

## SOLUTION OF THE 2D NAVIER-STOKES EQUATIONS WITH THE LBIE METHOD AND RBF CELLS

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**Abstract.** *This work presents the Local Boundary Integral Equation (LBIE) method for the solution of two dimensional incompressible fluid flow problems governed by the Navier-Stokes equations. The method uses, for its implementation, continuous and special discontinuous (RBF) cells over the analyzed domain for the interpolation of the interior and boundary variables. This technique leads to a fast and efficient approach, the locality of the method is maintained and the system matrices are banded with small bandwidth. The velocity - vorticity approach of the Navier-Stokes equations is adopted and the LBIEs are derived for the velocity and the vorticity field, resulting in a very stable and accurate implementation. The evaluation of the volume integrals is accomplished via a very efficient and accurate technique by parameterizing the local area of the nodal point and evaluating all integrals in the parametric cell plane. Numerical examples illustrate the proposed methodology and demonstrate its accuracy.*

## 1 INTRODUCTION

Many numerical methodologies have been developed so far to solve problems dealing with incompressible fluid flows. Here one can mention the Finite Differences (FDM), the Finite Elements (FEM), the Finite Volumes (FVM) and Boundary Elements (BEM) as the most widely used methods. However, in spite of their success, the FDM and FEM suffer of mesh problems, solution instabilities when the stability inf-sup conditions are not satisfied and difficulties associated with the treatment of the incompressibility condition [18]. Although many of these problems can be circumvented with the use of a velocity-vorticity approach, instead of the velocity - pressure formulation of Navier-Stokes equations, the problem of requiring good quality meshes remains ([6], [13], [14]). Many of the aforementioned problems with FEM have already been successfully solved with the development of advanced multiscale and related schemes. Someone here could mention the works of Masud and Khurram [21], Gravemeier et al. [8], the works of Codina and Soto [4] as well as the fractional step method [22]. In contrast with the FEM, the BEM as it is presented in the works of Alujevic et al [1], Skerget and Rek [27], Skerget et al [28] and Hribersek and Skerget [16] [35] has the distinct advantages over the FDM and FEM of treating without any problem the incompressibility condition and requiring less expensive discretizations. Nevertheless, the final system of linear algebraic equations taken by a BEM formulation leads to unsymmetric and full-populated matrices, the numerical treatment of which is very time consuming. Although some polyregion BEM formulations (Grigoriev and Dargush [9]) accelerate the solution process, the non local nature of the BEM affects negatively its efficiency. Recently, Zhu et al [32] [33] proposed a meshless method, called Local Boundary Integral Equation (LBIE) method, as an alternative to the BEM. This method seems to avoid the aforementioned problems associated with the conventional BEM, offering simultaneously the advantages of a meshless method where neither domain nor surface discretization is required. In this LBIE methodology properly distributed nodal points without any connectivity requirement are covering the domain of interest as well as the surrounding global boundary. All these nodal points are positioned in the center of regular sub-domains e.g. circles for two-dimensional problems. At each nodal point, the field is represented through the conventional integral equation used in a BEM which contains integrals defined on the regular boundary of the aforementioned subdomains. The field at the local and global boundaries as well as in the interior of the sub-domains are approximated by a Moving Least Square (MLS) scheme. The local nature of the sub-domains leads to a final linear system of equations, with a coefficient matrix which is sparse and not full-populated as in the case of the BEM.

In the present work the interpolation of the unknown field is achieved with the introduction of the Radial Basis Functions (RBF), for the following three reasons. The first reason is that RBFs maintain the local nature of the proposed LBIE methodology and are independent of any elemental connectivity. Thus it is possible to do a remeshing by adding, removing or moving any nodal point without any difficulty. The second reason is

that RBFs possesses the delta property, a fact that makes the imposition of the boundary conditions straightforward and simple. Also this important property eventually leads to the reduction in size of the system of equations for those points where  $u$  is known, like the FEM. In the present work, the size reduction is done in the transport integral equation. The third reason for the use of RBFs is their computational efficiency, since every nodal point is associated with a RBF matrix, with small dimension. The inverse of the matrix is computed only once for every nodal point, and the interpolation functions and their derivatives can easily be computed in a less costly way by a simple internal product. The paper is organized as follows. In the first section the local boundary-volume integral equations for both the kinematics and the transport kinetics are derived. The next section is devoted to the derivation of the discretized integral equations for the kinematics and kinetics and to the assembly of the system of equations. Finally, the method is applied to one well known benchmark for fluid flow problems and the accuracy of the proposed methodology is demonstrated.

## 2 LOCAL INTEGRAL EQUATION

Incompressible fluid flows are modeled by means of the conservation of mass and momentum equations, respectively [34]

$$\nabla \cdot \mathbf{u} = 0 \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{u} \quad (2)$$

where  $\mathbf{u}$  represents the velocity field,  $P$  is the pressure and  $\rho$ ,  $\nu$  stand for the density and the diffusion coefficient (viscosity), respectively. It is well known (see e.g. [16]) that numerical methods based on the weak formulations of Eqs (1) and (2) have numerical instabilities related to the presence of the pressure gradient term. This term can be circumvented by adopting the velocity-vorticity formulation, which is obtained by applying the curl operator in both Eqs (1) and (2), i.e.

$$\nabla^2 \mathbf{u} + \nabla \times \omega = \mathbf{0} \quad (3)$$

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{u} - \nu \nabla^2 \omega = \mathbf{0} \quad (4)$$

with  $\omega$  being the vorticity vector defined as

$$\omega = \nabla \times \mathbf{u} \quad (5)$$

In two dimensional flows, where the vorticity vector is always perpendicular to the plane of the flow ( $\omega \cdot \nabla \mathbf{u} = \mathbf{0}$ ), Eqs (3) and (4) are written in the following simplified form:

$$\frac{\partial^2 u_i}{\partial x_j \partial x_j} + e_{ij} \frac{\partial \omega}{\partial x_j} = 0 \quad (6)$$

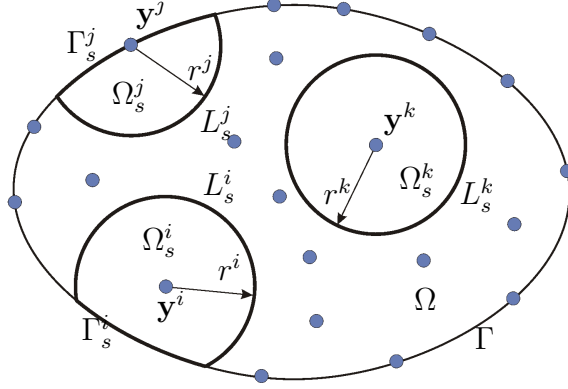


Figure 1: Local support domains  $r$ , boundary segments  $\Gamma_s$ , local circular surfaces  $L_s$  and local volumes  $\Omega_s$ .

$$\frac{\partial \omega}{\partial t} + u_j \frac{\partial \omega}{\partial x_j} - \nu \frac{\partial^2 \omega}{\partial x_j^2} = 0 \quad (7)$$

The coupled Eqs (6) and (7), known as velocity-vorticity formulation [12] of the Navier-Stokes equations (1) and (2), represent the kinematic and the kinetic dynamics of an incompressible Newtonian fluid. The corresponding initial boundary value problem must be complemented with the initial conditions

$$\begin{aligned} u_i(\mathbf{x}, 0) &= u_i^{(0)} \\ \omega_i(\mathbf{x}, 0) &= \omega_i^{(0)} \end{aligned} \quad \text{for } \mathbf{x} \in \Gamma \quad (8)$$

and one of the following boundary conditions

$$\begin{aligned} u_i(\mathbf{x}, t) &= \bar{u}_i \\ \omega_i(\mathbf{x}, t) &= \bar{\omega}_i \end{aligned} \quad \text{for } \mathbf{x} \in \Gamma \quad (9)$$

## 2.1 Local integral equations for flow kinematics

Consider a 2D domain  $\Omega$  with a smooth boundary  $\Gamma$ , for the flow of an incompressible Newtonian fluid (Fig 1). Eqs (6)-(7) with the initial and boundary conditions (8)-(9) form a well posed boundary value problem which admits a boundary-domain integral representation [28] of the form:

$$c(\mathbf{y}) \mathbf{u}(\mathbf{y}) + \int_{\Gamma} (\nabla u^* \cdot \mathbf{n}) \mathbf{u} d\Gamma = \int_{\Gamma} (\nabla u^* \times \mathbf{n}) \times \mathbf{u} d\Gamma + \int_{\Omega} (\omega \times \nabla u^*) d\Omega \quad (10)$$

Here  $c$  is a jump coefficient being equal to 1 for interior points and equal to 0.5 for points belonging to the smooth boundary  $\Gamma$  and  $u^*$  is the fundamental solution of the Laplace

operator having the form

$$u^* = \frac{1}{2\pi} \text{Log} \left( \frac{1}{r} \right) \quad (11)$$

with  $r = |\mathbf{y} - \mathbf{x}|$  representing the distance between the reference point  $\mathbf{y}$  and the source point  $\mathbf{x}$ . Here and in the sequel  $\mathbf{n}$  denotes the outward unit vector normal to the boundary. It is apparent that the fundamental solution (11) becomes singular only when the field point  $\mathbf{y}$  coincides with the source point  $\mathbf{x}$ . Thus, considering a local circular sub-domain  $\Omega_s$  with boundary  $L_s$  centered at point  $\mathbf{y}$  and applying Green's integral identity in the domain lying between the global and local boundaries  $\Gamma$  and  $L_s$ , respectively, it is easy to see that Eq (10) can be replaced by the following local boundary-volume integral equation

$$\begin{aligned} c(\mathbf{y}) \mathbf{u}(\mathbf{y}) + \int_{\Gamma_s \cup L_s} (\nabla u^* \cdot \mathbf{n}) \mathbf{u} d\Gamma = \\ \int_{\Gamma_s \cup L_s} (\nabla u^* \times \mathbf{n}) \times \mathbf{u} d\Gamma + \\ \int_{\Omega_s} (\boldsymbol{\omega} \times \nabla u^*) d\Omega \end{aligned} \quad (12)$$

where  $\Gamma_s$  is part of the global boundary intersected with the local sub-domain  $\Omega_s$ , as illustrated in Fig (1). When the unknown boundary condition is the tangential component of the velocity vector or the vorticity, the tangential form of Eq (12) is considered [16], [1]

$$\begin{aligned} c(\mathbf{y}) \mathbf{n}(\mathbf{y}) \times \mathbf{u}(\mathbf{y}) + \mathbf{n}(\mathbf{y}) \times \int_{\Gamma_s \cup L_s} (\nabla u^* \cdot \mathbf{n}) \mathbf{u} d\Gamma = \\ \mathbf{n}(\mathbf{y}) \times \int_{\Gamma_s \cup L_s} (\nabla u^* \times \mathbf{n}) \times \mathbf{u} d\Gamma + \\ \mathbf{n}(\mathbf{y}) \times \int_{\Omega_s} (\boldsymbol{\omega} \times \nabla u^*) d\Gamma \end{aligned} \quad (13)$$

Similarly, in cases where the normal component of the velocity is unknown, then the normal form of Eq (12) should be taken into account, i.e.

$$\begin{aligned} c(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \mathbf{u} + \mathbf{n}(\mathbf{y}) \cdot \int_{\Gamma_s \cup L_s} (\nabla u^* \cdot \mathbf{n}) \mathbf{u} d\Gamma = \\ \mathbf{n}(\mathbf{y}) \cdot \int_{\Gamma_s \cup L_s} (\nabla u^* \times \mathbf{n}) \times \mathbf{u} d\Gamma + \\ \mathbf{n}(\mathbf{y}) \cdot \int_{\Omega_s} (\boldsymbol{\omega} \times \nabla u^*) d\Gamma \end{aligned} \quad (14)$$

## 2.2 Local integral equations for flow kinetics

Since the first time derivative of the vorticity appears in Eq (7), it is convenient to employ the finite differences scheme

$$\frac{\partial \omega}{\partial t} = \frac{\omega - \omega_{t-1}}{\Delta t} \quad (15)$$

where  $\omega_{t-1}$  is the vorticity field at the previous time step and  $\Delta t$  is the considered time interval. In view of Eq (15), Eq (7) takes the form

$$\frac{\partial^2 \omega}{\partial x_j^2} - \frac{1}{\nu} u_j \frac{\partial \omega}{\partial x_j} - \frac{1}{\nu} \frac{\omega}{\Delta t} + \frac{1}{\nu} \frac{\omega_{t-1}}{\Delta t} = 0 \quad (16)$$

or

$$\frac{\partial^2 \omega}{\partial x_j^2} + b = 0 \quad (17)$$

where  $b$  represents the body forces

$$b = -\frac{1}{\nu} u_j \frac{\partial \omega}{\partial x_j} - \frac{1}{\nu} \frac{\omega}{\Delta t} + \frac{1}{\nu} \frac{\omega_{t-1}}{\Delta t} \quad (18)$$

Exploiting the Laplace fundamental solution (11) and applying Green's second identity for the scalars  $u^*$  and  $\omega$ , one obtains the following integral equation:

$$\begin{aligned} c(\mathbf{y}) \omega(\mathbf{y}) + \int_{\Gamma} \frac{\partial u^*}{\partial n} \omega \, d\Gamma &= \int_{\Gamma} u^* \left( \frac{\partial \omega}{\partial n} - \frac{1}{\nu} u_n \omega \right) \, d\Gamma + \frac{1}{\nu} \int_{\Omega} \frac{\partial u^*}{\partial x_j} u_j \omega \, d\Omega - \\ &\quad \frac{1}{\nu \Delta t} \int_{\Omega} u^* \omega \, d\Omega + \frac{1}{\nu \Delta t} \int_{\Omega} u^* \omega_{t-1} \, d\Omega \end{aligned} \quad (19)$$

where  $u_n = u_j n_j$ . For a point  $\mathbf{y}$  surrounded by the local boundary  $\Gamma_s \cup L_s$  Fig (1), Eq (19) takes the form

$$\begin{aligned} c(\mathbf{y}) \omega(\mathbf{y}) + \int_{\Gamma_s \cup L_s} \frac{\partial u^*}{\partial n} \omega \, d\Gamma &= \int_{\Gamma_s \cup L_s} u^* \left( \frac{\partial \omega}{\partial n} - \frac{1}{\nu} u_n \omega \right) \, d\Gamma + \\ &\quad \frac{1}{\nu} \int_{\Omega_s} \frac{\partial u^*}{\partial x_j} u_j \omega \, d\Omega - \frac{1}{\nu \Delta t} \int_{\Omega_s} u^* \omega \, d\Omega + \frac{1}{\nu \Delta t} \int_{\Omega_s} u^* \omega_{t-1} \, d\Omega \end{aligned} \quad (20)$$

In order to get rid of the term  $\left( \frac{\partial \omega}{\partial n} - \frac{1}{\nu} u_n \omega \right)$ , defined on the local boundaries  $L_s$ , a companion solution is employed [32]. This solution satisfies the Laplace equation and is equal to the fundamental solution  $u^*$  on the local boundary  $L_s$ . It has the form

$$u^c = \frac{1}{2\pi} \text{Log} \left( \frac{1}{r_0} \right) \quad (21)$$

with  $r_0$  representing the radius of the local domain  $\Omega_s$ . Applying Green's second identity to the scalars  $u^*$  and  $u^c$  and subtracting the resultant integral equation from Eq (20) one finally obtains

$$\begin{aligned}
 c(\mathbf{y})\omega(\mathbf{y}) + \int_{\Gamma_s \cup L_s} \frac{\partial u^*}{\partial n} \omega d\Gamma &= \int_{\Gamma_s} (u^* - u^c) \left( \frac{\partial \omega}{\partial n} - \frac{1}{\nu} u_n \omega \right) d\Gamma + \\
 \frac{1}{\nu} \int_{\Omega_s} \frac{\partial u^*}{\partial x_j} u_j \omega d\Omega - \frac{1}{\nu \Delta t} \int_{\Omega_s} (u^* - u^c) \omega d\Omega &+ \frac{1}{\nu \Delta t} \int_{\Omega_s} (u^* - u^c) \omega_{t-1} d\Omega
 \end{aligned} \tag{22}$$

It should be noted that in the local integral Eq (22) the diffusive term is expressed by the boundary integrals exclusively, and the convective term is expressed by the boundary and the volume integrals containing the velocity vector.

### 3 DISCRETIZATION AND NUMERICAL IMPLEMENTATION

#### 3.1 Kinematics integral equation

In the present formulation the kinematics system of equations consists of unknown velocities and vorticities for internal and boundary points.

For the boundary points the integral Eq (12) should be employed. However, due to the singularity that appears in the diagonal elements [28], the tangent (13) or the normal (14) forms of this equation should be used.

The discretized form of the integral Eq (13) for a boundary collocation node  $\mathbf{y}$  with the neighborhood nodal points  $\mathbf{y}_k$  is the following

$$\begin{aligned}
 c(\mathbf{y})\mathbf{n}(\mathbf{y}) \times \mathbf{u}(\mathbf{y}) + \mathbf{n}(\mathbf{y}) \times \int_{\Gamma_{su}} \frac{\partial u^*}{\partial n} \begin{bmatrix} u_x \\ u_y \\ 0 \end{bmatrix} d\Gamma &+ \\
 \mathbf{n}(\mathbf{y}) \times \int_{\Gamma_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma \begin{bmatrix} u_x \\ u_y \\ 0 \end{bmatrix}_{(\mathbf{y}_k)} + \mathbf{n}(\mathbf{y}) \times \int_{L_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma \begin{bmatrix} u_x \\ u_y \\ 0 \end{bmatrix}_{(\mathbf{y}_k)} &= \\
 \mathbf{n}(\mathbf{y}) \times \int_{\Gamma_{su}} \frac{\partial u^*}{\partial t} \begin{bmatrix} u_y \\ -u_x \\ 0 \end{bmatrix} d\Gamma + \mathbf{n}(\mathbf{y}) \times \int_{\Gamma_s} \frac{\partial u^*}{\partial t} \Phi(\mathbf{y}_k) d\Gamma \begin{bmatrix} u_y \\ -u_x \\ 0 \end{bmatrix}_{(\mathbf{y}_k)} &+ \\
 \mathbf{n}(\mathbf{y}) \times \int_{\Omega_s} \begin{bmatrix} -\frac{\partial u^*}{\partial y} \\ \frac{\partial u^*}{\partial x} \\ 0 \end{bmatrix} \Phi(\mathbf{y}_k) d\Omega [\omega]_{(\mathbf{y}_k)} &
 \end{aligned} \tag{23}$$

Note that the integral involving the  $\frac{\partial u^*}{\partial t}$  along the circular arc is always zero, since the vector  $\mathbf{r} = \mathbf{y} - \mathbf{x}$  representing the distance between the source and the reference point is always perpendicular to the tangential vector  $\mathbf{t}$ .  $u_x$  and  $u_y$  are the cartesian components

of the velocity field, defined either on the nodal point  $\mathbf{y}$  or on the neighborhood nodal points  $\mathbf{y}_k$ . The previous vector equation produces the following scalar equation

$$\begin{aligned}
 c(\mathbf{y})n_x(\mathbf{y})u_y(\mathbf{y}) - c(\mathbf{y})n_y(\mathbf{y})u_x(\mathbf{y}) + n_x(\mathbf{y})H_{\Gamma_s}^{\bar{u}_y} - n_y(\mathbf{y})H_{\Gamma_s}^{\bar{u}_x} + \\
 n_x(\mathbf{y})H_{\Gamma_s}^k u_y(\mathbf{y}_k) - n_y(\mathbf{y})H_{\Gamma_s}^k u_x(\mathbf{y}_k) + n_x(\mathbf{y})H_{L_s}^k u_y(\mathbf{y}_k) - n_y(\mathbf{y})H_{L_s}^k u_x(\mathbf{y}_k) = \\
 -n_x(\mathbf{y})T_{\Gamma_s}^{\bar{u}_x} - n_y(\mathbf{y})T_{\Gamma_s}^{\bar{u}_y} - n_x(\mathbf{y})T_{\Gamma_s}^k u_x(\mathbf{y}_k) - n_y(\mathbf{y})T_{\Gamma_s}^k u_y(\mathbf{y}_k) + \\
 [n_x(\mathbf{y})D_x^k + n_y(\mathbf{y})D_y^k]\omega(\mathbf{y}_k)
 \end{aligned} \tag{24}$$

Writing the above equation for every boundary nodal point, where either the vorticity or the tangential component of the velocity vector are unknowns, and imposing the boundary conditions accordingly, the line of the system of equations that corresponds to the nodal point  $\mathbf{y}$  is derived. When the normal component of the velocity field is unknown, then the following discretized form of Eq (14) is used for the boundary nodal point  $\mathbf{y}$

$$\begin{aligned}
 c(\mathbf{y})\mathbf{n}(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) + \mathbf{n}(\mathbf{y}) \cdot \int_{\Gamma_{su}} \frac{\partial u^*}{\partial n} \begin{bmatrix} u_x \\ u_y \end{bmatrix} d\Gamma + \\
 \mathbf{n}(\mathbf{y}) \cdot \int_{\Gamma_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{(\mathbf{y}_k)} + \mathbf{n}(\mathbf{y}) \cdot \int_{L_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{(\mathbf{y}_k)} = \\
 \mathbf{n}(\mathbf{y}) \cdot \int_{\Gamma_{su}} \frac{\partial u^*}{\partial t} \begin{bmatrix} u_y \\ -u_x \end{bmatrix} d\Gamma + \mathbf{n}(\mathbf{y}) \cdot \int_{\Gamma_s} \frac{\partial u^*}{\partial t} \Phi(\mathbf{y}_k) d\Gamma \begin{bmatrix} u_y \\ -u_x \end{bmatrix}_{(\mathbf{y}_k)} + \\
 \mathbf{n}(\mathbf{y}) \cdot \int_{\Omega_s} \begin{bmatrix} -\frac{\partial u^*}{\partial y} \\ \frac{\partial u^*}{\partial x} \end{bmatrix} \Phi(\mathbf{y}_k) d\Omega [\omega]_{(\mathbf{y}_k)}
 \end{aligned} \tag{25}$$

which gives the following scalar equation

$$\begin{aligned}
 c(\mathbf{y})n_x(\mathbf{y})u_x(\mathbf{y}) + c(\mathbf{y})n_y(\mathbf{y})u_y(\mathbf{y}) + n_x(\mathbf{y})H_{\Gamma_s}^{\bar{u}_x} + n_y(\mathbf{y})H_{\Gamma_s}^{\bar{u}_y} + \\
 n_x(\mathbf{y})H_{\Gamma_s}^k u_x(\mathbf{y}_k) + n_y(\mathbf{y})H_{\Gamma_s}^k u_y(\mathbf{y}_k) + n_x(\mathbf{y})H_{L_s}^k u_x(\mathbf{y}_k) + n_y(\mathbf{y})H_{L_s}^k u_y(\mathbf{y}_k) = \\
 n_x(\mathbf{y})T_{\Gamma_s}^{\bar{u}_y} - n_y(\mathbf{y})T_{\Gamma_s}^{\bar{u}_x} + n_x(\mathbf{y})T_{\Gamma_s}^k u_y(\mathbf{y}_k) - n_y(\mathbf{y})T_{\Gamma_s}^k u_x(\mathbf{y}_k) + \\
 [-n_x(\mathbf{y})D_y^k + n_y(\mathbf{y})D_x^k]\omega(\mathbf{y}_k)
 \end{aligned} \tag{26}$$

This equation should be rearranged according to the boundary conditions. Finally, for an internal point the following discretized form of Eq (12) is employed

$$\begin{aligned}
 \begin{bmatrix} u_x(\mathbf{y}) \\ u_y(\mathbf{y}) \end{bmatrix} + \int_{\Gamma_{su}} \frac{\partial u^*}{\partial n} \begin{bmatrix} u_x \\ u_y \end{bmatrix} d\Gamma + \int_{\Gamma_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{(\mathbf{y}_k)} + \\
 \int_{L_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{(\mathbf{y}_k)} = \int_{\Gamma_{su}} \frac{\partial u^*}{\partial t} \begin{bmatrix} u_y \\ -u_x \end{bmatrix} d\Gamma + \\
 \int_{\Gamma_s} \frac{\partial u^*}{\partial t} \Phi(\mathbf{y}_k) d\Gamma \begin{bmatrix} u_y \\ -u_x \end{bmatrix}_{(\mathbf{y}_k)} + \int_{\Omega_s} \begin{bmatrix} -\frac{\partial u^*}{\partial y} \\ \frac{\partial u^*}{\partial x} \end{bmatrix} \Phi(\mathbf{y}_k) d\Omega [\omega]_{(\mathbf{y}_k)}
 \end{aligned} \tag{27}$$



which can be written in the vector form

$$\begin{aligned} \begin{bmatrix} u_x(\mathbf{y}) \\ u_y(\mathbf{y}) \end{bmatrix} + \begin{bmatrix} H_{\Gamma_s}^{\bar{u}_x} \\ H_{\Gamma_s}^{\bar{u}_y} \end{bmatrix} + H_{\Gamma_s}^k \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{(\mathbf{y}_k)} + H_{L_s}^k \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{(\mathbf{y}_k)} = \\ \begin{bmatrix} T_{\Gamma_s}^{\bar{u}_y} \\ -T_{\Gamma_s}^{\bar{u}_x} \end{bmatrix} + T_{\Gamma_s}^k \begin{bmatrix} u_y \\ -u_x \end{bmatrix}_{(\mathbf{y}_k)} + \begin{bmatrix} -D_y^k \\ D_x^k \end{bmatrix} [\omega]_{(\mathbf{y}_k)} \end{aligned} \quad (28)$$

This equation needs also to be rearranged according to the boundary conditions. For the interior points the velocity vector is considered always unknown and the vorticity is known. Combining Eqs (24), (26) and (28) the final system of equations  $\mathbf{Ax} = \mathbf{b}$  is derived. It is in band form and is solved for the boundary vorticities or velocities and interior velocities, as unknowns. The involved boundary and volume integrals employed in the discretization of the kinematics integral equations are the following

$$H_{\Gamma_s}^{\bar{u}_x} = \int_{\Gamma_s^{u_x}} \frac{\partial u^*}{\partial n} \bar{u}_x d\Gamma \quad (29)$$

$$H_{\Gamma_s}^{\bar{u}_y} = \int_{\Gamma_s^{u_y}} \frac{\partial u^*}{\partial n} \bar{u}_y d\Gamma \quad (30)$$

$$H_{\Gamma_s}^k = \int_{\Gamma_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma \quad (31)$$

$$H_{L_s}^k = \int_{L_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma \quad (32)$$

$$T_{\Gamma_s}^{\bar{u}_x} = \int_{\Gamma_s^{u_x}} \frac{\partial u^*}{\partial t} \bar{u}_x d\Gamma \quad (33)$$

$$T_{\Gamma_s}^{\bar{u}_y} = \int_{\Gamma_s^{u_y}} \frac{\partial u^*}{\partial t} \bar{u}_y d\Gamma \quad (34)$$

$$T_{\Gamma_s}^k = \int_{\Gamma_s} \frac{\partial u^*}{\partial t} \Phi(\mathbf{y}_k) d\Gamma \quad (35)$$

and

$$D_j^k = \int_{\Omega_s} \frac{\partial u^*}{\partial x_j} \Phi(\mathbf{y}_k) d\Omega \quad (36)$$

### 3.2 Vorticity transport integral equation

In the discretization of the vorticity transport Eq (22), the integral with the unknown boundary flux  $\frac{\partial \omega}{\partial n}$  along the circular arc  $L_s$  is eliminated due to the companion solution as described in many MLPG/LBIE works (see [23], [29], [32] and [33]). By defining the local boundary portion  $\Gamma_s = \Gamma_{sq} \cup \Gamma_{s\omega}$  where  $\Gamma_{sq}$  is the part of the boundary with prescribed vorticity flux  $\bar{q} = \frac{\partial \omega}{\partial n}$  and  $\Gamma_{s\omega}$  is the boundary part with prescribed vorticity, then the discretized form of the Eq (22) is the following

$$\begin{aligned}
 c(\mathbf{y}) \omega(\mathbf{y}) + \int_{\Gamma_{sq}} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma [\omega]_{(\mathbf{y}_k)} + \int_{\Gamma_{s\omega}} \frac{\partial u^*}{\partial n} \bar{\omega} d\Gamma + \int_{L_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma [\omega]_{(\mathbf{y}_k)} = \\
 \int_{\Gamma_{sq}} (u^* - u^c) \bar{q} d\Gamma + \int_{\Gamma_{s\omega}} (u^* - u^c) \frac{\partial \Phi(\mathbf{y}_k)}{\partial n} d\Gamma [\omega]_{(\mathbf{y}_k)} - \\
 \frac{1}{\nu} \int_{\Gamma_{sq}} (u^* - u^c) u_n \Phi(\mathbf{y}_k) d\Gamma - \frac{1}{\nu} \int_{\Gamma_{s\omega}} (u^* - u^c) u_n \bar{\omega} d\Gamma + \\
 \frac{1}{\nu} \int_{\Omega_s} \frac{\partial u^*}{\partial x_j} u_j \Phi(\mathbf{y}_k) d\Omega [\omega]_{(\mathbf{y}_k)} - \frac{1}{\nu \Delta t} \int_{\Omega_s} (u^* - u^c) \Phi(\mathbf{y}_k) d\Omega [\omega]_{(\mathbf{y}_k)} + \\
 \frac{1}{\nu \Delta t} \int_{\Omega_s} (u^* - u^c) \Phi(\mathbf{y}_k) d\Omega [\omega_{t-1}]_{(\mathbf{y}_k)} \quad (37)
 \end{aligned}$$

Since the Radial Basis Functions possesses the delta property and the unknown boundary vorticity flux is expressed through the derivatives of the RBFs, only for the portions of the boundary where  $\frac{\partial \omega}{\partial n}$  is unknown, the above integral equation should be applied only to the nodal points with unknown vorticity. This leads to a reduction of the system of equations in the FEM sense. Writing the above equation for every boundary nodal point with vorticity unknown and for every interior nodal point, the following equation is obtained, which will be rearranged and assembled to the final kinetics system of equations. All unknowns are vorticity values.

$$c(\mathbf{y}) \omega(\mathbf{y}) + \mathbf{K}(\mathbf{y}, \mathbf{y}^k) [\omega]_{\mathbf{y}_k} = f(\mathbf{y}, \mathbf{y}^k) \quad (38)$$

where  $\mathbf{K}$  is a vector of integrals, that depend on the collocation node  $\mathbf{y}$  and the neighborhood nodal points  $\mathbf{y}^k$  and has the following form

$$\begin{aligned}
 \mathbf{K}(\mathbf{y}, \mathbf{y}^k) = \int_{\Gamma_{sq}} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma + \int_{L_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma - \\
 \int_{\Gamma_{s\omega}} (u^* - u^c) \frac{\partial \Phi(\mathbf{y}_k)}{\partial n} d\Gamma + \frac{1}{\nu} \int_{\Gamma_{sq}} (u^* - u^c) u_n \Phi(\mathbf{y}_k) d\Gamma - \\
 \frac{1}{\nu} \int_{\Omega_s} \frac{\partial u^*}{\partial x_j} u_j \Phi(\mathbf{y}_k) d\Omega + \frac{1}{\nu \Delta t} \int_{\Omega_s} (u^* - u^c) \Phi(\mathbf{y}_k) d\Omega \quad (39)
 \end{aligned}$$

or

$$\mathbf{K}(\mathbf{y}, \mathbf{y}^k) = [H_{\Gamma_{sq}}^k] + [H_L^k] - [G_{\Gamma_{s\omega}}^k] + \frac{1}{\nu} [F_{sq}^k] - \frac{1}{\nu} [D_j^k] [u_j^k] + \frac{1}{\nu \Delta t} [U_j^k] \quad (40)$$

and

$$f(\mathbf{y}, \mathbf{y}^k) = - \int_{\Gamma_{s\omega}} \frac{\partial u^*}{\partial n} \bar{\omega} d\Gamma + \int_{\Gamma_{sq}} (u^* - u^c) \bar{q} d\Gamma - \frac{1}{\nu} \int_{\Gamma_{s\omega}} (u^* - u^c) u_n \bar{\omega} d\Gamma + \frac{1}{\nu \Delta t} \int_{\Omega_s} (u^* - u^c) \Phi(\mathbf{y}_k) d\Omega [\omega_{t-1}]_{(\mathbf{y}_k)} \quad (41)$$

or

$$f(\mathbf{y}, \mathbf{y}^k) = - [H_{\Gamma_{s\omega}}] + [G_{\Gamma_{sq}}] - \frac{1}{\nu} [F_{\Gamma_{s\omega}}] + \frac{1}{\nu \Delta t} [U^k] [\omega_{t-1}^k] \quad (42)$$

It should be noted that, since the system of equations is reduced for the nodal points where the vorticity is known, Eq (40) should be rearranged again, and the terms with the known vorticities should be put to the right hand side. In this way the banded kinetics system of equations  $\mathbf{Ax} = \mathbf{b}$  is derived. The involved integrals for the transport integral equation are the following

$$H_L^k = \int_{L_s} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma \quad (43)$$

$$H_{\Gamma_{sq}}^k = \int_{\Gamma_{sq}} \frac{\partial u^*}{\partial n} \Phi(\mathbf{y}_k) d\Gamma \quad (44)$$

$$H_{\Gamma_{s\omega}} = \int_{\Gamma_{s\omega}} \frac{\partial u^*}{\partial n} \bar{\omega} d\Gamma \quad (45)$$

$$G_{\Gamma_{sq}} = \int_{\Gamma_{sq}} (u^* - u^c) \bar{q} d\Gamma \quad (46)$$

$$G_{\Gamma_{s\omega}}^k = \int_{\Gamma_{s\omega}} (u^* - u^c) \frac{\partial \Phi(\mathbf{y}_k)}{\partial n} d\Gamma \quad (47)$$

$$F_{\Gamma_{sq}}^k = \frac{1}{\nu} \int_{\Gamma_{sq}} (u^* - u^c) u_n \Phi(\mathbf{y}_k) d\Gamma \quad (48)$$

$$F_{\Gamma_{s\omega}} = \frac{1}{\nu} \int_{\Gamma_{s\omega}} (u^* - u^c) u_n \bar{\omega} d\Gamma \quad (49)$$

$$D_j^k = \int_{\Omega_s} \frac{\partial u^*}{\partial x_j} \Phi(\mathbf{y}_k) d\Omega \quad (50)$$

and

$$U_j^k = \int_{\Omega_s} (u^* - u^c) \Phi(\mathbf{y}_k) d\Omega \quad (51)$$

Attention should be drawn to the evaluation of the convective boundary integral (48), since the term  $u_n = u_j n_j$  is defined on the boundary  $\Gamma_{sq}$ , and the RBFs involve interior nodes. In this case, the velocity  $u_j$  is interpolated by boundary interpolation functions like those used in the BEM, while the field  $\omega$  is interpolated with the RBFs and eventually involve interior neighborhood nodes. Thus the convective boundary integral depends strongly on the velocity field, and it should be recalculated each time the boundary velocity is changing. i.e. in outflow regions. It should be noted that by dropping the integrals (51) in Eqs (40) and (42), the formulation of the static case is derived.

### 3.3 Solution approach

As it was already mentioned, two systems of equations should be solved in every iteration step. One system is related to the kinematics and the other to the kinetics transport equation. The local nature of the meshless methods is a very important characteristic, since it produces sparse systems of equations. Moreover, with the optimum positioning of the coefficients, the systems can be stored in band form [5]. This storage is very efficient, because the computer memory requirements are relatively small and the solution process is very fast due to the fact that the matrix is a narrow band around the main diagonal. The time domain solution approach is the following.

- Discretize the domain to a number of boundary and interior nodal points. Evaluate connectivity and for every nodal point invert the local RBF matrix [24], [25].
- Start the time domain loop.
- Initialize the vorticity field.
- Evaluate integrals (29) - (36) and (43)- (51) except (48) for outflow regions.
- Start the iteration loop.
  - Form kinetics system of equations (24), (26), (28) and solve it for boundary vorticities or velocity components and interior velocities.
  - Evaluate convective boundary integral for outflow regions (48).
  - Solve the transport equation for boundary and interior vorticities (38).
  - Check convergence of the vorticity field and quit the iteration loop if it is achieved.

- Relax vorticities and proceed to the next iteration step.
- End of iteration loop.
- End of time domain loop.

The initialization of the vorticity field is important for the solution approach. A good initial value reduces significantly the number of the iteration steps. In the examples the initial value for every time step is the vorticity field of the previous time step, and for the very first time step was equal to zero. The relaxation of the vorticity field is based on the following formula

$$\omega^{i+1} = \lambda\omega^i + (1 - \lambda)\omega^{i-1} \quad (52)$$

where  $0 < \lambda \leq 1$  is the relaxation parameter and  $i$  denotes the iteration step. This parameter is quite important in the solution process. A very large relaxation parameter can affect the convergence negatively. On the other hand if it is too small it can cause too many iteration steps.

The convergence norm of the problem is computed by the following formula

$$e = \frac{\sum_{j=1}^N (\omega_j^i - \omega_j^{i-1})^2}{\sum_{j=1}^N (\omega_j^i)^2} \quad (53)$$

where the index  $i$  denotes the iteration step and  $j$  denotes the nodal point.

#### 4 NUMERICAL EXAMPLE

This example concerns the well known lid driven cavity flow, widely used as a benchmark for incompressible flow codes. The fluid is contained in a squared unit cavity where the upper wall moves with a constant velocity  $\bar{u}_x = 1$  causing flow rotation (Fig. 2). Despite the simple geometry, driven cavity flows exhibit a variety of flow phenomena related to multiple counter rotating recirculations near the corners and singular solutions in the upper corners, both depending on the Reynolds number. The problem is solved with the proposed formulation for Reynolds number,  $Re = 5000$ . By taking into account the definition of the Reynolds number  $Re = u_x L / (\nu)$  the viscosity coefficient  $\nu$  is equal to 0.0002. The convergence tolerance is equal to  $10^{-6}$  and the time step is 0.5 sec. The problem is studied for the timeframe of 60 seconds. The cavity is discretized with 35x35 cells, giving a total of 8285 nodes. The results obtained with the present formulation are depicted in Figs (3), (4), (5) (6). They are in good agreement with the results obtained by Ghia et al [7] showing the validation of the present method.

#### 5 CONCLUSIONS

A meshless Local Boundary Integral Equation method for solving the two dimensional Navier-Stokes equations has been proposed. The main attractive features of the proposed

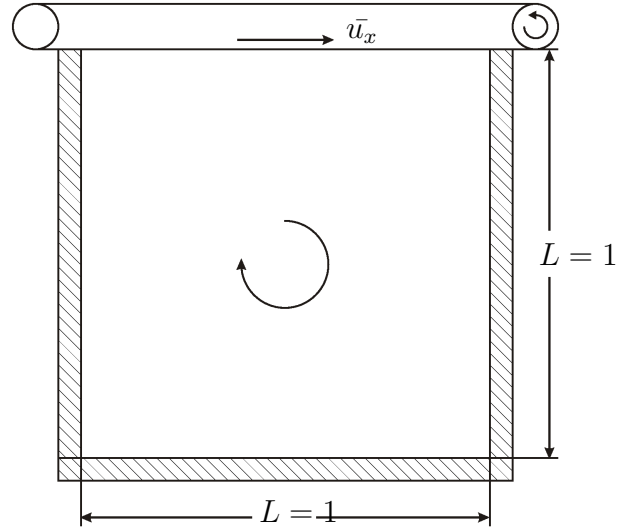


Figure 2: Lid driven cavity  $L \times L$ .

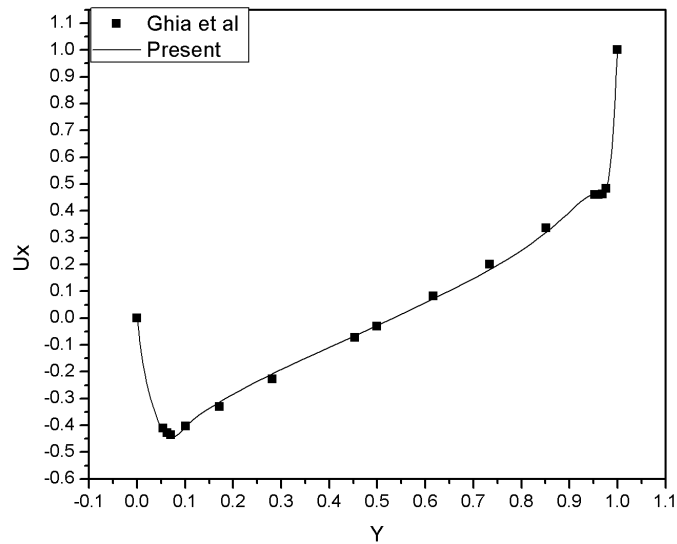


Figure 3: Profile of the velocity component  $u_x$  along a vertical line through the center of the cavity.

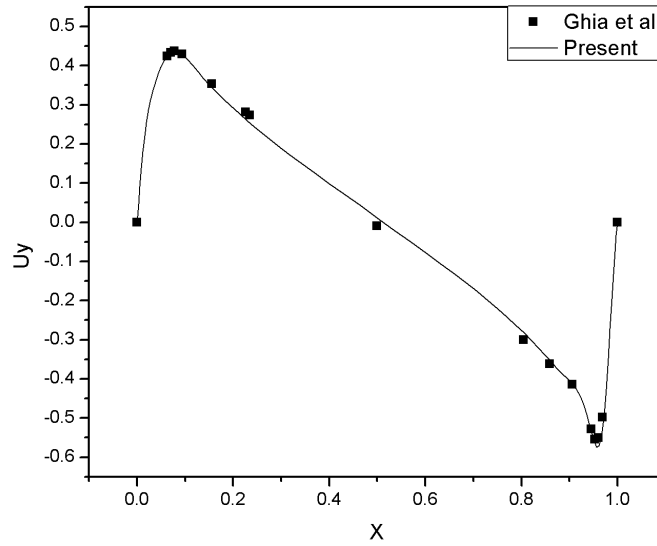


Figure 4: Profile of the velocity component  $u_y$  along a horizontal line through the center of the cavity.

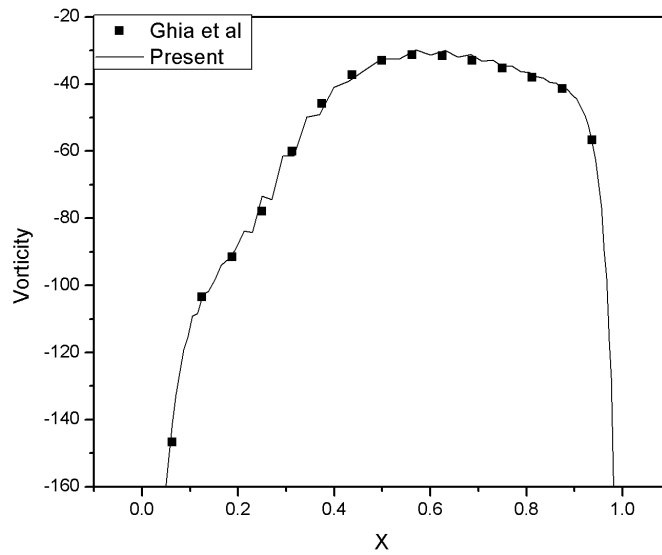


Figure 5: Lid-driven cavity flow, vorticity on the top wall

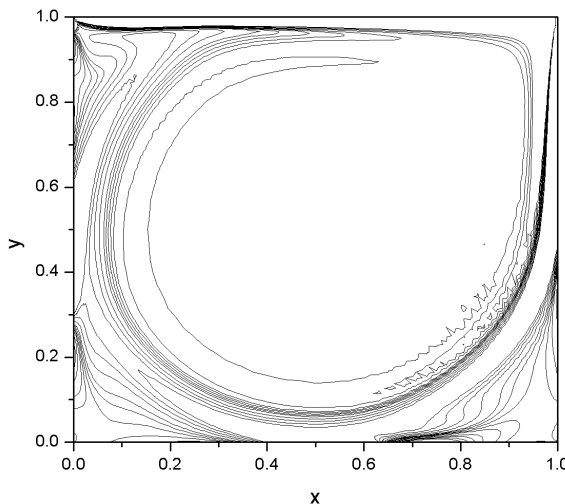


Figure 6: Lid-driven cavity flow, vorticity distribution (Re=5000)

methodology are the following:

- The unknown field is interpolated by the Radial Basis Functions, which posses the delta property, and the inversion of their matrix, is computed only once for every nodal point. This fact leads to a serious speed up in the computation of integrals.
- The derivatives of the Radial Basis Functions are used only in the transport equation for the boundaries with the unknown vorticity flux, leading to a very stable and accurate numerical scheme. The nodal points with prescribed vorticity are eliminated from the system of equations like in FEM.
- Every nodal point is connected with very few nodal points around it, leading to a banded system with very narrow bandwidth.

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