meshes. In this case, we can use the above described procedure (based on a relatively simple post-processing procedure for \mathbf{q}_h and τ_h but with one step retardation, i.e., averaging is performed on the mesh h_k but it is used in the error estimate for approximate solutions computed on the mesh h_{k-1} .

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