NEW UNSTEADY WALL LAWS FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Key words: wall law, Navier-Stokes, asymptotic analysis, unsteady flow, boundary layer

Abstract. This work is concerned with the construction of effective boundary conditions or wall laws for unsteady incompressible Navier-Stokes equations over rough domains containing periodic roughness elements. First and second order unsteady wall laws are proposed using two scale asymptotic techniques wherein the rough interface is replaced by an homogenized surface. The influence of the rough element geometry on the flow is incorporated through constants which are obtained by solving a steady Stokes problem in a local domain. A numerical test is presented to validate the proposed boundary conditions.

*The author was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil.
1 Introduction

Numerical simulation of flows over rough interfaces is a critical problem in CFD because of the highly oscillate micro structures near the wall. The direct solution of the Navier-Stokes equations in real 3D domains (thousand of roughness elements for computational treatment) becomes a difficult task, specially when one is interested in simulating unsteady viscous flows. There are many practical problems wherein unsteady flows over rough boundaries are relevant such as:

- In aerodynamics, space shuttles are covered with tiles and its walls have an array of periodic gaps between the tiles. In addition, in the drag control of an airplane wing, small injection jets are introduced over the wing in order to decrease the drag [4].

- In weather forecast the effects of hills, trees and buildings must be taken into account. In many climate applications, water waves should be included to properly simulate ocean-atmosphere interactions (see [11] and [10]).

- In optimal shape design, particularly in active control, the shapes are time dependent. In several cases it is possible to replace active shape control by a boundary control using wall laws or transpiration conditions [4].

- In hemodynamics, the cell surfaces of the endothelial has the property to modify the wall shear stress produced by the flow field [17]. Therefore, wall laws could be useful in order to simulate in an accurate fashion the cell geometry influence on the blood flow.

The problem of fluid flow simulation over rough boundaries has been first studied mathematically by Carrau-LeTallec [9] wherein a domain decomposition method was proposed to construct wall laws for laminar steady flows over periodic rough interfaces. This approach was extended to turbulent flows in Achdou et al. [7] wherein was considered some dependent geometric cases. Their argument was analyzed in Achdou and Pironneau [1] for the Laplace equation and in Achdou et al. [2] for the Stokes problem wherein developed a theoretical framework coupled with an convergence analysis showing good performance of the boundary conditions.

Recently, effective boundary conditions (wall laws) for the incompressible Navier-Stokes equations were stemmed within the framework of two-scale asymptotic expansion techniques [3]. Again it was considered a rough domain with periodic roughness elements. A survey of multiple scale expansions can be found in Bensoussan [8] and Sanchez-Palencia [14]. The two-scale analysis of wall laws was pursued by considering steady laminar flow dominated by viscous effects in the roughness elements. Under this assumption, the flow near the wall tended to be Stokes-like with corrections due to convection. This choice led to derivation of accurate numerical results as pointed out in [16].

Wall laws have been used for unsteady flows and conducted satisfactory numerical results as presented in [13] and [12]. However, the same wall laws have been used for
both steady and unsteady flows without any mathematical justification. In this work, we extend the asymptotic approach to consider time-dependent flow, assuming that the time variation is large compared to the time scale inherent to the roughness elements. A numerical experiment is performed to illustrate the conjecture. We introduce first and second order wall laws for unsteady flows and observe that for the first order approximation in fact the same wall law for both steady \[3\] and unsteady flows are obtained. On the other hand, this is not the case for the second order approximation which gives us more accurate numerical results than the first order as we shall present in the test case.

The outline of this paper is the following: the first section contains the description of the problem. The second section presents the first order approximation and the first order effective boundary condition. Second order wall law is derived in section three. The validation is performed by a numerical test in the section four. Finally, some conclusions are given in section five.

2 Definition of the Problem

We begin by describing a domain that is partly rough with periodic roughness elements. Let \((e_1, e_2)\) be an orthonormal basis of \(\mathbb{R}^2\), and let \(Y \subset \mathbb{R}^2\) a semi-infinite domain in the \(e_2\) direction, such that the boundary of \(Y\) is decomposed into three parts (figure 1):

\[
\partial Y = \partial Y_1 \cup \partial Y_2 \cup \partial Y_3,
\]

where

\[
\partial Y_1 = \{0\} \times [0, \infty[, \quad \partial Y_2 = \{2\pi\} \times [0, \infty[, \quad \partial Y_3 = \{(2\pi, 0)\},
\]

and \(\partial Y_3\) is a connected Lipschitz bounded curve such that

\[
\partial Y_1 \cap \partial Y_3 = \{(0, 0)\}, \quad \partial Y_2 \cap \partial Y_3 = \{(2\pi, 0)\}.
\]

Let \(\varepsilon\) be a small positive real number, and let \(Y^\varepsilon\) be the image of \(Y\) by a dilatation of ratio \(\varepsilon^{-1}\) and center \((0,0)\). Further let \(\Theta^\varepsilon\) be the semi-infinite domain of \(\mathbb{R}^2\) obtained by merging together all the images of \(Y^\varepsilon\) by the translations by \(2\pi k \varepsilon e_1\) where \(k\) takes
all the integer values. The infinite $\Theta^\varepsilon$ is contained in the half plane $x_2 > 0$. Let $\Omega$ be a bounded domain of $\mathbb{R}^2$ intersecting the domain \{\(x_2 \geq 0\)\}. For simplicity, we suppose that this intersection is connected. Thus, for $\varepsilon$ small enough, $\Theta^\varepsilon \cap \Omega$ has a fast oscillating rough boundary, with wavelength of order $\varepsilon$ (figure 2).

The amplitude of the roughness elements is also of order $\varepsilon$. We denote $\Omega^\varepsilon = \Theta^\varepsilon \cup \Omega$ and $\Gamma^\varepsilon$ the rough part of $\partial \Omega^\varepsilon$. We also denote by $\Omega^\circ = \{x \in \Omega : x_2 > 0\}$ and $\Gamma^\circ = \partial \Omega \cap \{x_2 = 0\}$, thus $\Gamma^\varepsilon \subset \Gamma^\circ$. When $\varepsilon \to 0$, $\Omega^\varepsilon$ converges to $\Omega^\circ$ in the sense of Hausdorff.

As usual, we use notation $(x_1, x_2)$ and $(y_1, y_2) = \left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$ for the macroscopic and for the microscopic variables respectively. We consider an unique time scale $t \in [0, T]$, where $T$ is the final time of the process.

Let us introduce the space $L^2_{\text{per}}(Y)$ of functions in $Y$, $2\pi$-periodic in the $y_1$ variable, and square integrable in $Y$, and the space $H^1_{\text{per}}(Y) \subset L^2_{\text{per}}(Y)$ of the functions whose first derivatives belong to $L^2_{\text{per}}(Y)$. We also introduce $S_{\text{per}}(Y)$ as the space of functions in $Y$ decaying fast in the variable $y_2$ and $2\pi$-periodic in the variable $y_1$. Without loss of generality we have chosen to work in two dimensions. Nonetheless, all that follows can be generalized to the three dimensional case.

We consider an unsteady fluid flow over a rough interface modeled by the usual unsteady incompressible Navier-Stokes equations

\begin{align*}
\frac{\partial u^\varepsilon}{\partial t} + u^\varepsilon \cdot \nabla u^\varepsilon - \nu \Delta u^\varepsilon + \nabla p^\varepsilon &= f \text{ in } \Omega^\varepsilon \times (0, T), \\
\nabla \cdot u^\varepsilon &= 0 \text{ in } \Omega^\varepsilon \times (0, T), \\
u^\varepsilon &= 0 \text{ on } \Gamma^\varepsilon \times (0, T), \quad \text{ and for simplicity, we assume that the support of the source term } f \text{ does not intersect } \Gamma^\varepsilon. \quad (1) \\
u^\varepsilon &= w \text{ in } \Omega^\varepsilon \text{ at } t = 0, \quad \text{Of course, it is possible to assume other boundary conditions. The initial velocity } w \text{ is given and defined in } \Omega^\circ. \\
\end{align*}

The coefficient $\nu$ is the viscosity. When $\nu$ is small, the flow exhibits boundary layers near the walls. Thus, the problem has three characteristic lengths namely the macroscopic scale (linked to $\Omega^\varepsilon$ and $f$), the Prandtl’s boundary layer scale (of order $\sqrt{\nu}$ for laminar
flows), and the roughness element scale $\varepsilon$. We are concerned with the case where these scales are well separated, specially when $\sqrt{\nu} \gg \varepsilon$. Under this assumption, it is reasonable to expect a viscous sublayer of size $O(\varepsilon)$ due to the roughness element inside the Prandtl’s boundary layer. Thus, we set $\nu = \mu \varepsilon$. This choice is convenient and permit to cover several practical applications, as numerically proved in [16]. Of course, other regimes with other asymptotic expansions are possible, but one has to keep in mind that asymptotic expansions are rather artificial since for practical applications, the viscosity and the geometry are both given and fixed. Therefore, adding this scaling law the problem can be rephrased as

$$\frac{\partial \mathbf{u}^\varepsilon}{\partial t} + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon - \varepsilon \mu \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f} \text{ in } \Omega^\varepsilon \times (0, T),$$

$$\nabla \cdot \mathbf{u}^\varepsilon = 0 \text{ in } \Omega^\varepsilon \times (0, T),$$

$$\mathbf{u}^\varepsilon = \mathbf{0} \text{ on } \Gamma^\varepsilon \times (0, T),$$

$$\mathbf{u}^\varepsilon = \mathbf{w} \text{ in } \Omega^\varepsilon \text{ at } t = 0.$$  \hspace{1cm} (2)

We shall assume enough regularity on the data such that all the Navier-Stokes problems introduced below have isolated branches of solutions corresponding to laminar regimes [15].

In the following, we shall denote $\mathcal{L}^\varepsilon$ the partial differential operator

$$\mathcal{L}^\varepsilon(u, p) = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \varepsilon \Delta \mathbf{u} + \nabla p$$

and make use of the important assumption that the mean flow is not strongly affected by the roughness elements, i.e., the solution of (1) is a perturbation of the solution of the following problem:

$$\frac{\partial \mathbf{u}^\circ}{\partial t} + \mathbf{u}^\circ \cdot \nabla \mathbf{u}^\circ - \varepsilon \mu \Delta \mathbf{u}^\circ + \nabla p^\circ = \mathbf{f} \text{ in } \Omega^\circ \times (0, T),$$

$$\nabla \cdot \mathbf{u}^\circ = 0 \text{ in } \Omega^\circ \times (0, T),$$

$$\mathbf{u}^\circ = \mathbf{0} \text{ on } \Gamma^\circ \times (0, T),$$

$$\mathbf{u}^\circ = \mathbf{w} \text{ in } \Omega^\circ \text{ at } t = 0.$$  \hspace{1cm} (3)

In the case of the linear unsteady Stokes equations, the above assumption can in fact be rigorously proved. We also assume that the solution of the above system describes a laminar flow, in such a way that we have the following Prandtl’s length scales on $\Gamma^\circ$:

$$\frac{\partial u^\circ_1}{\partial x_2} = O(\nu^{-1/2}), \quad \frac{\partial^2 u^\circ_1}{\partial x_2^2} = O(\nu^{-1}), \quad \frac{\partial^2 u^\circ_1}{\partial x_1 \partial x_2} = O(\nu^{-1/2}),$$

$$\frac{\partial u^\circ_1}{\partial t} = O(1), \quad \frac{\partial^2 u^\circ_1}{\partial x_2 \partial t} = O(\nu^{-1}).$$  \hspace{1cm} (4)

As we shall see in the numerical test, the zeroth order approximation (3) fails to correctly predict the velocity, pressure and the friction coefficient on the rough wall, as these variables are influenced by the roughness elements. Therefore, we are interested
in constructing higher order approximation problems based on the asymptotic expansion techniques.

3 The First Order Asymptotic Expansion

3.1 The Ansatz

We assume the following approximation for $u^\varepsilon$ and $p^\varepsilon$:

$$u^\varepsilon(x,t) \approx u^1(x,t) + \varepsilon u^1_{BL}(x, \frac{x}{\varepsilon}, t),$$

$$p^\varepsilon(x,t) \approx p^1(x,t) + \varepsilon p^1_{BL}(x, \frac{x}{\varepsilon}, t).$$

The terms $u^1_{BL}(x, \frac{x}{\varepsilon}, t)$ and $p^1_{BL}(x, \frac{x}{\varepsilon}, t)$ are called boundary layer correctors because they correct at first order the fast oscillating error when replacing $u^\varepsilon$ by $u^\circ$ in (1). The influence of the correctors is restricted to the boundary layer and decay exponentially fast as the variable $\frac{x^2}{\varepsilon}$ tends to infinity. The terms $u^1(x,t)$ and $p^1(x,t)$ are called the macroscopic first order corrections of $u^\circ$ and $p^\circ$ because they do not depend on the fast variable $\frac{x}{\varepsilon}$. In order to construct the boundary layer correctors and the macroscopic first order approximation, let us first examine the error when $u^\varepsilon$ is replaced by $u^\circ$ in (2).

Since $\Omega^\varepsilon \subset \Omega^\circ$, the error arises from the fact that $u^\circ$ does not satisfy the no-slip boundary conditions on $\Gamma^\varepsilon$. However, since $u^\circ$ vanishes on $\Gamma^\circ$ and since $\Gamma^\varepsilon$ is close to $\Gamma^\circ$, the error should be small and is given by a Taylor expansion in the $x_2$ variable:

$$u^\circ(x,t) = -\varepsilon \frac{\partial u^\circ}{\partial n}(x_1,0,t) \frac{x_2}{\varepsilon} + \varepsilon^2 \frac{\partial^2 u^\circ}{\partial n^2}(x_1,0,t)(\xi(x)\frac{x_2}{\varepsilon})^2, \quad \forall (x,t) \in \Gamma^\varepsilon \times (0,T), \quad 0 < \xi < 1.$$  

(6)

Here, the assumption that (2) describes a laminar flow implies that, at leading order,

$$u^\circ(x,t) \approx -\varepsilon \frac{\partial u^\circ}{\partial n}(x_1,0,t) \frac{x_2}{\varepsilon}$$

$$= -\frac{\partial u^\circ}{\partial n}(x_1,0,t) \frac{x_2}{\varepsilon} e_1, \quad \forall (x,t) \in \Gamma^\varepsilon \times (0,T),$$

where we have used the fact that $u^\circ$ is divergence free and the no-slip boundary condition on $\Gamma^\circ$. The term (6) is the error at leading order which is expressed in term of the product between a function of the macroscopic variable and a fast oscillating periodic term.

Therefore, it is natural to look for correctors in the form:

$$\tilde{u}^1(x,t) = \chi^1(\frac{x}{\varepsilon}) \frac{\partial u^\circ}{\partial n}(x_1,0,t), \quad \tilde{p}^1(x,t) = \pi^1(\frac{x}{\varepsilon}) \frac{\partial u^\circ}{\partial n}(x_1,0,t),$$

6
where $\chi^1$, (resp. $\pi^1$) is a function with value in $\mathbb{R}^2$, (resp. $\mathbb{R}$), periodic in $y_1$ direction. We may note that these correctors are not time dependent because only one time scale is considered here. We impose that there exists $\chi^1 \in \mathbb{R}^2$ such that $\chi^1 - \chi^T$ and $\pi^1$ decay exponentially fast as $y_2$ goes to infinity. This condition is rigorously discussed in [3]. Thus, $\tilde{u}^1(x, t)$ will have to be modified in a second step to yield the corrector $u^1_{BL}(x, \frac{x}{\varepsilon}, t)$, and the first order macroscopic approximations $u^1(x, t), p^1(x, t)$.

Therefore, the leading order of $\mathcal{L}^\varepsilon (u^0 + \varepsilon \tilde{u}^1, p^0 + \varepsilon \tilde{p}^1) - f$ resemble of

$$\mathcal{L}^\varepsilon (u^0 + \varepsilon \tilde{u}^1, p^0 + \varepsilon \tilde{p}^1) - f \approx \frac{\partial u^0_1}{\partial n}(x_1, 0, t)(-\mu \Delta_y \chi^1 + \nabla_y \pi^1). \tag{7}$$

The convective term $\frac{\partial u^0_1}{\partial n}(x_1, 0, t)u^0 \cdot \nabla_y \chi^1$ looks like of the same order, but since $\nabla_y \chi^1$ decays exponentially fast as $y_2$ goes to infinity and since $u^0$ vanishes on $\Gamma^0$, this term is actually smaller. Similarly, the term $\varepsilon \frac{\partial^2 u^0_1}{\partial n \partial t}(x_1, 0, t)(\chi^1 - \chi^T)$ was dropped at this order because $(\chi^1 - \chi^T)$ decreases exponentially to zero with $y_2$ and from the the Prandtl length scale.

Likewise,

$$\nabla \cdot (u^0 + \varepsilon \tilde{u}^1) = \frac{\partial u^0_1}{\partial n}(x_1, 0, t)\nabla_y \cdot \chi^1 + \varepsilon \frac{\partial^2 u^0_1}{\partial n \partial x_1}(x_1, 0, t)\chi^1_1. \tag{8}$$

Therefore the correctors $\chi^1, \pi^1$ and the constant vector $\chi^T$ must satisfy:

$$-\mu \Delta_y \chi^1 + \nabla_y \pi^1 = 0 \text{ in } Y, \quad \nabla_y \cdot \chi^1 = 0 \text{ in } Y, \quad \chi^1 = y_2 \mathbf{e}_1 \text{ on } \partial Y_3, \quad \chi^1 - \chi^T \in S_{per}^2, \quad \pi^1 \in S_{per}. \tag{9}$$

It is possible to prove that the problem (9) is well-posed and that $\chi^T = \chi^T_1 \mathbf{e}_1$ from the incompressibility condition (see [3]). One may observe that even for an unsteady problem one has to solve just a steady Stokes problem in the cell $Y$.

It is now possible to define $u^1_{BL}(x, \frac{x}{\varepsilon}, t)$ and $p^1_{BL}(x, \frac{x}{\varepsilon}, t)$ as well as $u^1(x, t)$ and $p^1(x, t)$ from $\tilde{u}^1$ and $\tilde{p}^1$. The boundary layer correctors are defined as

$$u^1_{BL}(x, \frac{x}{\varepsilon}, t) = \frac{\partial u^0_1}{\partial n}(x_1, 0, t)(\chi^1(\frac{x}{\varepsilon}) - \chi^T), \quad p^1_{BL}(x, \frac{x}{\varepsilon}, t) = \frac{\partial u^0_1}{\partial n}(x_1, 0, t)\pi^1(\frac{x}{\varepsilon}). \tag{10}$$

Remark: Even for periodic roughness elements, here we allow general non-periodic assumption for the boundary layer correctors. In other words, we do not impose the
periodicity in the microscopic scales which is the main limitation of the Carrau-LeTallec approach [9].

Therefore, from the definitions of the correctors (10) we have

$$\mathbf{u}^\circ + \varepsilon \mathbf{u}_{BL}^1 = -\varepsilon \chi^1 \frac{\partial u^1_{\circ}}{\partial n}(x_1, 0, t) \text{ on } \Gamma^\varepsilon \times (0, T),$$

which shows that adding $\varepsilon \mathbf{u}_{BL}^1$ to $\mathbf{u}^\circ$ does not improve the error since it remains of the same order. However, a closer inspection shows that now, the error is no longer fast oscillating. Therefore, it can be corrected by solving the macroscopic problem: find $(\mathbf{u}^1, p^1)$ such that

$$\begin{align*}
\frac{\partial \mathbf{u}^1}{\partial t} + \mathbf{u}^1 \cdot \nabla \mathbf{u}^1 - \mu \varepsilon \Delta \mathbf{u}^1 + \nabla p^1 &= \mathbf{f} \text{ in } \Omega^\circ \times (0, T), \\
\nabla \cdot \mathbf{u}^1 &= 0 \text{ in } \Omega^\circ \times (0, T), \\
\mathbf{u}^1 &= 0 \text{ on } \partial \Omega^\circ \setminus \Gamma^\circ \times (0, T), \\
\mathbf{u}^1 &= \varepsilon \chi^1 \frac{\partial u^1_{\circ}}{\partial n} \text{ on } \Gamma^\circ \times (0, T), \\
\mathbf{u}^1 &= \mathbf{w} \text{ in } \Omega^\circ \text{ at } t = 0.
\end{align*}$$

(11)

### 3.2 The Related Effective Boundary Conditions

In practice, the computation of $(\mathbf{u}^\circ, p^\circ)$ and $(\mathbf{u}^1, p^1)$ require to solve two Navier-Stokes problems in $\Omega^\circ$. Alternatively, one may notice that near $\Gamma^\circ$, $\mathbf{u}^1 \approx \mathbf{u}^\circ$, which shows that the boundary condition on $\Gamma^\circ$ can be replaced by the Navier boundary condition

$$\mathbf{u}^1 = \varepsilon \chi^1 \frac{\partial u^1_{\circ}}{\partial n} \text{ on } \Gamma^\circ \times (0, T).$$

However, (11) may be ill-posed if the constant $\chi^1_{\circ}$ is positive since the variational formulation contains the term

$$-\frac{\mu}{\chi^1_{\circ}} \int_{\Gamma^\circ} u^1_1 v_1.$$

To avoid this difficulty, we introduce $\Omega^\delta$ (see figure 2):

$$\Omega^\delta = \Omega^\circ \cap \{x_2 > \delta \varepsilon\},$$

and $\Gamma^\delta = \partial \Omega^\delta \cap \{x_2 = \delta \varepsilon\}$, and we shall solve (11) in $\Omega^\delta$ rather than on $\Omega^\circ$.

The following Taylor expansion on $\mathbf{u}^\circ$
\[ u^\circ(x_1, 0, t) = u(x_1, \varepsilon \delta, t) + \varepsilon \delta \frac{\partial u^\circ}{\partial n}(x_1, \varepsilon \delta, t) + \frac{\varepsilon^2 \delta^2}{2} \frac{\partial^2 u^\circ}{\partial n^2}(x_1, \theta \varepsilon \delta, t) , \]

(12)

\[ \frac{\partial u^\circ}{\partial n}(x_1, \varepsilon \delta, t) = \frac{\partial u^\circ}{\partial n}(x_1, 0, t) + \varepsilon \delta \frac{\partial^2 u^\circ}{\partial n^2}(x_1, \theta \varepsilon \delta, t) , \]

gives the first order effective boundary conditions

\[ \varepsilon \mu \frac{\partial u_1^1}{\partial n} - \frac{\mu}{\chi_1} u_1^1 = 0 \text{ on } \Gamma^\circ \times (0, T) \]

(13)

\[ \Rightarrow \varepsilon \mu \frac{\partial u_1^1}{\partial n} - \frac{\mu}{\chi_1 - \delta} u_1^1 = 0 \text{ on } \Gamma^\delta \times (0, T). \]

**Remark:** Despite considering the problem unsteady, the first order wall law (13) is exactly the same as the one obtained from the steady problem [16]. This provide a good explanation for the fact that classical steady wall laws have been used for unsteady flows with reasonable success [13]. We shall see that is not the case for second order wall laws.

4 The Second Order Effective Boundary Condition

4.1 The Ansatz

In order to improve the approximation of \( u^\varepsilon \) and \( p^\varepsilon \), we propose the ansatz

\[ u^\varepsilon \approx u^2 + \varepsilon u_{BL}^1(x, \frac{x}{\varepsilon}, t) + \varepsilon^2 u_{BL}^2(x, \frac{x}{\varepsilon}, t), \]

(14)

\[ p^\varepsilon \approx p^2 + \varepsilon p_{BL}^1(x, \frac{x}{\varepsilon}, t) + \varepsilon^2 p_{BL}^2(x, \frac{x}{\varepsilon}, t), \]

where the first order boundary layer terms has already been computed. Let us evaluate the error made when we substitute \( u^1 + \varepsilon u_{BL}^1, p^1 + \varepsilon p_{BL}^1 \) in (1). To compute \( L^\varepsilon(u^1 + \varepsilon u_{BL}^1, p^1 + \varepsilon p_{BL}^1) \), we need to give an asymptotic expansion of the following three terms:

\[ u^1 \cdot \nabla u_{BL}^1 \approx \varepsilon \left( \frac{\partial u_1^1}{\partial n} \right)^2 (x_1, 0, t)(\chi^T - y_2 e_1) \cdot \nabla_x \chi^1 \left( \frac{x}{\varepsilon} \right), \]

\[ u_{BL}^1 \cdot \nabla u^1 = \frac{\partial u_1^1}{\partial n}(x_1, 0, t)(\chi^1(\frac{x}{\varepsilon}) - \chi^T) \cdot \nabla_x u^1 \]

\[ \approx - \left( \frac{\partial u_1^1}{\partial n} \right)^2 (x_1, 0, t) \chi^2(\frac{x}{\varepsilon}) e_1, \]

(15)

\[ u_{BL}^1 \cdot \nabla u_{BL}^1 \approx \frac{1}{\varepsilon} \left( \frac{\partial u_1^1}{\partial n} \right)^2 (x_1, 0, t)(\chi^1(\frac{x}{\varepsilon}) - \chi^T) \cdot \nabla_y \chi^1 \left( \frac{x}{\varepsilon} \right). \]
Therefore, the leading order term of $\mathcal{L}^\varepsilon(u^1 + \varepsilon u^1_{BL}, p^1 + \varepsilon p^1_{BL}) - f$ is

$$\mathcal{L}^\varepsilon(u^1 + \varepsilon u^1_{BL}, p^1 + \varepsilon p^1_{BL}) - f \approx \varepsilon \left( \frac{\partial u^1}{\partial n} \right)^2(x_1, 0, t) \left( -y_2 e_1 \cdot \nabla_y \chi^1(\frac{x}{\varepsilon}) - \chi^1_2(\frac{x}{\varepsilon}) e_1 + \chi^1(\frac{x}{\varepsilon}) \cdot \nabla_y \chi^1(\frac{x}{\varepsilon}) + \varepsilon \frac{\partial^2 u^1}{\partial n \partial t}(x_1, 0, t) \left( \chi^1(\frac{x}{\varepsilon}) - \chi^1 \right) \right).$$

The error on the divergence is

$$\nabla \cdot (u^1 + \varepsilon u^1_{BL}) \approx \varepsilon \frac{\partial^2 u^1}{\partial n \partial x_1}(x_1, 0, t) \left( \chi^1 - \chi^1 \right).$$

Since $\left| \varepsilon \frac{\partial^2 u^1}{\partial n \partial x_1}(x_1, 0, t) \right| = O(\sqrt{\varepsilon})$, this error does not need to be corrected at the leading order.

The error on the boundary condition is:

$$(u^1 + \varepsilon u^1_{BL}) \approx \frac{x^2_2}{2\varepsilon^2} \frac{\partial^2 u^1}{\partial n^2}(x_1, 0, t) \varepsilon^2 e_1 \text{ on } \Gamma^\varepsilon \times (0, T).$$

As in §3, we notice that the errors are given by products of fast oscillating periodic terms by slow varying functions. Therefore, it is natural to look for correctors in the form:

$$\tilde{u}^2(x, t) = \chi^2(\frac{x}{\varepsilon}) \left( \frac{\partial u^1}{\partial n} \right)^2(x_1, 0, t) + \chi^2_2(\frac{x}{\varepsilon}) \frac{\partial^2 u^1}{\partial n^2}(x_1, 0, t) + \chi^2_2(\frac{x}{\varepsilon}) \frac{\partial^2 u^1}{\partial n \partial t}(x_1, 0, t),$$

$$\tilde{p}^2(x, t) = \pi^2(\frac{x}{\varepsilon}) \left( \frac{\partial u^1}{\partial n} \right)^2(x_1, 0, t) + \pi^2(\frac{x}{\varepsilon}) \frac{\partial^2 u^1}{\partial n^2}(x_1, 0, t) + \pi^2_2(\frac{x}{\varepsilon}) \frac{\partial^2 u^1}{\partial n \partial t}(x_1, 0, t),$$

where $\chi^2, \chi^2_2, \chi^2_2$ (resp. $\pi^2, \pi^2_2, \pi^2_2$) are functions taking values in $\mathbb{R}^2$, (resp. $\mathbb{R}$) and periodic in the $y_1$ direction. Furthermore, we assume that there exists $\overline{\chi^2}, \overline{\chi^2_2}, \overline{\chi^2_2}$ such that $\chi^2 - \overline{\chi^2}, \chi^2_2 - \overline{\chi^2_2}, \chi^2 - \overline{\chi^2_2}$ decay exponentially fast as $y_2$ goes to infinity. The pair $(\chi^2, \pi^2)$ satisfies:

$$-\mu \Delta_y \chi^2 + \nabla_y \pi^2 = -\left( \chi^1(\frac{x}{\varepsilon}) - y_2 e_1 \right) \cdot \nabla_y \chi^1(\frac{x}{\varepsilon}) - \chi^1_2(\frac{x}{\varepsilon}) e_1 \text{ in } Y,$$

$$\nabla_y \cdot \chi^2 = 0 \text{ in } Y,$$

$$\chi^2 = 0 \text{ on } \partial Y_3,$$

$$\chi^2 - \overline{\chi^2} \in S_{\text{per}}^2, \quad \pi^2 \in S_{\text{per}}.$$
whereas the pair \((\chi^2', \pi^2')\) satisfies:

\[
-\mu \Delta_y \chi^2' + \nabla_y \pi^2' = 0 \text{ in } Y,
\]

\[
\nabla_y \cdot \chi^2' = 0 \text{ in } Y,
\]

\[
\chi^2' = -\frac{y_2}{2} \mathbf{e}_1 \text{ on } \partial Y_3,
\]

\[
\chi^2' - \chi^2' \in \mathcal{S}_{\text{per}}^2, \quad \pi^2' \in \mathcal{S}_{\text{per}},
\]

and the pair \((\chi^2'', \pi^2'')\) satisfies:

\[
-\mu \Delta_y \chi^2'' + \nabla_y \pi^2'' = -\left(\chi^1 - \chi^1\right) \text{ in } Y,
\]

\[
\nabla_y \cdot \chi^2'' = 0 \text{ in } Y,
\]

\[
\chi^2'' = 0 \partial Y_3,
\]

\[
\chi^2'' - \chi^2'' \in \mathcal{S}_{\text{per}}^2, \quad \pi^2'' \in \mathcal{S}_{\text{per}}.
\]

**Remark:** The new corrector \((\chi^2'', \pi^2'')\) is associated with the time derivative term and therefore is not present in the steady second order approximation [3].

The problem (20) is well-posed, as showed in the following result:

**Theorem 1** There is a unique pair of functions \((\chi^2'', \pi^2'')\) and a unique vector \(\chi^2'' \in \mathbb{R}^2\) such that \(\chi^2'' - \chi^2'' \in H^1_{\text{per}}(Y) \cap \mathcal{S}_{\text{per}}(Y), \pi'' \in L^2_{\text{per}}(Y) \cap \mathcal{S}_{\text{per}}(Y),\) and (20) is satisfied in a weak sense.

Proof: The proof follows the same technique presented in [3] to demonstrate equivalent result for the problem (9). For details see [6].

It is now possible to define \(u_{BL}^2(x, \frac{x}{\varepsilon}, t), u_{BL}'(x, \frac{x}{\varepsilon}, t), p_{BL}^2(x, \frac{x}{\varepsilon}, t)\) and \(p_{BL}'(x, \frac{x}{\varepsilon}, t)\) as well as \(u^2(x, t), p^2(x, t)\) from \(\chi^2, \chi^2', \chi^2'', \pi^2, \pi^2'\) and \(\pi^2''\). In fact, the boundary layer correctors corresponding to the leading order error term in the partial differential equation are defined by

\[
u_{BL}^2(x, \frac{x}{\varepsilon}, t) = \left(\frac{\partial u_1^1}{\partial \mathbf{n}}\right)^2(x_1, 0, t)(\chi^2(\frac{x}{\varepsilon}) - \chi^2) + \frac{\partial^2 u_1^1}{\partial \mathbf{n} \partial t}(x_1, 0, t)(\chi^2''(\frac{x}{\varepsilon}) - \chi^2''),
\]

\[
p_{BL}^2(x, \frac{x}{\varepsilon}, t) = \left(\frac{\partial n_1}{\partial \mathbf{n}}\right)^2(x_1, 0, t)\pi^2(\frac{x}{\varepsilon}) + \frac{\partial^2 u_1^1}{\partial \mathbf{n} \partial t}(x_1, 0, t)\pi^2''(\frac{x}{\varepsilon}),
\]

whereas the boundary layer correctors corresponding to the leading order error term in the boundary conditions are

\[
u_{BL}'(x, \frac{x}{\varepsilon}, t) = \frac{\partial^2 u_1^1}{\partial \mathbf{n}^2}(x_1, 0, t)(\chi^2'\frac{x}{\varepsilon} - \chi^2'),
\]

\[
p_{BL}'(x, \frac{x}{\varepsilon}, t) = \frac{\partial^2 u_1^1}{\partial \mathbf{n}^2}(x_1, 0, t)\pi^2'\frac{x}{\varepsilon}.
\]
Note that by integrating by parts the divergence free conditions on $\chi^2$, $\chi^2'$ and $\chi^2''$, we obtain that

$$\chi^2 = 0, \chi^2' = 0, \chi^2'' = 0.$$

The second order macroscopic approximations of $u^\varepsilon, p^\varepsilon$ can now be found by solving the follow boundary value problem: find $(u^2, p^2)$ such that

$$\frac{\partial u^2}{\partial t} + u^2 \nabla u^2 - \mu \varepsilon \Delta u^2 + \nabla p^2 = f \text{ in } \Omega^0 \times [0, T),$$

$$\nabla \cdot u^2 = 0 \text{ in } \Omega^0 \times (0, T),$$

$$u^2 = 0 \text{ on } \partial \Omega^0 \setminus \Gamma^0 \times (0, T),$$

$$u^2_1 = \varepsilon \chi_1 \frac{\partial u^1_1}{\partial n} + \varepsilon^2 \left( \chi_1 \frac{\partial u^1_1}{\partial n} \right)^2 + \chi_1 \frac{\partial^2 u^1_1}{\partial n^2} + \chi_1'' \frac{\partial^2 u^1_1}{\partial n \partial t} \text{ on } \Gamma^0 \times (0, T),$$

$$u^2_2 = 0 \text{ on } \Gamma^0 \times (0, T),$$

$$u^2 = \omega \text{ in } \Omega^0 \text{ at } t = 0.$$

4.2 The Related Effective Boundary Conditions

As above, it is more convenient to compute $(u^2, p^2)$ by changing the boundary conditions on $\Gamma^0$ slightly. Indeed, if $\chi_1^2 \neq 0$ one may use the second order effective boundary conditions

$$\varepsilon \mu \frac{\partial u^1_1}{\partial n} \chi_1 - \mu \chi_1^2 u^2 + \mu \chi_1^2 \varepsilon^2 \left( \chi_1^2 \frac{\partial u^1_1}{\partial n} \right)^2 + \chi_1 \frac{\partial^2 u^1_1}{\partial n^2} + \chi_1'' \frac{\partial^2 u^1_1}{\partial n \partial t} = 0,$$

$$u^2_2 = 0 \text{ on } \Gamma^0 \times (0, T). \quad (24)$$

Moreover, by using the first order approximation of $\frac{\partial u^1_1}{\partial n}$ given by (11),

$$\varepsilon^2 \frac{\mu}{\chi_1^2} \chi_1 \left( \frac{\partial u^1_1}{\partial n} \right)^2 \approx \frac{\chi_1^2 \mu}{\chi_1^2} (u^1_1)^2,$$

$$\varepsilon^2 \frac{\mu}{\chi_1^2} \chi_1'' \frac{\partial^2 u^1_1}{\partial n \partial t} \approx \varepsilon \frac{\mu}{\chi_1^2} \chi_1'' \frac{\partial u^1_1}{\partial t}. \quad (25)$$

This enables to have the nonlinearity on a zeroth order term rather than a derivative and only first order derivative term in time. Furthermore, the Navier-Stokes equations restrict at the boundary, indicate that at leading order,
\[ \mu \varepsilon \frac{\partial^2 u_1^2}{\partial n^2}(x_1, 0, t) \approx \frac{\partial p^2}{\partial x_1}(x_1, 0, t) + \frac{\partial u_1^2}{\partial t}(x_1, 0, t). \]

Therefore it is possible to obtain the following set of boundary conditions:

\[ \varepsilon \mu \frac{\partial u_1^2}{\partial n} - \frac{\mu}{\chi^1_1} u_1^2 + \varepsilon \frac{\chi^2_1}{\chi^1_1} \frac{\partial p^2}{\partial x_1} + \varepsilon \left( \frac{\chi^2_1}{\chi^1_1} + \mu \frac{\chi^2_1}{\chi^1_1} \right) \frac{\partial u_1^2}{\partial t} + \mu \frac{\chi^2_1}{\chi^1_1} (u_1^2)^2 = 0, \]

\[ u_2^2 = 0 \text{ on } \Gamma^0 \times (0, T). \quad (26) \]

**Remark:** The wall law (24) is different to (26), but the approximation error associated with each boundary law is smaller than the actual leading error.

As for the first order approximation, the boundary value problem in \( \Omega^0 \) may be ill-posed. However, it is possible to construct a well-posed problem in \( \Omega^\delta \). From a Taylor expansion, we obtain that

\[ u_1^2(x_1, 0, t) = u_1^2(x_1, \varepsilon \delta) + \varepsilon \delta \frac{\partial u_1^2}{\partial n}(x_1, \varepsilon \delta) + \frac{\varepsilon^2 \delta^2}{2} \frac{\partial^2 u_1^2}{\partial n^2}(x_1, \varepsilon \delta) + O(\varepsilon^3 \delta^3), \]

\[ \frac{\partial u_1^2}{\partial n}(x_1, 0, t) = \frac{\partial u_1^2}{\partial n}(x_1, \varepsilon \delta) + \varepsilon \delta \frac{\partial^2 u_1^2}{\partial n^2}(x_1, \varepsilon \delta) + O(\varepsilon^2 \delta^2), \]

\[ \left( \frac{\partial u_1^2}{\partial n} \right)^2(x_1, 0, t) = \left( \frac{\partial u_1^2}{\partial n} \right)^2(x_1, \varepsilon \delta) + O(\varepsilon \delta), \]

\[ \frac{\partial^2 u_1^2}{\partial n \partial t}(x_1, 0, t) = \frac{\partial^2 u_1^2}{\partial n \partial t}(x_1, \varepsilon \delta) + O(\varepsilon \delta). \quad (27) \]

The second order effective boundary condition on \( \Gamma^\delta \) reads

\[ \varepsilon \mu \frac{\partial u_1^2}{\partial n} - \frac{\mu}{\chi^1_1 - \delta} u_1^2 + \varepsilon^2 \frac{\mu}{\chi^1_1 - \delta} \left( \left( \delta \frac{\chi^1_1 - \delta^2}{2} + \frac{\chi^2_1}{\chi^1_1} \right) \frac{\partial u_1^2}{\partial n^2} + \frac{\chi^2_1}{\chi^1_1} (\frac{\partial u_1^2}{\partial n})^2 + \frac{\chi^2_1}{\chi^1_1} \frac{\partial^2 u_1^2}{\partial n \partial t} \right) = 0, \]

\[ u_2^2 = 0 \text{ on } \Gamma^\delta \times (0, T), \quad (28) \]

or,
\[ \varepsilon \mu \frac{\partial u^2}{\partial n} - \frac{\mu}{\chi^1_1 - \delta} u^2_1 + \frac{1}{\chi^1_1 - \delta} \left( \varepsilon \left( \delta \chi^1_1 - \frac{\delta^2}{2} + \chi^2_1 \right) \frac{\partial p^2}{\partial x_1} + \varepsilon \left( \delta \chi^1_1 - \frac{\delta^2}{2} + \chi^2_1 \right) \frac{\partial u^2_1}{\partial t} + \mu \frac{\chi^2_1}{(\chi^1_1 - \delta)^2} (u^2_1)^2 \right) = 0, \]

\[ u^2_2 = 0, \text{ on } \Gamma^\delta \times (0, T). \quad (29) \]

**Remark:** In the two sets of boundary conditions above, we have neglected the corrections from the Taylor expansion of \( u^2_2 \), as this term is of a smaller order.

### 4.2.1 Summary: the Proposed Effective Problems

The first order initial boundary value problem in \( \Omega^\delta \) is the following:

\[ \frac{\partial u^1}{\partial t} + u^1 \cdot \nabla u^1 - \mu \varepsilon \Delta u^1 + \nabla p^1 = f \text{ in } \Omega^\delta \times [0, T), \]
\[ \nabla \cdot u^1 = 0 \text{ in } \Omega^\delta \times [0, T), \]
\[ u^1 = 0 \text{ on } \partial \Omega^\delta \setminus \Gamma^\delta \times (0, T), \]

\[ \varepsilon \mu \frac{\partial u^1}{\partial n} - \frac{\mu}{\chi^1_1 - \delta} u^1_1 = 0 \text{ on } \Gamma^\delta \times (0, T), \]
\[ u^1 = w \text{ in } \Omega^\delta \text{ at } t = 0, \quad (30) \]

where \( \overline{\chi^1_1} \) is calculated from (9).

The second order initial boundary value problem in \( \Omega^\delta \) is:

\[ \frac{\partial u^2}{\partial t} + u^2 \cdot \nabla u^2 - \mu \varepsilon \Delta u^2 + \nabla p^2 = f \text{ in } \Omega^\delta \times (0, T), \]
\[ \nabla \cdot u^2 = 0 \text{ in } \Omega^\delta \times (0, T), \]
\[ u^2 = 0 \text{ on } \partial \Omega^\delta \setminus \Gamma^\delta \times (0, T), \]

\[ \varepsilon \mu \frac{\partial u^2}{\partial n} - \frac{\mu}{\chi^1_1 - \delta} u^2_1 + \frac{1}{\chi^1_1 - \delta} \left( \varepsilon \left( \delta \chi^1_1 - \frac{\delta^2}{2} + \chi^2_1 \right) \frac{\partial p^2}{\partial x_1} + \varepsilon \left( \delta \chi^1_1 - \frac{\delta^2}{2} + \chi^2_1 \right) \frac{\partial u^2_1}{\partial t} + \mu \frac{\chi^2_1}{(\chi^1_1 - \delta)^2} (u^2_1)^2 \right) = 0 \text{ on } \Gamma^\delta \times (0, T), \]
\[ u^2 = w \text{ at } t = 0, \quad (31) \]

where \( \overline{\chi^2}, \overline{\chi^2}' \) and \( \overline{\chi^2}'' \) are calculated from (18), (19) and (20) respectively.

For \( \delta \) large enough, (30) and (31) may have a solution. We shall not discuss this subject in this paper, but we shall solve (30) and (31) numerically in §5, and compare its solution with one obtained with the original flow.
Remark: It is easily seen that the effective first order condition is equivalent (at this order of approximation) to a non-slip condition on a flat wall at an average height, (computed by the asymptotic expansion). This is not the case in general for the second order effective condition.

Remark: It would be interesting to pursue the approach to construct higher order wall laws when the flow is almost turbulent. In this case, the next laws may present non-zero second component of velocity which improves the numerical results [16]. Another possibility to treat higher Reynolds number flow is to change the viscosity scale assumption ($\nu = \varepsilon \mu$) and to perform a different asymptotic expansion.

5 Channel with Two Types of Roughness

The numerical validation of the procedure proposed herein is performed using a 2D unsteady incompressible Navier-Stokes code. The discretization is accomplished within a stabilized finite element method using isoparametric bilinear elements as presented in [5] coupled with an implicit Euler scheme in time. The direct computation of (1) is done with high accuracy, with a large number of elements close to the rough wall. We investigate the first order (30) and second order (31) wall laws which allows to use a coarser mesh. Before employing the wall laws, we must solve the cell Stokes problems (9), (18), (19) and (20).

Figure 3: Second order correctors associated with the transient term for the sinusoidal and asymmetric roughness elements.
Because the solution decays fast as $y_2$ grows, the cell is truncated in the $y_2$–direction where a homogeneous Neumann condition is imposed for the first component whereas homogeneous Dirichlet for the second one. This decay of the solutions allows to use coarser finite element meshes.

The bottom boundary is of mixed type and is delimited by the coordinates $(0, 1) \times (0, 0.5)$. The subboundary delimited by the coordinates $(0, 0.25)$ is flat, whereas the other interval has asymmetric periodic roughness in the subinterval $(0.25, 0.75)$ and has sinusoidal periodic roughness elements in the subinterval $(0.75, 1)$. The period of the sinusoidal (resp. asymmetric) roughness element is $0.04$ (resp. $0.05$) and its amplitude is $0.01$ (resp. $0.025$).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{isolines.png}
\caption{Isolines of $|u|$ at $t = 0.04$.}
\end{figure}

A no-slip condition is imposed on the wall $x_2 = 0.5$ and at the bottom boundary.
A parabolic profile of velocity is imposed at the entry of the channel. The boundary conditions at $x_1 = 1$ are $u_2 = 0$ and $p = 0$.

The magnitude of the viscosity is chosen $10^{-2}$, with $\mu = 0.4$ and $\varepsilon = 0.025$. The direct computation is performed with 12000 elements.

For the sinusoidal (resp. asymmetric) roughness element, the horizontal size of the cell is 1.6 (resp. 2), and the cell is artificially truncated at $y_2 = 5$. The amplitude of the sinusoidal (resp. asymmetric) roughness element in the microscopic coordinates is 0.4 (resp. 1). The mesh used for the sinusoidal cell problems has 2000 elements and 1600 for the asymmetric ones.

**DIRECT COMPUTATION - TIME = 0.14**

**ORDER 0 - TIME = 0.14**

ORDER 1 - TIME = 0.14

ORDER 2 - TIME = 0.14

Figure 5: Isolines of $|u|$ at $t = 0.14$. 
The values of the computed constants for the asymmetric roughness element are
\[ \chi_1^1 = 0.84, \quad \chi_1^2 = -0.036, \quad \chi_1^2' = 4.47 \times 10^{-3}, \quad \chi_1^2'' = 2.63 \times 10^{-5}, \]
and for the sinusoidal ones are
\[ \chi_1^1 = 0.3, \quad \chi_1^1 = -0.05, \quad \chi_1^2 = 3.676 \times 10^{-3}, \quad \chi_1^2 = 9 \times 10^{-5}. \]

Figure 6: Cross section of tangential velocity on \( x_2 = 0.035 \) at \( t = 0.08 \) and friction coefficient on \( x_2 = 0.05 \) at \( t = 0.14 \).
For the first and second order effective boundary value problems, the wall laws are imposed on the line \( x_2 = \varepsilon \delta = 0.025 \). The number of elements used is 2400. The figure 3 depicts the contour lines of the third corrector \( \chi''_{x_1} \) for the sinusoidal and asymmetric geometry as well as a plot of their values on a cross-section (see [16] for the solution of the others corrector problems). One may note the fast convergence to a constant.

We also present the isolines of \(|u|\) in two different periods of time, where we can see the solutions reached from the direct computation and from the three approximation orders (see figures 4 and 5). Here the zeroth order condition is not accurate enough, whereas the other orders give us a substantial improvement. In particular, the second order effective boundary condition allows us to have a better approximation than the first order, as showed in the figure 6 in which we compare cross-sections of the tangential velocity and friction coefficient \( C_f \). In figure 7 the tangential velocity is presented in several periods of time on the point \((x_1, x_2) = (0.93, 0.035)\). One may clearly observe the improvement of the approximation when the second order wall law is adopted. The time derivative term and its associate cell problem are responsible to capture unsteadiness of the boundary layer.

![Comparison between the different order of approximations of the tangential velocity at \((x_1, x_2) = (0.93, 0.035)\).](image)

**Figure 7:** Comparison between the different order of approximations of the tangential velocity at \((x_1, x_2) = (0.93, 0.035)\).

6 Conclusions

New wall laws were presented to simulate unsteady flows over periodic rough interface. The asymptotic expansion consists of a powerful technique to construct real unsteady wall laws. In contrast to the first order wall law which is the same as adopted for steady flow, the second order approximation has shown to play an important role in the numerical results. The adoption of only one time-scale was able to capture the unsteadiness features of the flow as showed by the numerical results. It would be interesting to compare this
approach with the two time scale in the high Reynolds number context. In fact, the next aim would be the extension of the current framework to construct wall laws for turbulence models. This will be subject of future works.

References


