NUMERICAL SIMULATION OF ICE FORMATION AROUND CYLINDERS IN A FLOW WITH OVERLAPPED GRIDS

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Abstract. When plural coordinate systems are used for solving the whole region, the physical values are exchanged between two coordinate systems. In this process, the interpolation/extrapolation procedures are needed. In this paper, the numerical technique of dynamic interpolation is proposed. The interpolation accuracy is changed according to the ongoing solution. The referred points are dynamically changing. Furthermore, to evaluate the effects of dynamic interpolation, some numerical examples are shown including the ice formation around cylinders in a flow. In this calculation for instance, the main flow is solved with Cartesian coordinate system, while moving BFC is used around cylinders. In the traditional interpolation method, the mesh size of the Cartesian system should be almost the same as that of the BFC to keep the numerical accuracy. It means that the interpolation between two different sizes of grids causes the poor accuracy in overlapped region. However, when the same size of grid in overlapped region to avoid this problem, the mesh number of either coordinate system is unexpectedly large, which leads to a long computation time. By using the dynamic interpolation technique, the number of the Cartesian system can be reduced, which leads to a shorter computation time. It is important to know how the final solution changes according to the grid size ratio in overlapped region. The effects of choosing reference points on the final solution are also discussed.
1 INTRODUCTION

The main point of this paper is the interpolation/extrapolation technique in an overlapped region. When plural coordinate systems are used for solving the whole region, the physical values are exchanged between two coordinate systems. When overlapped grids are used in particular, the position where the physical value is defined in one coordinate system is not always the same as that in another coordinate system. It means that the interpolation/extrapolation procedures are needed when values are exchanged between the two systems. Usually the linear interpolation of low order accuracy is used. To keep the accuracy when the interpolation is done on overlapped grids, the size of each grid that is contained in overlapped region in one coordinate system should be almost the same as that of the other coordinate system. It means that the interpolation between two different sizes of grids causes the poor accuracy in overlapped region. However, when we use the same size of grids in the overlapped region to avoid this problem, the mesh number of either coordinate system is unexpectedly large, which leads to a long computation time. If the size of grids that belong to one coordinate system can be larger than that of the other system in overlapped region without losing the interpolation accuracy, it gives a shorter computation time. However, few discussions can be found on the effects of the interpolation between two fairly different grid sizes in each coordinate system on the final solution.

In this paper, the numerical technique of dynamic interpolation is proposed. In this technique, the interpolation accuracy is changed according to the ongoing solution. The number of referred points for interpolation of one point is dynamically changing. Furthermore, to evaluate the effects of dynamic interpolation, some numerical examples are shown including the ice formation around cylinders in a flow. In this calculation for instance, the main flow is solved with Cartesian coordinate system, while the moving boundary fitted coordinate system is used around cylinders. In the traditional interpolation method, the mesh size of the Cartesian system should be almost the same as that of the BFC to keep the numerical accuracy. By using the dynamic interpolation technique, we can reduce the number of the Cartesian system grids, which leads to a shorter computation time. However, it is important to know how the final solution changes according to the grid size ratio in overlapped region. The effects of grid size in each coordinate system on the final solution are also discussed.

2 NUMERICAL TECHNIQUE

2.1 Problems in linear interpolation

In numerical simulations using FDM, a lot of interpolation procedures are repeated. Usually a linear interpolation is applied, which causes no problems in a single coordinate system. When the geometry is complex and cannot be presented with a single coordinate system, plural coordinate systems should be used. The calculated data should be exchanged between two plural coordinate systems by using some interpolation in one grid system to another. In each coordinate system, for instance, high-order upwind difference is used for
keeping numerical accuracy. However, when the numerical accuracy of exchanging data in two coordinate systems is poor, the accuracy of the whole results is also poor. In an overlapped region that contains two kind of grids, when the location where dependent variables are defined in one coordinate system is exactly the same as the location in another coordinate system, the dependent variables are exchanged directly, which leads to keeping good accuracy. Generally when plural coordinate systems are used, the locations of numeric nodes in one grid system are not equal to that in another grid system in the overlapped region. The interpolation procedures are needed in the overlapped region to exchange the node information to each other.

For exchanging data in two grid systems, linear interpolation is conventionally used. In two dimensional problem for instance, when a node value in the coordinate system I is transferred to the system II in Figure 1, the four node locations in the system I that surround the targeted location in the system II are detected. Then the value in the system II is linearly interpolated from the four values. However, this procedure contains inconsistency in some cases. Figure 2 shows the one-dimensional case. The node locations in the system I are different from those in II. The linear interpolation from I to II is given as the following equation.

\[
\phi = \phi_i^I \frac{\phi_{i+1}^I - \phi_i^I}{x_{i+1}^I - x_i^I} + \phi_{i+1}^I \frac{\phi_i^I - \phi_{i+1}^I}{x_i^I - x_{i+1}^I} 
\]

When the values in I are linearly re-interpolated from the values in II that are obtained from the equation (1), however, the new values in I are different from the old values. Although the values that are transferred into II are usually used as the boundary condition in the system II, the linear interpolation contains some possible inconsistency as expressed in the equation (1), which brings poor accuracy around the overlapped region.

Figure 1: Conventional interpolation on overlapped region
To keep the accuracy high when the values are interpolated from I to II, it is important to increase the number of the referred points in I. Discussions on which and how many points should be referred come in section 2.3. General expressions of interpolation are given as the following equations based on Taylor expansion.

\[
\begin{bmatrix}
\phi'_{i-1} \\
\phi'_{i} \\
\phi'_{i+1} \\
\vdots
\end{bmatrix} = \begin{bmatrix}
1 & \Delta_{i-1} & \frac{1}{2} \Delta_{i-1}^2 & \frac{1}{6} \Delta_{i-1}^3 & \cdots \\
1 & \Delta_{i} & \frac{1}{2} \Delta_{i}^2 & \frac{1}{6} \Delta_{i}^3 & \cdots \\
1 & \Delta_{i+1} & \frac{1}{2} \Delta_{i+1}^2 & \frac{1}{6} \Delta_{i+1}^3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \begin{bmatrix}
\phi''_{i} \\
(\phi''')_{i} \\
(\phi''')_{i} \\
\vdots
\end{bmatrix}
\]  

(2)

The left hand side of equation (2) is a vector of referred points, while the matrix in right hand side is a coefficient matrix composed of the distance between each referred point and the target point. The vector in right hand side is differential coefficients at the target point. When the referred points are given under a suitable algorithm, the differential coefficients including zero-order coefficients are calculated from the equation (2). This expression also gives the possibility of extrapolation, which means that the target point is not necessarily located between two referred (known) points.

### 2.2 Multi-dimensional interpolation/extrapolation

A generalized interpolation (equation (2)) is applied to two-dimensional problem. The
Taylor expansion form is expressed as the following equation.

\[
\begin{bmatrix}
\phi_i \\
\phi_x \\
\phi_y \\
\phi_{xx} \\
\phi_{yy} \\
\phi_{xy} \\
\end{bmatrix} = \begin{bmatrix}
1 & \Delta x & \Delta y & \frac{1}{2} \Delta x^2 & \Delta x \Delta y & \frac{1}{2} \Delta y^2 & \cdots & \end{bmatrix} \begin{bmatrix}
\phi \\
\phi_x \\
\phi_y \\
\phi_{xx} \\
\phi_{yy} \\
\phi_{xy} \\
\end{bmatrix}
\]

(3)

Any types of combinations of the referred points are possible in the equation (3). For example, mathematical second-order accuracy is given when six points are referred. Mathematical third-order accuracy is given when ten points are referred as well. In three-dimensional problems, the same type of Taylor expansion expression is possible. However, it is important that the mathematical accuracy does not always give the physical accuracy. How to determine the referred points is then important. As Figure 2 shows, when the distribution of the physical value around the overlapped region is linear, the traditional linear interpolation gives a good prediction. On the contrary when the distribution is not linear, the linear interpolation does not return a good result. Which type of interpolation should be used is determined according to second-order differential coefficients of dependent values.

### 2.3 Dynamic procedure

It is important to decide which points should be referred to know the interpolated value at the target point with equation (3). Second-order differential coefficients give good suggestions for the decision. Here lower accuracy is enough for calculating the second-order differential coefficients than the accuracy of general interpolation. Figure 3 shows how to estimate the second-order differential coefficients. There are four kinds of coefficients, which are,

\[
\phi''(+a,+a) = \frac{2[\Delta a^- (\phi_{i+1,j+1} - \phi_i) - \Delta a^+ (\phi_i - \phi_{i-1})]}{\Delta a^- \Delta a^+ (\Delta a^- + \Delta a^+)}
\]

\[
\phi''(+b,+b) = \frac{2[\Delta b^- (\phi_{j+1} - \phi_j) - \Delta b^+ (\phi_j - \phi_{j-1})]}{\Delta b^- \Delta b^+ (\Delta b^- + \Delta b^+)}
\]

\[
\phi''(+a,+b) = \frac{2[\Delta c^- (\phi_{i+1,j} - \phi_{i,j}) - \Delta c^+ (\phi_{i,j} - \phi_{i-1,j})]}{\Delta c^- \Delta c^+ (\Delta c^- + \Delta c^+)}
\]

\[
\phi''(+a,-b) = \frac{2[\Delta d^- (\phi_{i,j+1} - \phi_{i,j}) - \Delta d^+ (\phi_{i,j} - \phi_{i,j-1})]}{\Delta d^- \Delta d^+ (\Delta d^- + \Delta d^+)}
\]

(4)

The eight points around the main point that is the closest point in system I to the target point in system II are used for calculating equation (4). The largest coefficient in equation (4) shows
the most effective direction for interpolation. When many points in the effective direction are referred for calculating equation (3), the physical accuracy of the interpolation is expected to be high. Some kinds of masks should be prepared according to the value of equation (4).

![Figure 3: Four kinds of second-order differential coefficients](image)

2.4 Application

The generalized formation as equation (3) is prepared as a subroutine in a source code. It is useful because FDM is a series of interpolation. This generalized formation is also applied not only to the overlapped region but also to the calculation of metrics that is essential in BFC and to upwind difference in convection terms as well. In particular, metrics should be important values for numerical simulation using BFC because they give effects on all terms that appear in basic equations. When upwind difference with generalized formation is used, the number of the points that locate in both upwind and downwind side from the target point can be chosen as any value. In multi-directional upwind difference technique, when a convection velocity that comes from the oblique angle is strong, the transported value is interpolated in oblique direction. However in this case, the transported value is calculated with physically high accuracy when the equation (3) and (4) are used.

3 HEAT TRANSFER AND FLUID FLOW AROUND A CYLINDER

3.1 Basic equations

In the problem of this type, two kinds of coordinate systems are needed. One is the Cartesian coordinate system for main flow, while the other is a curvilinear coordinate system for closer region to the cylinder. In this problem for example, fine meshes are needed for
calculating the region around the cylinder, while fine meshes for the Cartesian system are not necessarily needed for calculating the main flow because physical values are not changing significantly in the main flow. However in the traditional technique, the mesh size in the Cartesian system should be unnecessarily small to keep the accuracy in the overlapped region, because the mesh size of the two systems is recommended to be nearly equal in the overlapped region to keep the accuracy. The present interpolation technique is effective when values in sparse mesh are interpolated to values in fine mesh. When the number of the Cartesian system is small, the calculation time can be reduced.

![Figure 4: Coordinate systems around one cylinder](image)

The coordinate system for calculation is shown in Figure 4. One cylinder is located in a uniform flow. In the Cartesian coordinate system (system I), the physical values in the overlapped region are interpolated from the values in the curvilinear coordinate system (system II). It means that the values in system I are given as Dirichlet boundary condition. While in the out-boundary region in system II, the values are interpolated from the values in system I, which is also regarded as Dirichlet condition. By using the present dynamic interpolation method, suggestions on setting the overlapped region and also on minimum requirements of the mesh size in Cartesian system.

The basic equations in the Cartesian system are

\[
\begin{align*}
\frac{\partial u'^i}{\partial x'^i} &= 0 \\
\frac{\partial u'^i}{\partial t} + \frac{\partial}{\partial x'^j}(u'^ju'^i - \frac{1}{R^e} \frac{\partial u'^i}{\partial x'^j}) &= -\frac{\partial p}{\partial x'^j} \\
\frac{\partial T}{\partial t} + \frac{\partial}{\partial x'^j}(u'^jT - \frac{1}{R^e P^e} \frac{\partial T}{\partial x'^j}) &= S. \\
\end{align*}
\]
The basic equations in the curvilinear coordinate system are expressed as

\[
\frac{\sqrt{g_{jj}}}{J} \frac{\partial}{\partial \zeta^{(j)}} \left( \frac{J}{\sqrt{g_{jj}}} u^{(j)} \right) = 0
\]

\[
\frac{\partial u^{(j)}}{\partial t} + \frac{\sqrt{g_{jj}}}{J} \frac{\partial}{\partial \zeta^{(j)}} \left( \frac{J}{\sqrt{g_{jj}}} (u^{(i)} u^{(j)} - \tau^{(ij)}) \right) = -g^{(ij)} \frac{\partial p}{\partial \zeta^{(j)}} - C_{mj}^{i} (u^{(m)} u^{(j)} - \tau^{(mj)})
\]

\[
\frac{\partial T}{\partial t} + \frac{\sqrt{g_{jj}}}{J} \frac{\partial}{\partial \zeta^{(j)}} \left( \frac{J}{\sqrt{g_{jj}}} (u^{(i)} T - \frac{1}{\rho P} g^{(jm)} \frac{\partial T}{\partial \zeta^{(m)}}) \right) = S .
\]

(6)

Here, \(C_{mj}^{i}\) show the physical Christoffel symbols that are defined as

\[
C_{mj}^{i} = \frac{g_{ii}}{g_{mm} g_{jj}} (M_{mj}^{i} - \delta_{m}^{i} M_{mj}^{k} M_{mk}^{i})
\]

\[
M_{mj}^{i} = \frac{1}{2} g^{ik} \left( \frac{\partial g_{jk}}{\partial \zeta^{m}} + \frac{\partial g_{mk}}{\partial \zeta^{j}} - \frac{\partial g_{mj}}{\partial \zeta^{k}} \right) : \text{Mathematical Christoffel symbols.}
\]

(6-1)

The viscous shear stresses are expressed as

\[
\tau^{(i)} = \frac{1}{\rho P} \left( g^{(im)} \nabla_{(m)} u^{(j)} + g^{(jm)} \nabla_{(m)} u^{(i)} \right)
\]

\[
\nabla_{(j)} u^{(i)} = \frac{\partial u^{(i)}}{\partial \zeta^{(j)}} + C_{mj}^{i} u^{(m)}
\]

(6-2)

Other metrics that appear in the equations above are

\[
r = (x, y, z), \quad \mathbf{r}_{j} = \frac{\partial \mathbf{r}}{\partial \zeta^{j}}, \quad g_{ij} = \mathbf{r}_{i} \cdot \mathbf{r}_{j}
\]

\[
J^{2} = |g_{jj}|
\]

\[
g^{(ij)} = \sqrt{g_{ii}} \sqrt{g_{jj}} g^{(i) (j)} , \quad \tau^{(i)} = (g_{ij})^{-1} .
\]

(6-3)

In the equations above, the dependent variables are the physical contravariant components [1-2].

3.2 Dynamic procedure

The algorithm for solving equation (5) and (6) is HSMAC. Data exchange on overlapped
region occurs at boundary conditions shown in Figure 5. Some kinds of masks are prepared for reference points according to the value of the second-order differential coefficients as described in the section 2.3. To keep the mathematical third-order accuracy, for instance, at least ten points should be referred for the interpolation procedure. Figure 6 shows some possible masks for the interpolation. The numerical procedure for Figure 6(a) is,

\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \\
\phi_5 \\
\phi_6
\end{pmatrix} = \begin{pmatrix}
1 & a_1 & b_2 & \frac{1}{2} a_1^2 & a_1 b_2 & \frac{1}{2} b_2^2 \\
1 & a_2 & b_2 & \frac{1}{2} a_2^2 & a_2 b_2 & \frac{1}{2} b_2^2 \\
1 & a_3 & b_2 & \frac{1}{2} a_3^2 & a_3 b_2 & \frac{1}{2} b_2^2 \\
1 & a_2 & b_3 & \frac{1}{2} a_2^2 & a_2 b_3 & \frac{1}{2} b_3^2 \\
1 & a_2 & b_1 & \frac{1}{2} a_2^2 & a_2 b_1 & \frac{1}{2} b_1^2 \\
1 & a_1 & b_1 & \frac{1}{2} a_1^2 & a_1 b_1 & \frac{1}{2} b_1^2
\end{pmatrix} \begin{pmatrix}
\phi_a \\
\phi_b \\
\phi_{aa} \\
\phi_{ab} \\
\phi_{bb}
\end{pmatrix}
\] (7)

For a further order of accuracy, 10, 15 or 21 points should be referred. Too many kinds of possible masks are there, however, they should be carefully examined with a lot of numerical examples for choosing the best or the better mask for the given interpolation.

Figure 5: HSMAC algorithm
Masaaki Ishikawa, Shigeho Noda, and Tetsuo Hirata

4 SOLIDIFICATION AROUND A CYLINDER

4.1 Basic equations

In solid-liquid phase change problems, the first-order differential coefficient that is normal to the solid-liquid boundary is important because it appears in the moving boundary conditions[3]. The coordinate system near the solid-liquid interface is shown in Figure 7, and
the boundary condition on the interface is expressed as the following equations.

\[ \rho_s L \frac{ds}{dt} = q_s - q_l \]

\( \rho_s \): Density

\( L \): Latent heat

Here \( q_s \) and \( q_l \) are the heat flux that are normal to the interface and are defined as

\[ q_s = -\lambda \frac{\partial T}{\partial n} \]

\[ q_l = -\lambda \frac{\partial T}{\partial n} \]

\( \lambda \): Conductivity.

In the traditional numerical procedures, linear interpolation technique is usually used for expressing the first-order differential coefficient at the moving boundary. However, by using the present interpolation method, the physical accuracy is kept high. Not only the Dirichlet boundary condition but also the Neumann boundary condition is well calculated with the present interpolation method. The equation (3) gives the first-order differential coefficient.

4.2 Dynamic procedure

The chapter 3 shows the interpolation technique at the overlapped region in a flow. The possible masks for calculating equation (8) are shown in Figure 8. Basically the interpolations are one-side interpolation, so the masks should be set in each phase. In this case also, the second-order differential coefficients are examined for choosing suitable masks, which leads to physical high accuracy. The algorithm is shown in Figure 9.
5 SOLIDIFICATION AROUND CYLINDERS

As an example for all the present techniques, solidification around staggered-arranged cylinders is a good problem.

The application of this problem is the ice storage system. In a thermal energy storage system using latent heat of ice, the ice is formed around cooled cylinders when low temperature energy is charged and is melted when the stored low temperature energy is discharged. During the solidification process, ice-bridge is observed under some conditions. As the ice around one cylinder grows and becomes large, it meets the ice that is formed around the next cooled cylinder. Then the block ice includes more than one cylinder, which is called ice-bridge. The ice-bridge turns down the efficiency of ice formation because of its large thermal resistance. Furthermore, the ice-bridge might sometimes break the cylinders. To know the crisis limit of ice-bridge outbreak related to the conditions such as flow, temperature
and so on is important for designing thermal energy storage systems. In this problem, the final target is to know the margin for ice-bridge outbreak with numerical simulation.

The cooled cylinders are located in a uniform flow. The longitudinal axis of each cylinder is normal to the flow direction and the cylinders are staggered arranged. The Cartesian coordinate system is used for solving the main flow, while the boundary fitted coordinate system is used for solving the close regions to the cylinders. BFC is also used for solving the heat conduction in the ice that is formed around cylinders. The Cartesian coordinate system and each BFC make overlapped coordinate systems. The BFC around each cylinder deforms according to the ice growth, which enables to solve the heat conduction inside the ice and the heat transfer outside the ice simultaneously.

6 CONCLUSIONS

The dynamic interpolation technique is proposed. The present method is useful for exchanging the values on the overlapped region, which reduces the number of the unnecessary meshes. This method is also applied to both the Dirichlet and Neumann boundary conditions, and the numerical estimation of metrics in BFC as well. FDM is a repeated procedure of interpolations. The present method can be applied to general interpolation problems. It can also be applied to three-dimensional problems, however the possible masks should be examined in advance.

REFERENCES

