Numerical Study for Flow States of Viscoelastic Fluids in Sinusoidal Channels

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abstract: A numerical analysis is conducted in order to study flow states and characteristics of a viscoelastic fluid in two-dimensional sinusoidal channels. In the present study, four types of sinusoidal channels are used as model channels for simulating an industrial corrugated flow passage. The finite difference method with an adaptive mesh system, using the generalized coordinate system, is used for the numerical analysis. The stream function and vorticity are used in formulating the system of flow equations. The constitutive equation used in the present study for modeling viscoelastic fluid is the Giesekus model. From results of the numerical calculation it was found that vortex zones (flow circulation zones) appear firstly by increasing the Reynolds number due to the flow separation at the divergence part after each throat. The critical phenomena occur when the Reynolds number is further increased and flow becomes asymmetric at the center line of channel. It was found that the viscoelasticity plays a dominant role for occurrence of the asymmetric flow, showing that the critical Reynolds number \(Re_c\) becomes smaller for one channel when the Deborah number is increased.

The overall pressure drops \(\Delta P^*\) across the channel were obtained at the same time. The results indicate that the viscoelastic fluids show higher \(\Delta P^*\) compared with the Newtonian fluid, even at low Deborah number.
1 INTRODUCTION

Non-Newtonian fluids with the viscoelasticity show interesting behaviors and flow characteristics, depending on configurations of flow channels. To date, for viscoelastic flows many works are focused on flow characteristics of the entry flow\textsuperscript{1,2} or the exit flow\textsuperscript{3} in a channel geometry with sudden contraction or expansion for polymer concentrated solutions or melts in very low Reynolds numbers. However, a smoothly varying flow channel such as a Venturi type tube, where a fluid enters at the converging or diverging passage and leaves through passing nozzle parts, is often encountered in an installation of fluid transporting line or in an actual measurement device of flow rate. In relation to a study for the axially varying flow channel, James at al.\textsuperscript{4} studied the pressure loss characteristics in a packed bed for dilute polymer solutions and reported the substantial loss could be caused by the viscoelastic effect on the solutions. It is now widely understood that a representative model geometry for a packed bed or a porous medium would be a channel with alternate converging and diverging sections. Huzarewicz at al\textsuperscript{5} has measured the total pressure drop for a variety of extremely viscoelastic fluids flowing through channels having a sinuously varying channel wall and showed the conditions for appearance in the viscoelastic effect on the pressure drop in geometry. Most of the works concerning such flow behaviors of non-Newtonian fluids are focused on a single cycle channel with alternate converging and diverging sections. Davidson at al\textsuperscript{6,7}, among the limited works for the higher cycle of the diverging-converging channel, reported some results on velocity and stress distribution for very slow flows in two-dimensional sinuously varying channel using a highly viscoelastic polymer solution. Despite of its physical significance in an engineering application, the studies on flow characteristics in a high cycle of the channel with alternate converging and diverging sections are not fully accomplished, particularly for the flow states and pressure characteristics in relatively high Reynolds numbers.

In the present study, a numerical investigation is carried out for a viscoelastic fluid through two dimensional periodically varying wall geometries of 2 cycle converging-diverging or diverging-converging channels. The numerical method adopted in the present work is based on the finite difference method with the wall adaptive mesh system. The transient solutions are calculated on the basis of the implicit finite difference formulation, using the stream function and vorticity. The constitutive equation for modeling viscoelastic fluids is the Giesekus model\textsuperscript{8}. The flow fields and wall pressures are obtained as the numerical
solutions, varying the Reynolds numbers for four different channel geometries of two cycle converging-diverging or diverging-converging channels with different channel expansion ratios.

2 Numerical analysis

2.1 Governing equations

The flow is assumed as incompressible and isothermal. The equations governing the flow are as follows:

$$\nabla \cdot v = 0$$  \hspace{1cm} (1)

$$\rho \frac{Dv}{Dt} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho g,$$  \hspace{1cm} (2)

where Eq.(1) is the equation of continuity and Eq.(2) is the momentum equation (Cauchy’s equation); $v$ the velocity vector, $\rho$ the density, $t$ the time, $p$ the pressure, $g$ the acceleration due to gravity, $D/Dt$ the substantial derivative and $\boldsymbol{\tau}$ the extra-stress tensor derived form the following constitutive equation.

$$\tau + \lambda_1 \nabla \cdot \tau + \alpha \frac{\lambda_1}{\chi_0} (\nabla \cdot \tau) - \alpha \lambda_2 (\nabla \cdot \dot{\gamma} + \dot{\gamma} \cdot \tau) = \chi_0 (\dot{\gamma} + \dot{\gamma} - \alpha \frac{\lambda_2}{\lambda_1} (\dot{\gamma} \cdot \dot{\gamma}))$$  \hspace{1cm} (3)

$$\nabla \cdot \epsilon = \frac{D \epsilon}{Dt} - \nabla \cdot v^T \epsilon - \epsilon \cdot \nabla v$$  \hspace{1cm} (4)

$$\chi_0 = \chi_s + \chi_p$$  \hspace{1cm} (5)

$$\lambda_2 = \lambda_1 \frac{\chi_s}{\chi_s + \chi_p},$$  \hspace{1cm} (6)

where $\dot{\gamma}$ is the rate of strain tensor, $\dot{\gamma} = \frac{1}{2}(\nabla v + \nabla v^t)$, $\chi_0$ is the zero shear viscosity, $\chi_p$ is the polymer contribution index, $\chi_s$ is the solvent contribution index and $\nabla \cdot \epsilon$ indicate the upper convective derivative.

In the present analysis the following non-dimensional parameters are used,

$$x^* = \frac{x}{H}, y^* = \frac{y}{H}, t^* = \frac{t v}{H}, \dot{\gamma}^* = \frac{\dot{\gamma} H}{v}$$
\[ v_x^* = \frac{v_x}{v}, v_y^* = \frac{v_y}{v}, \tau_{ij}^* = \tau_{ij} \frac{H}{\eta_0 v} \]

\[ P^* = \frac{P}{\rho v^2}, \psi^* = \frac{\psi}{v H^2}, \omega^* = \omega \frac{H}{v} \]

\[ Re = \frac{\rho v H}{\eta_0}, De = \lambda_1 \frac{v}{H}, M = \lambda_2 \frac{v}{H}, \quad (7) \]

where \( \phi^* \) is the stream function, \( \omega^* \) is the vorticity, \( Re \) is the Reynolds number based on the zero shear viscosity, \( De \) is the Deborah number and \( M \) is the dimensionless retardation time constant. The representative velocity \( v \) is the mean velocity at the developing flow section of channel, whose width is \( H \) (the representative length). \( v_x, v_y \) are respectively axial and its traversal velocity component of the velocity vector \( \mathbf{v} \). In the numerical analysis the following equations are solved,

\[ \frac{\partial \omega}{\partial t} + \frac{\partial \phi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \phi}{\partial x} \frac{\partial \omega}{\partial y} = \frac{1}{Re} \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} - \frac{\partial^2 \tau_{xx}}{\partial x \partial y} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \tau_{yy}}{\partial x \partial y} \right), \quad (8) \]

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\omega, \quad (9) \]

where Eq.(8) is the vorticity transport equation and Eq.(9) is the Poisson equation for the stream function. It is noted that the rotation of the non-dimensional parameters appearing in Eq.(7) is omitted in Eqs.(8) and (9) for the sake of clarity.

It is further noted that the stress components appeared in Eq.(8) are,

\[ \tau_{xx} = \tau_{xx} - 2 \frac{\partial v_x}{\partial x}, \tau_{xy} = \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right), \tau_{yy} = \tau_{yy} - 2 \frac{\partial v_y}{\partial y}, \quad (10) \]

and the velocity components \( v_x \) and \( v_y \) are obtained as follows,

\[ v_x = \frac{\partial \phi}{\partial y}, v_y = -\frac{\partial \phi}{\partial x}. \quad (11) \]

In Eq.(10) the stress tensor is decoupled so that the elliptic form of the vorticity transport equation (8) is recaptured, and thus Eq.(8) can be solved as in the Newtonian case \((De=0,M=0)\) by the successive over relaxation (SOR) iterative method.
2.2 Boundary condition

Boundary conditions are such that at the inlet of channels the flow is assumed fully developed Poiseuille flow so that the following conditions are imposed,

\[ v_x = 1.5 \left(1 - \left( \frac{y}{H/2} \right)^2 \right), \quad v_y = 0, \quad \phi = \int_{-H/2}^{y} v_x \, dy, \quad \omega = -\frac{\partial v_x}{\partial y}. \]  

(12)

The non-slip condition at channel walls is considered as follows, where the constitutive equation (3) and vorticity transport equation (8) are solved at channel walls with the wall boundary conditions

\[ v_x = 0, \quad v_y = 0, \quad \phi = 1(\text{upper wall}), \quad \phi = 0(\text{lower wall}). \]

(13)

At the channel exit of channels the outlet flow conditions are imposed as follows,

\[ \frac{\partial \tilde{v}}{\partial x} = 0, \quad \frac{\partial^2 \phi}{\partial x^2} = 0, \quad \frac{\partial^2 \omega}{\partial x^2} = 0, \quad \frac{\partial \tau}{\partial x} = 0. \]

(14)

2.3 Numerical method and procedure

Eqs.(8),(9) and the constitutive equation (3) are discretized in order that they can be solved by the finite different method. In Fig.1 the mesh configurations of channels used in the finite difference calculations are depicted with their basic configuration parameters. In the present investigation, based on the representative length \( H \) (the inlet width) the inlet section \( L_1/H \), the outlet section \( L_2/H \), two cycle wave length \( 2\lambda/H \) of sinusoidal wall are fixed as 1.0,1.0 and 6.28 respectively for all channel types. It is noted that \( L_2/H \) is taken enough length compared to \( L_1/H \) so that the flow, after the sinusoidal section, can be sufficiently developed before reaching the outlet of the channel. In differentiating the channel configuration we adopted four types channel geometries, where for channel A and channel B the expansion ratio \( S_2/H \) is 1.66 and 1.0 respectively, and the contraction ratio \( S_3/H \) is 1.0 and 0.334 for channel A and B respectively. Namely in channel A the sinusoidal wall starts with the diverging passage and ends with the converging passage in the sinusoidal test section, while in channel B the configuration is opposite, showing that the datum line of the sinusoidal wall \( S_1/H \) is outside of the inlet channel wall for the channel A while it is inside of the inlet channel wall for channel B. For channel C and channel D the expansion ratio \( S_2/H \) and the contraction ratio \( S_3/H \) are the same as...
1.33 and 0.66 for both channels, and the datum line $S_1/H$ is same as the inlet channel wall width for both channels. The chief difference between channel C and D is whether the sinusoidal wall starts with the diverging passage (for channel C) or the converging passage (for channel D). The size of the mesh used in the numerical simulation is $141 \times 61$ for $i$ and $j$ respectively. In order to ensure resolution for the numerical solutions, some trial runs were performed using different size of meshes. Representative calculations for a Newtonian fluid showed that results obtained with a $242 \times 122$ mesh system made an approximately 0.01% difference to the total pressure drop $\Delta P^*$ compared with a $141 \times 61$ mesh system. Since the difference due to the mesh size was minimal, a $141 \times 61$ mesh system was used in all calculations. The finite difference equations for Eqs.(8) and (9) are formulated with a finite difference of the second-order accuracy in space, and implicit method is used to solve the transient equations. In each time step, Eqs.(8) and (9) are solved using the (SOR) technique. The convergence of the Poisson equations is determined with a maximum relative error of less than $1.0 \times 10^{-5}$.

In the present study, the maximum range of $Re$ and $De$ number is $Re \leq 90$ and $De \leq 0.02$ respectively. The total pressure drop across the sinusoidal wall section is calculated after the flow field (velocity field) being converged at each time step. The total pressure drop $\Delta P^*$ in non-dimensional form is defined as,

$$\Delta P^* = \frac{P_1 - P_2}{\left(\frac{\rho u^2}{2}\right)},$$

where $P_1 - P_2$ is the apparent pressure drop and is defined as,

$$P_1 - P_2 = (-p + \tau_{yy})_1 - (-p + \tau_{yy})_2,$$

where the index 1 and 2 indicate the beginning point of the sinusoidal wall and the ending point of the sinusoidal wall respectively.

3 Results and discussion

Prior to study of the flow characteristics, which are represented by a steady flow field and a steady state total pressure drop $\Delta P^*$, the transient behavior of the system response which is presented by the time variation of the total pressure drop $\Delta P^*$ after start up is displayed in the case of channel B in Fig.2. As shown in Fig.2 typically, where the
Newtonian case is compared for two different Deborah numbers \( \text{De} = 0.04 \) and 0.02 at \( \text{Re} = 30 \), there is some time delay immediately after the start-up in the case of viscoelastic fluid. This is due to the retardation of the response in the viscoelastic model. After the short delay of the response, a local maximum is observed and the value then relaxes, reaching almost steady state as time elapses. The trend becomes more evident as De is increased, as seen in Fig.2, whereas no such transient behavior is seen for the Newtonian case and rather \( \Delta P^* \) tends to undershoot after the start-up. It is noted that \( \Delta P^* \) for time infinity decreases when De is increased as shown in Fig.2. In the present study, it should be noted that results presented in the proceeding figures are those obtained for the steady state (at time infinity).

In Fig.3 typical flow fields for channel A,B,C and D are displayed respectively. In each figure, furthermore, flow fields for the case of viscoelastic fluid (which is modeled by Giesekus model) and of Newtonian are compared respectively at Reynolds number \( \text{Re} = 60 \) (with \( \text{De} = 0.02 \) in the case of viscoelastic fluid). As seen in Fig.3-A and in comparison with the viscoelastic(a) and Newtonian fluid(b), there is only smooth wall flow, in which fluid flows along the wall configuration. A series of calculation results reveals that there is only smooth wall flow for \( \text{Re} \leq 90 \) and \( \text{De} \leq 0.02 \). However, as it will be discussed in details later in this section, there is a difference in the total pressure drop \( \Delta P^* \) between the viscoelastic and Newtonian fluid. In Fig.3-B for channel B, it was found that there were some different flow patterns (flow models) appearing in the sinusoidal channel section, depending upon the choice of Re and De. As seen in Fig.3-B for the viscoelastic and Newtonian (a) and (b), there are vortex zones at each diverging passage in the channels. However, the flow in (a) is asymmetric at the center line of the channel while in (b) the flow is still symmetric. From a series of calculation for channel B, it was found that flows (both the viscoelastic and Newtonian) are the smooth wall flow at low Reynolds number and by increasing Reynolds number the vortex zones are generated simultaneously at each diverging passage, keeping the flow in the channel still symmetric. The effect of viscoelasticity (which is represented by De) was minimal for the first appearance of these vortex zones within the range of De (\( \text{De} \leq 0.02 \)). It should be noted that the vortex zones are generated by the flow separation along the channel wall in the diverging passage due to the inverse pressure gradient along the flow direction. Further increase of the Reynolds number causes the flow to become asymmetric at a critical Reynolds number Re and the
Rec has the strong dependence on the Deborah number. These critical phenomena will be discussed later in this section. The flow patterns displayed Fig.3-B(b) is still symmetric while by the viscoelasticity (De=0.02) the flow in Fig.3-B(b) becomes asymmetric at the same Reynolds number. Similarly in Fig.3-C flow patterns for the viscoelastic and Newtonian fluids are depicted in (a) and (b). It was found, in the case of this channel, that there are two flow modes, i.e. the symmetric smooth wall flow and the symmetric flow with vortex zones. The effect of the viscoelasticity on the flow patterns are that vortex zones are slightly bigger and the motion of circulating flow in the vortex zones is stronger than Newtonian case. It is interesting to note that vortex zones at the diverging passage tend to appear in the second diverging passage in the channel, where there was no vortex zone generated at the first diverging passage adjacent to the inlet of the sinusoidal section. This would be due to the reason that the inverse pressure gradient, which causes the flow separation, is higher in the second and third diverging passage due to higher contraction of the nozzle part. Similar flow patterns are observed in Fig.3-D(a)and(b) for channel D. Again, there was no asymmetric flow within the range of Re and De.

In Fig.4 as it was mentioned previously, the critical phenomena of the asymmetric flow mode are depicted for a change of $v_y$ at the center line of the channel in terms of the Reynolds number. The representative location, where the $v_y$ is calculated, is at the second nozzle of the sinusoidal section(Fig.1(b) or Fig.3-B(a) and (b)). From Fig.4 in the case of Newtonian fluid it is explained that the flow is symmetric before $Rec=60.2$ since $|v_y|$ =0 at the center line of the channel. However, when the Reynolds number exceeds from $Rec=60.2$, $|v_y|$ tends to grow with the relation by $(Re - Rec)^{1/2}$. This is the critical phenomena caused by the pitchfork bifurcation. According to Fig.4, the pitchfork bifurcation is strongly affected by De and the critical Reynolds number becomes $Rec=59.4$. In the present study the symmetry braking pitchfork bifurcation is only observed in channel B, which has the biggest contraction ratio among others.

Finally in Fig.5 the total pressure drops $\Delta P^*$, which was defined in Eqs.(15) and (16), are depicted in (a) for channel A and B and in (b) for channel C and D. The higher pressure drop occurs in channel B along others. And for all channels(A,B,C and D) the viscoelastic fluid showed higher $\Delta P^*$ compared to those of Newtonian fluid. This is due to appearance $\tau_{yy}$ component in the stress tensor and particularly $\tau_{yy}$ becomes higher when the viscoelastic fluid flows through strong elongational fields$^{4,5}$ such as channel B.
The interesting results (particularly for engineering purpose) are that channel D shows higher $\Delta P^*$ s compare to the channel C in both the viscoelastic and Newtonian cases, despite that the wall configuration is identical except that the sinusoidal section begins by the diverging passage (channel C) or the converging passage (channel D). In view of an engineering application it is more advantageous (in terms of $\Delta P^*$) that the bellows types tubes must be connected like channel C configuration. It is noted that the obvious change of the gradient of each curves in Fig.5 (a) and (b) indicate the occurrence of the flow separation which leads to generate the vortex zones.

In the report, we have shown the validity and effectiveness of the present numerical scheme for treating viscoelastic fluids. However, no convergent solution was obtained for the case of high Deborah number with high Reynolds number, specifically over De=0.02. With the condition of high Deborah number for high Reynolds number, the flow field (as well as stress) tends to oscillate after some time steps, resulting in overshooting velocities and divergence of numerical solutions. This is due to numerical difficulties caused by an excessive local concentration of stress tensor, whichever the finer mesh system and however small the steps used, resulting in a numerical breakdown$^{11}$. Further investigation is required to overcome these difficulties and to proceed to three-dimensional simulation. Detailed experimental data may also be of value to compare with numerical results. However, this is not within the scope of our present study. We hope to report further works solving these problems in future publications.

4 Conclusion

A numerical simulation is conducted for the viscoelastic fluids which are modeled by the Giesekus model and the Newtonian fluid. Using the numerical technique based on the finite method, the numerical solutions were obtained for four different types of two-dimensional sinusoidal wall channels. Results obtained in the present numerical simulation yield the following conclusions.

1. The channel with the highest contraction ratio shows various flow modes, i.e. the smooth wall flow, symmetric flow with vortex zones and asymmetric flow with vortex zones. Particularly the asymmetric flow is caused by the symmetry-breaking pitchfork bifurcation and the critical Reynolds number is strongly affected by the elastic effect.
2. The total pressure drop becomes lower when the sinusoidal passage begins with the diverging passage in an identical sinusoidal channel geometry. The total pressure drops are affected by the viscoelasticity, showing greater values.

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6 Reference

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