V European Conference on Computational Fluid Dynamics
ECCOMAS CFD 2010
J. C. F. Pereira and A. Sequeira (Eds)
Lisbon, Portugal,14-17 June 2010

BLOCK PRECONDITIONING WITH SCHUR COMPLEMENTS FOR MONOLITHIC FLUID-STRUCTURE INTERACTIONS

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Key words: Fluid-Structure Interactions, Finite Elements, Preconditioning, Multigrid

Abstract. *We introduce a block preconditioner for monolithic 'arbitrary Lagrangian-Eulerian' (ALE) fluid-structure interactions based on finite elements. While using this framework all equations arising from fluid and structure are treated in a common coordinate system. After spatial discretization of the continuous problem by the Galerkin method, the arising system is non-linear and solved by Newton's method.*

For large problems with many unknowns, at least for 3d problems, good candidates to solve the inner linear equation system are Krylov space methods, e.g. GMRES. This solver works efficiently with suitable preconditioners for the Jacobian matrix which is very ill-conditioned in the case of fluid-structure interactions.

In this contribution we present a block preconditioner which is based on the Schur complement technique for the Jacobian matrix, utilizing geometric multigrid approaches. After a theoretical discussion, the performance of this block preconditioner is studied in numerical computations for a stationary fluid-structure interaction problem.

1 INTRODUCTION

We investigate efficient numerical solution techniques for a nonlinear coupled problem describing the interaction of a fluid and a solid structure. General situations of fluid-structure interactions (FSI) arise in many problems in engineering and biomechanics, e.g. flow around elastic structures (airplanes) and flow in elastic structures (haemodynamics).

Typically, fluid and structure are given in different coordinate systems making a common solution approach challenging: Fluid flows are given in Eulerian coordinates whereas the structure is treated in a Lagrangian framework. The coupling of these subproblems leads to highly nonlinear behaviour and the development of robust numerical solution techniques is subject of intensive research; see, e.g., Heil,[10] and Badia et al.,[1] and references cited therein.

Two main approaches to solve these problems are used in practice. In the *partitioned approach*, separate equations for the fluid and for the structure problem are solved alternatingly, using the calculated forces of the one to drive the other via boundary conditions (velocity and normal stress). See the book of Bungartz and Schäfer[5] for various articles on the partitioned approach. If used as an explicit method it requires very small time steps. Alternatively, for an implicit scheme an iteration between the two subproblems in every time step is necessary.

A second possibility is the *monolithic approach*. Here, one of the subproblems is transformed to an artificial coordinate system that matches the other. In the well-known 'arbitrary Lagrangian Eulerian' approach (ALE) the fluid problem is rewritten on an *arbitrary* transformed 'reference' domain. Usually, this is achieved by some auxiliary unknown coordinate transformation for the fluid domain. For detailed derivation of this approach we refer to Hron and Turek,[11] and Dunne.[7]

Both approaches, partitioned and monolithic, are handled with different solution strategies. A partitioned approach utilizes separate solvers for the fluid and solid domains. Then, the solution procedure is coupled via a fixed-point iteration. This approach is relatively easy to implement and the major advantage is that efficient specialized solvers for the fluid and structure parts are available. However, partitioned methods also have serious drawbacks. Fixed point iteration techniques tend to converge slowly and the accuracy of the interface conditions is less clear. It is very costly to achieve higher accuracy, at least for large time dependent problems.

In the monolithic approach all equations are solved at once. Here, the interface conditions, continuity of velocity and normal stresses, become implicit and are automatically satisfied. However, spatial discretization of the variational equations results in a large system of coupled nonlinear algebraic equations. It can be solved by a Newton-like method whose robust and rapid convergence make it very attractive. The major drawback of the monolithic approach is the ill-conditioning of the Jacobian matrix.

For two-dimensional problems direct solvers are a good choice when dealing with manageable amount of unknowns. However, for systems with more than 10^6 unknowns in

2d and at least for three-dimensional problems the systems should be solved by Krylov space methods, e.g. GMRES (see Saad[17]). Due to ill-conditioning of the Jacobian it is indispensable to use appropriate preconditioners with multigrid acceleration. The development and application of such multigrid-based preconditioners for adaptively refined meshes is the subject of this paper. We will base our work on preconditioners with local smoothing for adaptively refined meshes, see, e.g., Janssen and Kanschat.[13]

For fluid problems an efficient preconditioner is based on the Schur complement technique which can be derived by formal Gauss elimination of the system matrix. Then, the Schur complement is approximated by the mass matrix on the pressure space, see Turek.[18] We extend this idea to the 3×3 system which arises for the monolithic fluid-structure interaction under consideration.

The outline of this paper is as follows: In the second section, the monolithic problem in ALE formulation is briefly introduced. The fluid is assumed to be Newtonian and laminar, whereas the structure is described by classical (geometrically) nonlinear compressible Saint Venant-Kirchhoff material (STVK). In Section 3 we briefly describe the spatial discretization of the monolithic variational formulation. Furthermore we present the block preconditioner for the Jacobi matrix of Newton's method. Finally, in Section 5, we evaluate the proposed preconditioner for the GMRES scheme with a widely used stationary fluid-structure interaction benchmark, and compare it to a direct solver. All computations for the numerical tests are done using the finite element library *deal.II*.[2]

2 THE COUPLED PROBLEM IN ALE COORDINATES

In this section we briefly introduce the coupled problem in ALE coordinates. Detailed discussion of the derivation can be found, for instance, in Dunne.[7]

A coupled system for fluid-structure interaction problems must be formulated on the common domain $\hat{\Omega}$ which is called the *reference domain*. The subproblems are given on the reference domains $\hat{\Omega}_f$ and $\hat{\Omega}_s$. To distinguish between fluid and the structure domains we use a subsript 'f' or 's', respectively.

Using the reference domains $\hat{\Omega}_f$ and $\hat{\Omega}_s$ lead to the well-established ALE coordinates. To gain the monolithic formulation we need to specify the transformation $\hat{T}_f : \hat{\Omega}_f \to \Omega_f$ from the reference state into the physical fluid-domain.

Inside $\hat{\Omega}_f$ the transformation should be as smooth and regular as possible, but apart from that it is arbitrary. On the interface $\hat{\Gamma}_i$ between fluid and structure, as shown in Figure 1 this transformation is given by the structure displacement:

$$\hat{T}_f(\hat{x})\big|_{\hat{\Gamma}_i} = \hat{x} + \hat{u}_f(\hat{x})\big|_{\hat{\Gamma}_i}. \tag{1}$$

On the outer boundary of the fluid domain $\partial\hat{\Omega}_f \setminus \hat{\Gamma}_i$ there holds $\hat{T}_f = I$. One possibility is to harmonically extend $\hat{u}_s|_{\hat{\Omega}_s}$ to the fluid domain $\hat{\Omega}_f$:

$$(\hat{\nabla}\hat{u}_f, \hat{\nabla}\phi^u)_{\hat{\Omega}_f} = 0, \quad \hat{u}_f = \hat{u}_s \text{ on } \hat{\Gamma}_i, \quad \hat{u}_f = 0 \text{ on } \partial\hat{\Omega}_f \setminus \hat{\Gamma}_i. \tag{2}$$

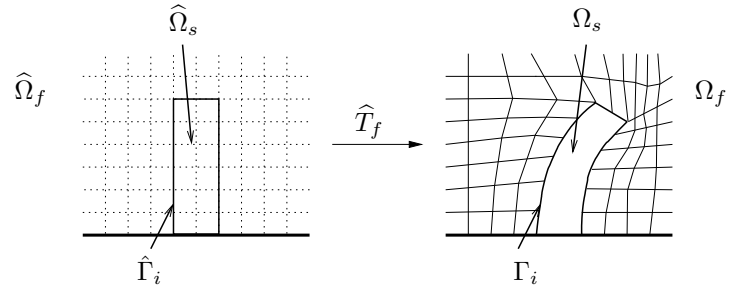

Figure 1: Fluid mesh transformation by $\hat{T}_f$ of the ALE transformation

This way we can define a continuous variable $\hat{u}$ on the whole domain Ω defining the deformation in $\hat{\Omega}_s$ and supporting the transformation in $\hat{\Omega}_f$. Thus, by skipping the subscripts and since the definition of $\hat{T}_f$ coincides with the definition of $\hat{T}_s : \hat{\Omega}_s \to \Omega_s$ with

$$\hat{T}_s(\hat{x}) = \hat{x} + \hat{u}_s(\hat{x}), \tag{3}$$

we define on $\hat{\Omega}$:

$$\hat{T} := \hat{x} + \hat{u}, \quad \hat{F} := \hat{\nabla}\hat{T} = I + \hat{\nabla}\hat{u}, \quad \hat{J} := \det(\hat{F}). \tag{4}$$

The transformed variables for fluid and pressure are denoted by $\hat{v}$ and $\hat{p}$. Their defintions are as follows:

$$\hat{v}(\hat{x},t) = v(x,t) = v(\hat{T}(\hat{x},t),x), \quad \hat{p}(\hat{x},t) = p(x,t) = p(\hat{T}(\hat{x},t),x). \tag{5}$$

Then, with (4) we get the relations:

$$\nabla v = \hat{\nabla}\hat{v}\hat{F}^{-1}, \quad \partial_t v = \partial_t \hat{v} - (\hat{F}^{-1}\partial_t \hat{T}\cdot\hat{\nabla})\hat{v}, \quad \int_{\Omega} f(x)\mathrm{d}x = \int_{\hat{\Omega}} \hat{f}(\hat{x})\hat{J}\mathrm{d}\hat{x}. \tag{6}$$

With help of these relations we can formulate the Navier-Stokes equations in artificial coordinates, and subsequently define the fully coupled fluid-structure interaction problem. For details of derivation we refer to Dunne.[7]

Let now $\hat{V}^0$ be a subspace of $H^1(\hat{\Omega})$ with trace zero on Dirichlet boundaries $\hat{\Gamma}^D := \hat{\Gamma}_f^D \cup \hat{\Gamma}_s^D$. Further, let $\hat{v}^D, \hat{u}^D \in H^1(\hat{\Omega})$ be prolongations of the Dirichlet data for velocity and deformation into the domain. Finally, with $\hat{L} := L^2(\hat{\Omega})/\mathbb{R}$ we find $\hat{v} \in \hat{v}^D + \hat{V}^0$, $\hat{u} \in \hat{u}^D + \hat{V}^0$ and $\hat{p} \in \hat{L}$ by:

Problem 2.1 (Variational stationary fluid-structure interaction, ALE framework)**.** Find $\{\hat{v},\hat{u},\hat{p}\} \in \{\hat{v}^D + \hat{V}^0\} \times \{\hat{u}^D + \hat{V}^0\} \times \hat{L}$, for the equations

$$\begin{aligned}
(\hat{J}\hat{\rho}_f(\hat{F}^{-1}\hat{v}\cdot\hat{\nabla})\hat{v}),\hat{\phi}^v)_{\hat{\Omega}_f} + (\hat{J}\hat{\sigma}_f\hat{F}^{-T},\hat{\nabla}\hat{\phi}^v)_{\hat{\Omega}_f} + (\hat{J}\hat{\sigma}_s\hat{F}^{-T},\hat{\nabla}\hat{\phi}^v)_{\hat{\Omega}_s} \\
-\langle\hat{g},\hat{\phi}^v\rangle_{\hat{\Gamma}_N} - (\hat{\rho}_f\hat{J}\hat{f}_f,\hat{\phi}^v)_{\hat{\Omega}_f} - (\hat{\rho}_s\hat{J}\hat{f}_s,\hat{\phi}^v)_{\hat{\Omega}_s} = 0 \quad \forall\hat{\phi}^v \in \hat{V}^0, \\
(\hat{v},\hat{\phi}^u)_{\hat{\Omega}_s} + (\hat{\nabla}\hat{u},\hat{\nabla}\hat{\phi}^u)_{\hat{\Omega}_f} - \langle\hat{n}_f\hat{\nabla}\hat{u},\hat{\phi}^u\rangle_{\hat{\Gamma}_i} = 0 \quad \forall\hat{\phi}^u \in \hat{V}^0, \\
(\widehat{\mathrm{div}}\,(\hat{J}\hat{F}^{-1}\hat{v}_f),\hat{\phi}^p)_{\hat{\Omega}_f} + (\hat{p}_s,\hat{\phi}^p)_{\hat{\Omega}_s} = 0 \quad \forall\hat{\phi}^p \in \hat{L},
\end{aligned} \tag{7}$$

where $\hat{\rho}_f$ and $\hat{\rho}_s$ describe density of fluid and structure, respectively. Exterior volume forces are represented by $\hat{f}_f$ and $\hat{f}_s$, and $\hat{g}$ is a function which is given at the Neumann boundary. Further, the viscosity of the fluid is given by $\hat{\nu}_f$, and the structure is characterized by the Lamé coefficients $\hat{\lambda}_s$ and $\hat{\mu}_s$. The Cauchy stress tensors for fluid and structure are given by:

$$\hat{\sigma}_f := -\hat{J}\hat{p}_f I\hat{F}^{-T} + \hat{\rho}_f\hat{\nu}_f(\hat{\nabla}\hat{v}_f\hat{F}^{-1} + \hat{F}^{-T}\hat{\nabla}\hat{v}_f^T),$$
$$\hat{\sigma}_s := \hat{J}^{-1}\hat{F}(\hat{\lambda}_s(tr\hat{E})I + 2\hat{\mu}_s\hat{E})\hat{F}^{-T}.$$

The boundary term on the interface for the harmonic continuation of $\hat{u}$ is necessary to prevent a spurious feedback of the extended (artificial) displacement variable in the flow domain $\hat{\Omega}_f$ to the structure domain $\hat{\Omega}_s$, for more details on this we refer to Richter and Wick.[16]

Due to integration by parts in both subdomains, this formulation includes the natural boundary condition leading to the correct equilibrium in variational formulation:

$$(\hat{J}\hat{\sigma}_f\hat{F}^{-T}\hat{n}_f, \hat{\phi}^v) = (\hat{J}\hat{\sigma}_s\hat{F}^{-T}\hat{n}_s, \hat{\phi}^v) \text{ on } \hat{\Gamma}_i. \tag{8}$$

3 DISCRETIZATION

In this section, we focus on the spatial discretization of the fluid-structure interaction problem (7). Our method of choice is a Galerkin finite element method. In case of monolithic ALE fluid-structure interactions the computations are done on the reference configuration $\hat{\Omega}$. We use two-or three dimensional shape-regular meshes[6] that consists of quadrilateral cells $\hat{K}$. They perform a non-overlapping cover of the computation domain $\hat{\Omega} \subset \mathbb{R}^d$, $d = 2, 3$. The discretization parameter $\hat{h}$ is cellwise constant and is given by the diameter $\hat{h}_{\hat{K}}$ of the cell $\hat{K}$.

3.1 Spatial discretization

In order to discretize the spatial variables, we define the following semilinear form on a continuous level in an abstract setting: Find $\hat{U} = \{\hat{v}, \hat{u}, \hat{p}\} \in \hat{X}$, where $\hat{X} := \{\hat{v}^D + \hat{V}^0\} \times \{\hat{u}^D + \hat{V}^0\} \times \hat{L}$, such that

$$\hat{A}(\hat{U})(\hat{\Phi}) = \hat{G}(\hat{U})(\hat{\Phi}) \quad \forall\hat{\Phi} \in \hat{X}.$$

Here the semilinear forms $\hat{G}(\hat{U})(\hat{\Phi})$ and $\hat{A}(\hat{U})(\hat{\Phi})$ are defined as

$$\hat{G}(\hat{U})(\hat{\Phi}) = (\hat{\rho}_f\hat{J}\hat{f}_f, \hat{\phi}^v)_{\hat{\Omega}_f} + (\hat{\rho}_s\hat{J}\hat{f}_s, \hat{\phi}^v)_{\hat{\Omega}_s} + \langle\hat{g}, \hat{\phi}^v\rangle_{\hat{\Gamma}_N} \tag{9}$$

and

$$\begin{aligned}\hat{A}(\hat{U})(\hat{\Phi}) = {} & (\hat{J}\hat{\rho}_f(\hat{F}^{-1}\hat{v}\cdot\hat{\nabla})\hat{v}), \hat{\phi}^v)_{\hat{\Omega}_f} + (\hat{J}\hat{\sigma}_f\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^v)_{\hat{\Omega}_f} + (\hat{J}\hat{\sigma}_s\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^v)_{\hat{\Omega}_s} \\ & + (\hat{v}, \hat{\phi}^u)_{\hat{\Omega}_s} + (\hat{\nabla}\hat{u}, \hat{\nabla}\hat{\phi}^u)_{\hat{\Omega}_f} - \langle\hat{n}_f\hat{\nabla}\hat{u}, \hat{\phi}^u\rangle_{\hat{\Gamma}_i} \\ & + (\widehat{\text{div}}\,(\hat{J}\hat{F}^{-1}\hat{v}_f), \hat{\phi}^p)_{\hat{\Omega}_f} + (\hat{p}_s, \hat{\phi}^p)_{\hat{\Omega}_s}.\end{aligned} \tag{10}$$

The semilinear form is discretized by the Galerkin finite element method. Thus, we construct a conforming finite subspace $\hat{X}_h \subset \hat{X}$, with $\hat{X}_h := Q_2^c \times Q_2^c \times P_1^{dc}$. Here, Q_2^c denotes the continuous space of piecewise bi- or tri-quadratic polynomials, whereas P_1^{dc} denotes the discontinuous space of piecewise linear polynomials. These spaces satisfy the usual compatibility conditions for incompressible fluids (see Girault and Raviart[9]), as well as corresponding conditions for coupling fluid and solid.

The discrete problem is formulated as: Find $\hat{U}_h = \{\hat{v}_h, \hat{u}_h, \hat{p}_h\} \in \hat{X}_h$, such that

$$\hat{A}_h(\hat{U}_h)(\hat{\Phi}) = \hat{G}_h(\hat{U}_h)(\hat{\Phi}) \quad \forall \hat{\Phi} \in \hat{X}_h.$$

3.2 Linearization

The whole discretization results in a nonlinear algebraic problem which is solved using a Newton-like method. Here, the linear defect-correction problem

$$\begin{aligned} \hat{A}'_h(\hat{U}_h^j)(\hat{\delta}\hat{U}_h, \hat{\Phi}) &= \hat{G}_h(\hat{U}_h)(\hat{\Phi}) - \hat{A}_h(\hat{U}_h^j)(\hat{\Phi}) \\ \hat{U}_h^{j+1} &= \hat{U}_h^j + \lambda \hat{\delta}\hat{U}_h \end{aligned} \tag{11}$$

has to be solved, where $\lambda \in (0, 1]$ describes a damping parameter. Assembling the system (11) requires the evaluation of directional derivatives. For more details we refer to the work by Richter and Wick,[16] and Dunne.[7] Due to the large size of the Jacobian and the strongly nonlinear behaviour of the complete FSI problems in the monolithic ALE framework, calculating the Jacobian matrix can be cumbersome. Nevertheless, in this context we use the exact Jacobian matrix to get optimal convergence properties of Newton's method.

3.3 Development of the block preconditioner

In the interest of brevity, we will omit hats indicating the reference configuration in the following. After spatial discretization and linearization, in each Newton step a linearized problem is solved to achieve the solution of the (originally) non-linear problem.

The contribution within the fluid subdomain to the global problem (11) has the following structure

$$\begin{pmatrix} K + L_{vv} & S_{vu} & B \\ 0 & L_{uu} & 0 \\ B^T & S_{pu} & 0 \end{pmatrix} \begin{pmatrix} \delta v_f \\ \delta u_f \\ \delta p_f \end{pmatrix} = \begin{pmatrix} b_{f,v} \\ b_{f,u} \\ b_{f,p} \end{pmatrix}, \tag{12}$$

whereas the contribution of the STVK material is

$$\begin{pmatrix} 0 & S_{vu} & 0 \\ M_{uv} & 0 & 0 \\ 0 & 0 & M_{pp} \end{pmatrix} \begin{pmatrix} \delta v_s \\ \delta u_s \\ \delta p_s \end{pmatrix} = \begin{pmatrix} b_{s,v} \\ b_{s,u} \\ b_{s,p} \end{pmatrix}. \tag{13}$$

Consequently, the Jacobian of the full problem has the structure

$$\begin{pmatrix} K + L_{vv} & S_{vu} & B \\ M_{uv} & L_{uu} & 0 \\ B^T & S_{pu} & M_{pp} \end{pmatrix} \begin{pmatrix} \delta v \\ \delta u \\ \delta p \end{pmatrix} = \begin{pmatrix} b_v \\ b_u \\ b_p \end{pmatrix}. \tag{14}$$

Here, the introduced matrices are characterized as follows. In the fluid domain, we have the convection term K, the Laplacians L_{vv} and L_{uu}, the gradient matrix B and the (negative) transposed divergence matrix B^T on a discrete level. The remaining terms S_{vu} and S_{pu} arise from coupling with structure variables. In the structure domain, we find two coupling terms S_{vu} and M_{vu} with fluid variables, and the pressure mass matrix M_{pp}.

In each Newton step, see equation (11), we are concerned with a linear system

$$Ax = b, \tag{15}$$

where A is a block matrix which represents (24), $x = \{\delta u, \delta v, \delta p\}$, and b the right-hand side. To solve system (15) we precondition by a matrix P^{-1} and arrive at

$$P^{-1}Ax = P^{-1}b. \tag{16}$$

If we find appropriate entries for P^{-1} such that the condition number of $P^{-1}A$ is moderate, then the whole systems will converge in a few iterations.

Ideally, a decomposition of A would be the perfect preconditioner. In practice, we will be content with the following block decomposition:

$$\begin{aligned} A &= \begin{pmatrix} K + L_{vv} & S_{vu} & B \\ M_{uv} & L_{uu} & 0 \\ B^T & S_{pu} & M_{pp} \end{pmatrix} \\ &= \begin{pmatrix} \Sigma_2^{-1} & -\Sigma_2^{-1}\Sigma_1 L_{uu}^{-1} & -\Sigma_2^{-1} B M_{pp}^{-1} \\ 0 & L_{uu}^{-1} & 0 \\ 0 & 0 & M_{pp}^{-1} \end{pmatrix} \begin{pmatrix} \Sigma_2 & \Sigma_1 & B \\ 0 & L_{uu} & 0 \\ 0 & 0 & M_{pp} \end{pmatrix}. \end{aligned} \tag{17}$$

The first matrix is used as (exact) preconditioner for A and is denoted by P^{-1}. Two Schur complement matrices have been introduced in (17):

$$\Sigma_1 = S_{vu} - B M_{pp}^{-1} S_{pu}, \tag{18}$$

$$\Sigma_2 = K + L_{vv} - B M_{pp}^{-1} B^T - \Sigma_1 L_{uu}^{-1} M_{uv}. \tag{19}$$

Remark 3.1. Since the matrices of L_{uu} and M_{pp} 'live' only on one part of the domain it is very difficult to compute the inverse of these matrices. The term L_{uu} is given on the fluid domain and not on the structure domain, whereas for M_{pp} it is vice versa. To be able to invert these matrices we add their corresponding parts on the other domain. These are 'penalty' terms of lower order. However, this procedure is critical when using iterative solvers since their performance depends on the condition number of the matrices.

Due to the fact that we use P^{-1} in a Krylov method we only have to perform matrix-vector multiplications, such as

$$\begin{pmatrix} X_{new} \\ Y_{new} \\ Z_{new} \end{pmatrix} = \begin{pmatrix} \Sigma_2^{-1} & -\Sigma_2^{-1}\Sigma_1 L_{uu}^{-1} & -\Sigma_2^{-1} B M_{pp}^{-1} \\ 0 & L_{uu}^{-1} & 0 \\ 0 & 0 & M_{pp}^{-1} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{20}$$

Writing these terms explicitly, we have to compute (in reverse order):

$$Z_{new} = M_{pp}^{-1} Z, \tag{21}$$
$$Y_{new} = L_{uu}^{-1} Y, \tag{22}$$
$$\begin{aligned} X_{new} &= \Sigma_2^{-1} X - \Sigma_2^{-1} \Sigma_1 L_{uu}^{-1} Y - \Sigma_2^{-1} B M_{pp}^{-1} Z \\ &= \Sigma_2^{-1} [X - \Sigma_1 Y_{new} - B Z_{new}]. \end{aligned} \tag{23}$$

In these subsequent steps, one has to compute M_{pp}^{-1} and L_{uu}^{-1}, where the first one can be treated by the cg-algorithm. The inverse of L_{uu} can be computed by a direct solver or with geometric multigrid solver. Such an algorithm with local smoothing for adaptive finite elements with hanging nodes is extensively discussed for the Laplace equation in Janssen and Kanschat.[13]

The third step (23) is the most difficult one. As part of the procedure, use the first Schur complement Σ_1 as derived in (18) without any approximation. To this end, we also use Σ_2 as proposed in (19) and subsequently compute its inverse.

To the authors knowledge, there is no efficient approximation to Σ_2^{-1} proposed in the literature so far. Therefore, we simple compute the inverse by an inner GMRES iteration which is cumbersome due to its high ill-conditionedness. It is still an open question and current research to find a good preconditioner for Σ_2^{-1}.

Theoretical results

Other precondioning strategies can be found in Heil[10] , and Badia et al.[1] In the first article, three block-triangular approximations of the Jacobian matrix are introduced, to separate the coupling terms of fluid and structure. For the fluid problem, the 'normal' Schur complement iteration technique is used. The solution of the fully coupled system remains in a number of sub-steps for each of the three preconditioners. The application of the preconditioners results in the solution of four linear systems and (up to) three matrix-vector products.

In the article by Badia et al.[1] , the authors investigate semi-implicit solution schemes for fluid-structure interactions. The key idea is, to decouple the computation of fluid's velocity from the whole system where then only structure- and pressure variables remain. The advantage of this approach is to reduce computational cost and ensure stability of the solution algorithm. The authors apply explicit-implicit splitting derived from algebraic factorization splitting methods which are based on inexact factorization of the system matrix. This method is then used as preconditioner which results in a predictor-corrector method.

4 Numerical Example

We validate the proposed preconditioner with the Fluid Structure Interaction (FSI) benchmark configurations which are given in Hron and Turek.[12]

Configuration

The computational domain has length $L = 2.5$, height $H = 0.41$. The circle center is positioned at $C = (0.2, 0.2)$ with radius $r = 0.05$. The elastic flag has length $l = 0.35$ and height $h = 0.02$. The right lower end is positioned at $(0.6, 0.19)$ and the left end is attached to the circle. We evaluate the displacement at the trailing edge of the structure, with $A(0) = (0.6, 0.2)$.

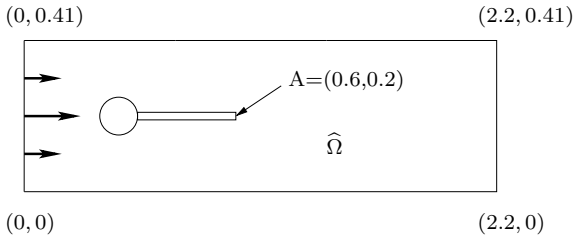

Figure 2: Flow around cylinder with elastic flag with circle-center $C = (0.2, 0.2)$ and radius $r = 0.05$.

Boundary conditions

A parabolic inflow velocity profile is given at the left channel side by

$$v_f(0, y) = 1.5\bar{U}\frac{4y(H - y)}{H^2}.$$

At the right side (outlet) the 'do-nothing' outflow condition leads to zero-stress $\sigma \cdot n = 0$. This implicitly gives the mean value of the pressure to be zero. The structure displacement is zero at inflow- and outflow boundary. For the upper and lower boundaries the 'no-slip' conditions for velocity and structure displacement are given.

Material parameters

The fluid flow is considered to be a incompressible Newtonian fluid, the cylinder as fixed and rigid, and the elastic structure is given by (compressible) STVK material.

Parameters	FSI 1
$\varrho_s[10^3 kgm^{-3}]$	1
ν_s	0.4
$\mu_s[10^6 kgm^{-1}s^{-2}]$	0.5
$\varrho_f[10^3 kgm^{-3}]$	1
$\nu_f[10^{-3}m^2s^{-1}]$	1
$\bar{U}[ms^{-1}]$	0.2

Table 1: Parameters for the FSI 1 test case

Quantities of comparison

1. x- and y deflection of the flag at A

2. The forces exerted by the fluid on the whole body (rigid cylinder and elastic flag), i.e. drag- and lift forces which are denoted by F_D and F_L, respectively. They form a closed path such that the forces can be computed with help of line integration. The formula reads:

$$(F_D, F_L)^T = \int_S \sigma_{all} \cdot n \, ds = \int_{S(\text{circle})} \sigma_f \cdot n \, ds + \int_{S(\text{flag})} \sigma_f \cdot n \, ds.$$

Mesh refinement

The computations are done on globally refined meshes and (heuristicly) refined meshes based on zonal mesh refinement around the flag.

Solvers

The results shown below are computed with two different linear solvers. In the first set of computations we use a direct solver (UMFPACK) to solve the whole linear system at once and to validate the FSI code. The second set of computations is done by the proposed GMRES iteration with block preconditioner.

Properties of the iterative solvers

The lower bound for the Newton residuum is chosen as 10^{-8} whereas the outer GMRES finishes for an accuracy of 10^{-6}. The inner GMRES iteration to compute the inverse of Σ_2 is taken as 10^{-4}.

Results

We introduce the positive parameters α_p and α_v to control the influence of the 'penalty' terms as discussed in Remark 3.1. These parameters are chosen small enough such that their physicial influence is negligible. That means we solve the following linear problem:

$$\begin{pmatrix} K + L_{vv} & S_{vu} & B \\ \alpha_v M_{uv} & L_{uu} & 0 \\ B^T & S_{pu} & \alpha_p M_{pp} \end{pmatrix} \begin{pmatrix} \delta v \\ \delta u \\ \delta p \end{pmatrix} = \begin{pmatrix} b_v \\ b_u \\ b_p \end{pmatrix}. \tag{24}$$

If α_p and α_v are chosen close to one the inner iterations converge faster than taking $\alpha_p \ll 1$ and $\alpha_v \gg 1$.

Results for a test case where the linear systems were solved with a direct solver are shown in Table 2. The next results shown in Table 3 are computed using the proposed

Unknowns	$u_x(A)[\times 10^{-5}]$	$u_y(A)[\times 10^{-4}]$	F_D	F_L
5032	2.2631	8.8020	14.116	0.7676
8944	2.2592	8.8583	14.120	0.7661
19424	2.2493	8.7538	14.144	0.7612
(ref.)	2.2700	8.2090	14.294	0.7637

Table 2: Results for the FSI 1 test case computed by UMFPACK and $\alpha_p = 0$ and $\alpha_v = 10^4$

block preconditioner. In this test we chose the parameters $\alpha_p = 10^{-3}$ and $\alpha_v = 10^2$. The

Unknowns	$u_x(A)[\times 10^{-5}]$	$u_y(A)[\times 10^{-4}]$	F_D	F_L
5032	2.1045	0.1467	12.846	0.2494
8944	2.0780	0.0824	12.886	0.1848

Table 3: Results for the FSI 1 test case computed by outer GMRES iteration with block preconditioner and $\alpha_p = 10^{-3}$ and $\alpha_v = 10^2$

Unknowns	$u_x(A)[\times 10^{-5}]$	$u_y(A)[\times 10^{-4}]$	F_D	F_L
5032	2.1252	1.0023	12.855	0.5074
8944	2.1215	0.5614	12.858	0.4608

Table 4: Results for the FSI 1 test case computed by outer GMRES iteration with block preconditioner and $\alpha_p = 10^{-3}$ and $\alpha_v = 10^3$

test shown in Table 4 uses the block preconditioner as well but the parameters are chosen as $\alpha_p = 10^{-3}$ and $\alpha_v = 10^3$. The Table 5 displays the range of outer number of GMRES iteration steps needed to perform one Newton step.

Unknowns	GMRES it.	α_p	α_v
5032	$8-12$	10^{-3}	10^2
8944	$8-14$	10^{-3}	10^2
5032	$8-14$	10^{-3}	10^3
8944	$9-15$	10^{-3}	10^3

Table 5: Number of outer GMRES iterations within one mesh refinement cycle

5 CONCLUSION

In this article we presented a block preconditioner for the Jacobian matrix of a monolithic fluid-structure interaction problem. Due to the complex structure of the Jacobian matrix it is a challenging task to find appropriate preconditioners for iterative solvers. In this work we proposed a block preconditioner that treats the whole matrix. It is based on the Schur complement techniques which are well-known from fluid dynamics. Since, no approximation to the Schur complement matrix in case of fluid-structure interactions is known a simple approximation is used instead. In a next step, suitable approximations for the inverses of the Schur complement matrices will be investigated.

Then, we extend the ideas suggested in this contribution to time-dependent problems as well as for three dimensional test cases where usage of direct solvers is much more difficult and iterative solvers become indispensable for problems with many unknowns.

6 ACKNOWLEDGEMENTS

We thank Prof. Wolfgang Bangerth and Prof. Guido Kanschat from Texas A&M University for fruitful discussions and remarks on this work. Furthermore, we would like to thank Dr. Thomas Richter from University of Heidelberg for discussing the initial ideas of this work.

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