Nonlinear dynamics of multibody systems with flexible and rigid components and holonomic constraints

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Key words: multibody dynamics, holonomic constraints, integration schemes

Abstract. In this work we discuss an application of the finite element method to modeling of flexible multibody systems employing geometrically exact structural elements. Two different approaches to handle constraints, one based on the Lagrange multiplier procedure and another based on the use of release degrees of freedom, are examined in detail. The energy conserving time stepping scheme, which is proved to be well suited for integrating stiff differential equations, governing the motion of a single flexible link is appropriately modified and extended to nonlinear dynamics of multibody systems.
1 Introduction

Currently accepted trend for treating the flexible members rather than the traditional rigid links has raised the level of modeling sophistication and has widely opened the door to employing the finite element method in multibody dynamics. In that respect we depart from the previous practice (so-called floating or shadow frame of [8] or [22] e.g. see [21] and extend the nonlinear formulation of structural dynamics problems in [28] to multibody dynamics. The latter is set in a fixed inertia frame, which drastically simplifies the structure of the finite-element-based semi-discrete equations of motion in nonlinear structural dynamics resulting with a linear form of the inertia term and nonlinear form of the internal forces. Another key ingredient for development of such an approach to nonlinear structural dynamics problems are the so-called geometrically exact formulations of nonlinear structural theories (beams, plates and shells) capable of extracting the finite strain measures from the large overall motion, regardless of the size of displacements and rotations. The first formulations of this kind where provided for beams (e.g. see [26],[4],[15]), and subsequently extended to shells with drilling rotations (e.g. see [13]) and 3d solids with independent rotation field (e.g. see [14]). All these structural elements share the same configuration space, consisting of 3d displacement vectors and 3d finite rotation tensors, so that they can easily be combined within the same model of a complex mechanical system. The main difficulty remaining in such an approach to nonlinear structural dynamics, related to a non-vectorial character of 3d finite rotations (e.g. see [1] or [9]), is nowadays well under control, both in terms of a computationally convenient choice for rotation parameters (e.g. [17] or [16]) and in terms of a number of appropriate modifications (e.g. see [29] or [18]) of the standard time-stepping schemes for structural dynamics (e.g. Newmark or Wilson $\theta$, see [32]).

In this work we extend this approach to dynamics of flexible multibody systems. To that end, several novel features which have attracted the attention of the current research are thoroughly discussed:

(i) The first objective we have fixed in this work is to incorporate the holonomic constraints 1 in a multibody dynamics formulation relying on the geometrically exact structural theories. The constraints to be discussed encompass all the standard types of joints (e.g. revolute, prismatic, spherical, cylindrical, plane, etc.), as well as the constraints representing a rigid component in a flexible multibody system. The first approach tested is the classical one employing the method of Lagrange multipliers, which was previously proposed by Cardona et al. [6] for joint constraint and extended to rigid component constraint by Taylor and Chen [30] and Chen [7]. Development of the pertinent equations of motion of a constrained multibody system using the Lagrange multiplier method to handle constraints is quite straightforward, and can be carried out in a very systematic manner. However, this method does not necessarily lead to the most efficient implementation since the total number of unknowns is significantly increased by the presence of Lagrange multipliers. More

\[1\text{holonomic constraint is an equality constraint which is enforced on the configuration space}\]
importantly, as noted by Cardona and Geradin [5], the set of governing finite element semi-discrete equations of motion becomes of differential/algebraic type (with zero values of mass corresponding to constraint equations), which lead to unstable behavior of standard time-stepping schemes. A more efficient approach to handling constraints can be developed, as recently proposed by [19], where nonlinear kinematic relationship is employed to remove the constrained degrees of freedom from the global set of equations. The proposed procedure applies to all holonomic constraints (constraints on the configuration space) and includes all types of joints and rigid components.

(ii) Presence of constraints is not the only reason for which the standard time-stepping schemes are not well suited for flexible multibody systems, and often experience unstable behavior. The latter can also be the consequence of a large difference in stiffness of particular members of a flexible multibody system or even a large difference in stiffness of the same member in particular modes of deformations (e.g. bending vs. axial deformation), which leads to a set of so-called stiff differential equations characterized by a very large difference in maximal and minimal eigenvalues of the tangent operator. In that respect, a set of differential/algebraic equations, arising from handling a constrained multibody system by the Lagrange multiplier method, can be considered as an ultimate case of stiff system with some frequencies (the frequencies associated with zero mass) taking an infinite value. The rich mathematical literature dealing with time-integrations schemes for stiff equations (e.g. see [10]) proposes universal remedies that are often only partially satisfying, since the physical nature of a particular problem is ignored. Therefore, we prefer to follow a number of very recent works carried out in the computational mechanics community, which discovered that the robust time-stepping schemes for stiff equations in nonlinear structural dynamics can be derived by enforcing the conservation of salient motion properties, such as the total energy or angular momentum (see [27], [9], [23], [2] and [3], among others). The latter complements the earlier findings (e.g. see [12]) that enforcing energy conservation can also improve the performance of the standard time-stepping procedures, such as the Newmark scheme. Our second goal in this work is thus directed towards extending the domain of application of the energy-conserving methods to multibody dynamics problems.

The outline of the paper is as follows. In Section 2 we briefly recall the governing equations of nonlinear dynamics of a chosen model problem, a three-dimensional geometrically exact beam. Two different approaches to modeling the constraints in a flexible multibody system, based on the Lagrange multiplier technique and direct enforcement of nonlinear kinematic constraints are discussed in Section 3. In Section 4, we present the modification of the energy-conserving scheme of [27] pertaining to multibody systems. Several numerical simulations are presented in Section 5 in order to illustrate a very satisfying performance of the presented methodology. In section 6, we state some closing remarks.
beam. The model essentially represents a convenient reparametrization of the classical beam model of [24] and [25] proposed initially by [26] for straight beams and extended by [15] to space curved beams. The initial configuration of such a model is specified by the beam axis, a smooth curve which has a plane domain $A$ referred to as the cross section, attached at each point. The orientation of the cross section is specified by its exterior unit normal vector $t_{1,0}(s)$. Without loss of generality, we can choose that the cross section is initially normal to the beam axis

$$t_{1,0}(s) = \varphi_0'(s)$$

where $\varphi_0(s)$ is the position vector and $(\mathbf{x})' = \frac{\partial}{\partial t}(\mathbf{x})$. A convenient way to construct the base vector of this local Cartesian frame is simply by rotating the Euclidean base vectors, $e_i$, by rotation matrix $A_0$; i.e.

$$A_0 = t_{i,0} \otimes e_i \Rightarrow t_{i,0} = A_0 e_i$$

The key kinematic hypothesis of such a model is that each section is displaced with the beam axis remaining unchanged in shape. Therefore, the base vectors $t_{i,t}(s)$ still form a local Cartesian basis, which can be defined by an orthogonal matrix

$$A_t(s) = t_{i,t} \otimes e_i \Rightarrow t_{i,t} = A_t e_i$$

Due to shear deformation the plane section in the deformed configuration no longer remains orthogonal to the beam axis. In order to preclude extreme values of shear deformation we require that

$$\varphi_t'(s) \cdot t_{1,t}(s) > 0$$

In summary, the configuration space of the geometrically exact beam consists of one parameter family of 3d vectors and orthogonal matrices,

$$\mathcal{C} := \{ (\varphi_t, A_t) : [0, L] \times [0, T] \rightarrow \mathbb{R}^3 \times SO(3) ; \varphi_t' \cdot A_t e_1 > 0 \}$$

with time $t \in [0, T]$ as the evolution parameter. The central problem of computational structural dynamics is reduced to finding the time-history of the state variables by integrating their rate equations

$$\dot{\varphi}_t = v_t \quad \dot{A}_t = A_t \dot{W}_t \quad \dot{W}_t b = W_t \times b \quad \forall b \in \mathbb{R}^3$$

where $(\mathbf{x})' = \frac{\partial}{\partial t}(\mathbf{x})$ is the partial derivative with respect time. The computed values of state variables should satisfy the weak form of momentum balance equations, which can be written (e.g. see [26] or [15]) as

$$G(\varphi_t, A_t, v, w) := \int_0^L (v \cdot \dot{p}_t + w \cdot \dot{r}_t) ds + \int_0^L \{(v' - w \times \varphi_t') \cdot n_t + w' \cdot m_t\} ds + G_{ext} = 0$$
where \( w = \Lambda_t W \) and \( W_t \) and \( w_t \) are material and spatial form of angular velocity. In (7) above, \( p_t \) and \( r_t \) are linear and angular momenta, which can be written as

\[
p_t = A_p v_t \quad ; \quad r_t = i_p w_t
\]

where \( A_p \) is distributed mass per unit length of the reference configuration

\[
A_p = \int_A \rho \, dA
\]

whereas \( i_p \) is the moment of inertia

\[
i_p = \Lambda_t I_p \Lambda_t^T \quad ; \quad I_p = \int_A \sum_{i=2}^{3} \sum_{j=2}^{3} \rho \zeta_i \zeta_j [ (e_i \cdot e_j) I - e_i \otimes e_j ] dA
\]

In (7) we denoted as \( n_t \) and \( m_t \) the stress resultant force and moment which derive from the linear elastic constitutive law

\[
n_t := \Lambda_t N_t \quad ; \quad N_t = CE_t \quad ; \quad E_t = \Lambda_t^T \varphi' - \Lambda_0^T \varphi'_0
\]

and

\[
m_t := \Lambda_t M_t \quad ; \quad M_t = DK_t \quad ; \quad K_t = \Lambda^T \Lambda' - \Lambda^T_0 \Lambda_0'
\]

where \( C \) and \( D \) are constant \( 3 \times 3 \) matrices. If the local ortho-normal frame, \( t_i \), coincides with the principal axes of the cross-section, the constitutive matrices take a simple, diagonal forms,

\[
C = \text{diag}(EA, GA_2, GA_3) \quad ; \quad D = \text{diag}(GJ, EI_3, EI_3)
\]

where \( E \) is the Young modulus, \( G \) is the shear modulus, whereas \( A_i \) and \( I_i \) are section shear areas and moments of inertia.

It can easily be shown by using the Hamilton principle of least action that in the absence of external forces the total energy of the chosen model is conserved. To that end, first recall that in accordance with the constitutive model in (9) and (10) the strain energy can be written as a quadratic form in the chosen strain measures.

\[
W(E_t, K_t) = \frac{1}{2} (E_t \cdot CE_t + K_t \cdot DK_t)
\]

Second, due to choice of inertia reference frame the kinetic energy can also be written as a quadratic form in linear and angular velocities

\[
K(v_t, W_t) = \frac{1}{2} (v_t \cdot A_p v_t + W_t \cdot I_p W_t)
\]

Finally, by selecting the total energy as the corresponding Hamiltonian

\[
H(\varphi_t, \Lambda_t) = \int_0^L \{ K(v_t, W_t) + W(E_t, K_t) \} ds
\]
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It can readily be shown that its time derivative amounts to the weak form of equations of motion in (7) written for \( G_{\text{ext}} = 0 \). It thus follows that the Hamiltonian is conserved by the solution of (6) in that

\[
\frac{\partial}{\partial t} H(\bullet) = 0
\]

As shown subsequently, enforcing the energy conservation in the discrete approximation leads to a very robust time-stepping scheme.

2 Flexible multibody system with holonomic constraints

In this section we carry out the necessary development to make the presented model of geometrically exact beam applicable to modeling of multibody systems. More precisely, we discuss the joint constraints between flexible members modeled by the geometrically exact beams, as well as the connection of the flexible members to rigid components that might be integrated in the system. Two alternative methods to account for constraints are discussed, one using the Lagrange multipliers and the other employing nonlinear kinematic relations of master-slave type.

2.1 Joint constraints and Lagrange multiplier procedure

We consider a flexible member attached to linkage assembly by a joint forcing it to follow the motion of the mechanism. To make this idea more precise, the linkage assembly node is referred to as master and denoted with superscript "m", whereas the corresponding flexible member node is called slave, and denoted as "s".

In view of the configuration space in (5), the motions of the master and slave nodes are specified by their position vectors and orthogonal matrices

\[
\phi^m_t, \Lambda^m_t = [t^m_{1,t}, t^m_{2,t}, t^m_{3,t}] \quad \text{and} \quad \phi^s_t, \Lambda^s_t = [t^s_{1,t}, t^s_{2,t}, t^s_{3,t}]
\]

If the slave and master node are rigidly connected, then \( \phi^s_t = \phi^m_t \) and \( \Lambda^s_t = \Lambda^m_t \) and the motion of the slave node is identical to the motion of the master node.

However, if the slave and master node are connected by a joint constraint (e.g. revolute, prismatic, or spherical), then some motion components of the slave node need not necessarily follow the motion of the master node. Hence, these motion components have to be obtained as a part of the global solution procedure. It is important to note that by using the method of Lagrange multipliers to handle constraints each pair of slave-master nodes doubles the number of the corresponding motion components to be computed, as well as adding parameters for the multiplier itself. The role of the joint constraint is to enforce the equality of selected components. For example in the case of revolute joint constraint (see Figure 1) we can write that

\[
\begin{align*}
\phi^s_t - \phi^m_t &= 0 \\
t^m_{3,t} \cdot t^s_{1,t} &= 0 \\
t^m_{3,t} \cdot t^s_{2,t} &= 0
\end{align*}
\]
Imposing these constraints by the Lagrange multiplier procedure leads to a modified weak form in (7) which can be written as

$$G_{rev} = G + \lambda_v \frac{\partial}{\partial t} (\varphi^s_i - \varphi^m_i) + \lambda_{w1} \frac{\partial}{\partial t} (t^m_{3,t} \cdot t^s_{1,t}) + \lambda_{w2} \frac{\partial}{\partial t} (t^m_{3,t} \cdot t^s_{2,t}) = 0$$  (17)

where $\lambda_v, \lambda_{w1}$ and $\lambda_{w2}$ are the corresponding Lagrange multipliers. The latter takes a clear physical interpretation as the reactive forces and moments enforcing the constraints. Carrying on with this approach for other types of joints one can see that only two different types of elementary constraints are needed to build any standard joint constraint. These elementary constraints impose the equality of the corresponding components of translational and rotational motion, i.e

$$\varphi^s_{i,t} - \varphi^m_{i,t} = 0 ; \quad t^s_{i,t} \cdot t^m_{i,t} = 0$$  (18)

and each one introduces one Lagrange multiplier into the weak form. A complete list of elementary constraints that need to be imposed for several commonly used joints is given in [19].

### 2.2 Joint constraints and relative motion

In order to avoid an increase in number of global variables and resulting increase in computational effort which are inherent to the method of Lagrange multipliers, we propose an alternative approach to imposing the master-slave joint constraints, which introduces the relative motion. For example, going back to the revolute joint one can introduce the relative rotation angle $\theta^r$ (see Figure 1) orienting the slave triad with respect to the master one so that we can write

$$t^m_{1,t} \cdot t^s_{1,t} = \cos \theta^r_i ; \quad t^m_{2,t} \cdot t^s_{2,t} = \sin \theta^r_i$$  (19)

In view of constraints in (16), the last expression can thus be rewritten as

$$\varphi^s_i = \varphi^m_i ; \quad \Lambda^s_i = \Lambda^m_i \exp(\theta^r_i) ; \quad \theta^r_i = \begin{pmatrix} 0 \\ 0 \\ \theta^r_i \end{pmatrix}$$  (20)
By exploiting the last result the slave motion components can be completely eliminated from the global solution procedure, and recovered subsequently once this global computation, along with the local (element based) computation of the relative motion (see [19]), is completed. Such an approach does not increase the number of global equation and it leads to a very efficient computation.

Carrying on with this type of consideration, we can find out (see [19]) that all standard types of joint constraints can be described by a generalized joint featuring the relative motion vectors $d^r$ and $\theta^r$ such that

$$\varphi^s_t = \varphi^m_t + \Lambda^m_t d^r_t; \quad \Lambda^s_t = \Lambda^m_t \exp(\theta^r_t)$$

(21)

The particular choices of $d^r$ and $\theta^r$ for several most frequently used joints are given in [19].

Another type of kinematic constraint which we can also handle in a multibody system is the connection between a flexible and a rigid component (see [19]).

3 Energy conserving scheme

Applying the finite element methods to dynamics of flexible multibody systems often leads to a set of stiff differential equations; or, in the extreme case when constraints are enforced by the Lagrange multiplier method, to a set of algebraic/differential equations. It has been noted in [11] that many standard implicit time-stepping schemes, might experience a loss of stability for the case when very high (or infinite) frequencies of a set of stiff equations are excited. This kind of instability might occur even for implicit time-stepping schemes, the ones which are unconditionally stable in linear analysis. The latter indicates that one ought to define the notion of stability appropriate for nonlinear analysis, which is typically related to energy-conservation properties of the scheme. A number of schemes capable of enforcing the energy conservations have been proposed recently (e.g. see [27], [9], [23], [2] and [3], among others). In this work we choose the scheme proposed by [27], and devise the corresponding modification for the multibody dynamics problem on hand.

3.1 Energy conservation for multibody system with Lagrange multipliers

If the joint constraints are imposed by the Lagrange multiplier procedure, the governing set of differential equations is accompanied by the corresponding set of algebraic equations. We will keep the same notation as in the previous section for displacement and rotations of two nodes related by constraints as $\varphi^s_t, \varphi^m_t$ and $\Lambda^s_t, \Lambda^m_t$, respectively, although in this case these nodes should not be referred to as the slave and the master, since they both feature in the list of unknowns at the global level. This list also contains the corresponding number of Lagrange multiplier, i.e. the reactive forces enforcing the constraints. For example for a flexible multibody system with a revolute joint constraint described in Section 3.1, the weak form of the differential equation of motion can be written as

$$G(\varphi_t, \Lambda_t, v, w) + \lambda_{w1}(\dot{\varphi}^s_t - \dot{\varphi}^m_t) + \lambda_{w2}(\dot{t}^m_{3,t} \cdot \dot{t}^s_{1,t} + \dot{t}^m_{3,t} \cdot \dot{t}^s_{1,t}) + \lambda_{w2}(\dot{t}^m_{3,t} \cdot \dot{t}^s_{2,t} + \dot{t}^m_{3,t} \cdot \dot{t}^s_{2,t}) = 0$$

(22)
where $G(\cdot)$ is the weak form of the flexible components. The algebraic constraint equations are given as
\[
\mu_v \cdot (\varphi_i - \varphi_i^m) + \mu_{w1}(t_{3,t}^m \cdot t_{1,t}^1) + \mu_{w2}(t_{3,t}^m \cdot t_{2,t}^s) = 0
\]  
(23)

where $\mu_v = \hat{\lambda}_v$, $\mu_{w1} = \hat{\lambda}_{w1}$ and $\mu_{w2} = \hat{\lambda}_{w2}$

The mid-point approximation of the rate equations of the state variables, given in (23a) and (28), respectively, as well as the mid-point approximation of the weak form of the flexible part in (38), are constructed in the same way as already described for a single flexible component. For a multibody system it only remains to provide the corresponding mid-point approximation for the constraints. In particular, for the displacement constraint contribution to the weak form in (38) we can write
\[
\lambda_v \cdot (v_{n+\frac{1}{2}}^s - v_{n+\frac{1}{2}}^m) = \frac{1}{\Delta t} \lambda_v \cdot (\varphi_{n+1}^s - \varphi_{n+1}^m - (\varphi_n^s - \varphi_n^m))
\]  
(24)

where we used the result in (23a). Furthermore using the results in (3) and (28) we can write the mid-point approximation for the rotational constraint contributions to the weak form in (38) as
\[
\lambda_{w1}(t_{3,n+\frac{1}{2}}^m \cdot t_{1,n+\frac{1}{2}}^s + t_{3,n+\frac{1}{2}}^m \cdot t_{1,n+\frac{1}{2}}^s) = \lambda_{w1}(\Lambda_{n+\frac{1}{2}}^m e_3 \cdot \Lambda_{n+\frac{1}{2}}^s e_1 + \Lambda_{n+\frac{1}{2}}^m e_3 \cdot \Lambda_{n+\frac{1}{2}}^s e_1) \\
= \frac{\lambda_{w1}}{2\Delta t} [(\Lambda_{n+1}^m - \Lambda_n^m) e_3 \cdot (\Lambda_{n+1}^s + \Lambda_n^s) e_1 + (\Lambda_{n+1}^m + \Lambda_n^m) e_3 \cdot (\Lambda_{n+1}^s - \Lambda_n^s) e_1] \\
= \frac{\lambda_{w1}}{\Delta t} \left[ t_{3,n+1}^m \cdot t_{1,n+1}^s - t_{3,n}^m \cdot t_{1,n}^s \right]
\]  
(25)

We can also obtain by a similar consideration
\[
\lambda_{w2}(t_{3,n+\frac{1}{2}}^m \cdot t_{2,n+\frac{1}{2}}^s + t_{3,n+\frac{1}{2}}^m \cdot t_{2,n+\frac{1}{2}}^s) = \frac{\lambda_{w2}}{\Delta t} \left[ t_{3,n+1}^m \cdot t_{2,n+1}^s - t_{3,n}^m \cdot t_{2,n}^s \right]
\]  
(26)

We recall again that the results in (40), (41) and (42) provide the mid-point approximation to the constraint contribution to the weak form of the equations of motion for a multibody system. It is easy to see that by enforcing the constraint equations in (39) at $t_{n+1}$ and $t_n$ all the contributions in (40) to (42) will vanish, thus it will not perturb the energy conservation which was already enforced for the flexible part of the system.

### 3.2 Energy conservation for multibody system with master-slave constraints

For a multibody system with joint constraints modeled by nonlinear kinematic relationship accounting for relative motion between the master and slave nodes (see Section 3.3), the mid-point approximation given in (22a) and (22b) is used to compute the motion of each master node as well as the relative motion. In the presence of a revolute joint, the rate
equations for the motion of the slave node are obtained by differentiating \((21)\) with respect to time to get
\[
\dot{\varphi}_s^t = \dot{\varphi}_m^t; \quad \dot{\Lambda}_s^t = \dot{\Lambda}_m^t \Lambda^r_t + \Lambda^m_t \dot{\Lambda}^r_t
\] (27)
The mid-point approximation of the rate equation in \((43)\) then leads to
\[
\varphi_{s_n + 1}^n - \varphi_{s_n}^n = \frac{\varphi_{m_{n + 1}}^n - \varphi_{m_n}^n}{u^m_n}
\] (28)
where \(u^m_n\) are obtained by the global mid-point method based solution procedure. It follows from \((44)\) above that we can write \(u = u^m\), if the displacement constraint in \((20)\) are enforced both at \(t_{n+1}\) and \(t_n\)
\[
\varphi_{s_{n + 1}}^n = \varphi_{s_n}^n; \quad \varphi^s = \varphi_n^s
\] (29)
By the same token, the mid-point approximation of the rate equation for the rotational motion of the slave node, leads to
\[
\Lambda_{s_{n + 1}}^s - \Lambda_{s_n}^s = (\Lambda_{m_{n + 1}}^m - \Lambda_n^m)\Lambda_r^r + \Lambda_{m_{n + 1}}^m (\Lambda_r^r n_{n + 1} - \Lambda_n^r)
\]
\[
= \Lambda_{m_{n + 1}}^m \Lambda_r^r - \Lambda_n^m \Lambda_n^r
\] (30)
Again, we note that the last result is true only if the rotational constraint in \((20)\) is enforced both at \(t_n\) and \(t_{n+1}\), i.e.
\[
\Lambda_{s_{n + 1}}^s = \Lambda_{m_{n + 1}}^m \Lambda_r^r; \quad \Lambda_{s_n}^s = \Lambda_n^m \Lambda_n^r
\] (31)
In view of the result in \((27)\), it follows from \((47)\) that
\[
cay[\varphi] \Lambda_s^s = cay[\varphi^m] \Lambda_m^m cay[\varphi^r] \Lambda_{mT}^T \Lambda_{s}^s \Lambda_n^r
\] (32)
Finally, by making use of the Cayley transform property indicated in \((26)\), we can recast the last expression in the following form
\[
\varphi^s = cay^{-1}[cay[\varphi^m] cay[\Lambda_n^m \varphi^r]]
\] (33)
Thus, the energy conservation for a multibody system with nonlinear master-slave joint constraints can be demonstrated by following the same line of arguments as for a single flexible components, leading to
\[
\sum_{a=1}^{n_{np}} G_a(\varphi_{n+\frac{1}{2}}, \Lambda_{n+\frac{1}{2}}, u^a, \varphi^a) = H_{n+1} - H_n
\] (34)
where, for any slave node, \(s = a\), relationships in \((44)\) and \((49)\) should be exploited to simplify \((50)\) accordingly.
4 Numerical Examples

Two numerical simulations are presented in this section in order to illustrate a very satisfying performance of the proposed methodology for dynamics of flexible multibody systems with joint constraints and rigid links. All the computations are performed by an enhanced version of the computer program FEAP (e.g. see [31] and [32]) for the description of a basic version).

4.1 Spatial slider-crank mechanism

The multibody system we study in this example (see Figure 2) is a three-dimensional version of the standard slider-crank mechanism. The system consists of two rigid links, initially placed in vertical position, joined by a flexible connecting rod, which is initially horizontal (see Figure 2).

Another flexible link is attached to the center of the connecting rod at one end and to a sliding support at another end. The mechanical and geometric properties of the flexible link and connecting rod are chosen the same with axial and shear stiffness $EA = GA = 10^6$, bending and torsional stiffness $EI = GJ = 10^5$ and distributed mass $A_p = 1$ and inertia $I = diag(20, 10, 10)$.

The connecting rod is attached to the rigid links by spherical joints, whereas the flexible link and the connecting rod are rigidly connected between them. In this manner the sliding motion of the flexible link can be transferred to the motion of the rigid links around their support point. In order not to prevent truely three-dimensional motion, both support points of the rigid links are provided by the spherical joints.

We have first studied the motion transfer form a forced sliding motion under a horizontally applied force $F = 1000$, with a given sinusoidal time variation for the first $2sec$ of motion, $f(t) = \sin \pi t, t \in [0, 2]$. After that time the force is removed and the system undergoes free vibrations. The finite element model used in the analysis consists of 12 two-node elements, 6 for the connecting rod and 6 for the flexible link. The constraints due to the spherical joints and rigid components are handled by master-slave logic explained in Section 3.2.

Figure 2.- Spatial slider-crank mechanism: mechanical and geometric characteristics

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\[ for additional example, see [20] \]
The energy-conserving scheme is used to carry out the computation, with the time step $\Delta t = 10^{-2}\text{sec}$. As shown in Figure 3, where we give the plot of the total energy versus time, the energy-conserving scheme fulfills its role in enforcing the salient feature of the corresponding continuum problem and preserving the total energy in the free-vibration phase, $t > 2\text{sec}$. In Figure 4, we present the displacement components at node A, the mid-point of the flexible link, and at node B, the point where the link is attached to the connecting rod. It can be seen that during the period of forced vibration (first 2 sec.) and initial period of free vibration, the motion remains predominantly two-dimensional with essentially vanishing value of the out-of-vertical-plane displacement component. At approximately 5 sec of motion, the out-of-plane motion components start participating in the motion, increasing their values towards the end of the time interval of interest. Namely, due to the presence of spherical joints the rigid links can change the axis of rotation, and that is precisely what happens at the later stage. The deformed shapes in Figure 5 confirm this finding. It is important to note, as indicated in Figure 3, the total energy remains conserved through all the different stages of the free-vibration phase.

In the second part of the analysis we study the influence of the operating speed on the modeling assumption of the slider-crank mechanism as being either flexible or rigid. To that end we consider the second load case consisting of angular velocities applied at support point of one rigid link taking the value $\omega = 5, 10, 20\text{rad/sec}$ In order to be able to better represent different deformation modes, we have also refined the finite element model by increasing the total number of two-node beam elements to 15 for the flexible link and 15 for connecting rod. The analysis is carried out by the energy-conserving scheme.
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Figure 5- Spatial slider-crank mechanism: Deformed shapes at different stages of motion using the time step \( \Delta t = 10^{-2} \). The results computed for the time history of the bending deformations of the flexible link at point A for different operating speeds are presented in Figure 6. They clearly illustrate that simplifying modeling assumption on neglecting the mechanism deformations becomes questionable at higher operating speeds.

4.2 Closed loop multibody system

In this example we study the forced and free vibrations of a closed loop mechanism. The mechanism takes a form of a closed frame of square shape, with each side equal to 10. A couple of revolute joints and a spherical joint are placed at the corners of the frame. See Figure 7. Three sides of frame are considered rigid, whereas the fourth is composed of a flexible component (whose length is equal to 6) attached to two rigid ends each of length equal to 2.
The mechanical chosen characteristics of the flexible component are: Young’s modulus $E = 10^3$ and Poisson’s ratio $\nu = 0.25$. The flexible and all rigid components are chosen of unit cross-section and unit mass density.

The mechanism is initially placed in a vertical plane and it is acted upon at its upper right corner by an out-of-plane force, which peaks to 10 at 0.5 sec, and goes to zero at 1 sec. Thereafter, the mechanism undergoes free vibrations. We have explored two modeling assumptions regarding the flexible component: The first is constructed with 8 two-node beam elements, whereas in the second model the flexible link is presented by 3d solid elements. The FE mesh of the second model is constructed by replacing each beam of the first model by 4 solid elements, i.e. $8 \times 4 = 32$ elements for the whole component.

The analysis is performed by the presented energy-conserving scheme for the first 30 sec of motion, by using the time step $\Delta t = 0.25$. The deformed shapes computed with two FE models at every 10 sec of motion are presented in Figure 8.

We can see from Figure 8 that the motion is truly three-dimensional, with the considerable values of the induced deformation. The deformed shapes computed by two models are somewhat similar in nature. Far less agreement between the results obtained by two models is found when comparing the corresponding time history of the total energy, which
is given in Figure 9.

Figure 9.- Closed-loop mechanism : Total energy time history for flexible model

However, one can first note from Figure 9 that the presented energy conserving scheme is applicable to both models, in that it ensures the energy conservation in the free vibration phase for both models.

The difference in the total energy computed values between the two models can be explained by the different manners to account for deformation and the resulting strain energy between beam and solid model (e.g. no warping of the cross-section and reduced integration for beam). Indeed, we have repeated the simulation by selecting the higher value of Young’s modulus, $E = 10^6$, and thus significantly reducing the role played by the strain energy. The total energy computed by the two models presented in Figure 10 shows much better correlation between the two models.

Figure 10.- Closed-loop mechanism: Total Energy time history for rigid model

The latter is due to the fact that in this case the deformation remains negligible and that both models, either beam or 3d solid, have the same behavior representing the assembly of rigid bodies.

Acknowledgments

The financial supports from ABONDEMENT ANVAR and BFA program of French-Algerian cooperation are gratefully acknowledged.
5 Conclusions

The major thrust in this work was directed towards enhancing the capabilities of the finite element models in dealing with the nonlinear dynamics of flexible multibody systems with rigid links and joint constraints.

To that end, the most important modification of the standard finite element technology that we discussed pertain to handling the constraints between the flexible components. This can be accomplished either by introducing the method of Lagrange multipliers or by employing the nonlinear kinematic constraints and relative motions governed by the constraints. Although the Lagrange multipliers method is more general it is also less efficient, considering that the number of global equations should be increased to accommodate the multipliers. The method of relative motion is limited to holonomic constraints, but it can be rendered very efficient since it requires only the local (element-based) computations related to relative motion.

Nonlinear dynamics of multibody systems is often described by the stiff differential equations, arising either from the presence of the Lagrange multipliers or large difference in stiffness among different components of a multibody system, or even different deformation modes, which disqualifies the use of a great number of standard time-stepping schemes. It is shown that the proposed modification of the energy conserving scheme, which is constructed for both methods for handling the constraints, is quite capable of dealing successfully with the applications of this kind.

We have also shown by means of numerical examples that at high operating speeds the mechanism deformations can play a significant role, and that, in general, it should not be ignored. As illustrated by the last example, the finite element methodology greatly simplify building of the models for a flexible multibody systems of any desired level of complexity.

References


