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A POSITIVE MULTIGRID SCHEME FOR COMPUTATIONS WITH TWO-EQUATION TURBULENCE MODELS

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Abstract. *An efficient and robust method has been designed for the simulation of the compressible Navier-Stokes equations with two-equations turbulence models. The compressible Navier-Stokes equations are integrated explicitly to steady state and a novel approach is used to maintain the positivity of the turbulent quantities. The novel approach is based on a conservative estimate of the characteristics of the turbulent equations leading to a restriction of the local time step determined by the residual. As steady state is approached the residual and the restriction of the time step is reduced and hence the asymptotic rate of convergence is not affected. Multigrid is used to accelerate the convergence and a similar approach is used when adding the corrections to guarantee a positive scheme. A higher order restriction operator improves the robustness and the performance of the scheme. It is demonstrated that the higher order restriction can improve the rate of convergence, especially for 'bad' grids with large stretchings and locally large variations in grid size. Numerical results are presented for a model problem and on a RAE2822 airfoil.*

1 INTRODUCTION

The design of efficient and robust numerical algorithms is essential for the simulation of compressible turbulent flows, especially for engineering-type applications. The main objective of this paper is to demonstrate new approaches to gain increased robustness in the iteration of an explicit Navier-Stokes solver coupled with a two-equation turbulence model and accelerated with a multigrid method.

It is well known that positivity of the discrete turbulent variables at any time during iteration is crucial. Unphysical negative values rapidly lead to instability and positivity must therefore be ensured. Various methods exist to achieve this. Ad hoc methods like enforcing the positivity by taking the absolute values of the turbulent quantities often prevents the convergence or cause divergence of the iterative process.

A more classical approach for avoiding the generation of spurious oscillations is to impose TVD-like conditions[10]. Explicit TVD schemes for transport equations have been considered by Harten[8] and implicit schemes by Yee *et al.*[16]. When transport equations for turbulent variables are present additional constraints need to be made to the source term. Jongen & Marx have developed TVD-like criteria for a general advection-diffusion equation with a source term. Both explicit and implicit time integration are considered. They consider separately the convective, viscous and source term of the equation and give a positivity criterion for the advection term that is a weaker requirement of a TVD criterion. The central viscous term can not increase the total variation. The negative part of the source term is treated implicitly and the positive part explicitly. A criterion is given on how the linearization of the negative terms must be chosen to maintain positivity. The criterion for the source term often restricts the local time step more than does the stability criterion. In addition, Jongen & Marx consider only the update of the variables in the time integration. They give no criteria of how to maintain positivity of the variables in a multigrid procedure where the variables are updated from prolongated coarse grid corrections.

In this paper an alternative, novel approach is suggested to guarantee positive turbulent variables. The approach is based on an estimate of the spectral radius of the complete turbulent equations and produces an underrelaxation of the local time step based on the residual. The underrelaxation is only active in those regions where the residual is large compared to the positive dependent variable and where the sign of the residual is such that the variable is decreased. The underrelaxation does not affect the asymptotic rate of convergence since the relaxation is only active initially when the solution is far from being converged. This results in a method that is not required to fulfil the TVD criterion for the convection, nor from the positivity requirements setup for the source term to have a positive solution in the iterative time stepping. The only requirement is that a stable time step is chosen.

An explicit Runge-Kutta finite-volume method is used to solve the compressible, Reynolds averaged Navier-Stokes equation. Local time steps are used based on a stability analysis of the convective and viscous terms. A point-implicit treatment of the turbulent source terms is chosen to guarantee stability in the iteration to steady state of the turbulence.

The convergence is further accelerated by using FAS multigrid. To ensure positive variables when the prolongated coarse grid corrections are added, a similar procedure as in the time stepping is introduced. An additional parameter is introduced to scale the underrelaxation, the size of this parameter is investigated numerically.

A higher order restriction operator is used which is basically the transpose of a linear prolongation operator. For some reason, to the authors knowledge this restriction operator has not been used before for CFD calculations. It is demonstrated in this paper that a higher order operator is as important as a higher order (linear) prolongation operator used on a common basis and that a lower order operator may prevent convergence. It is shown that especially when the grid is ‘bad’ with local large variations and when the grid is stretched with high cell aspect ratios the combination of the higher order transfer operators are beneficial.

Numerical results are presented for a model problem and for a 2D RAE2822 airfoil (case 10). The computations are all started from free stream, with or without multigrid. The novel approach is not restricted to a specific turbulence model, results are presented for the k–ε equations with the near wall treatment by Chien[2]. The focus will not be on the predicted results, robustness and the rate of convergence are considered instead.

2 CONSTRUCTION OF POSITIVE SCHEME

Finite-volume discretization schemes for a general transport equation are considered integrated to a steady state. The purpose is to investigate the stability and positivity requirements in the time integration. A positive discretization of the advection may be obtained by a TVD like criterion, see e.g. Jongen & Marx[10] where it also shown that the diffusion does not increase the total variation. A positivity requirement of the source term is also given leading to a restriction of the local time step. This requirement is usually more restrictive than the requirement for stability. Below an alternative approach is suggested in which the stability requirement only is required. The positivity is ensured by a an underrelaxation of the time step in the initial stage of the computation.

2.1 1-D model problem

To investigate the positivity requirements a simple model problem is considered. A scalar, linear semi-discrete model problem with diffusion and a source term is investigated

$$\frac{dq_i}{dt} = R(q) = \frac{\nu}{\Delta x^2}(q_{i+1} - 2q_i + q_{i-1}) + aq_i \qquad (1)$$

where q represents a positive, transported quantity (e.g. k or ε), ν represents a positive viscosity, Δx the step length and a is a constant multiplying the source term. The advection term is left out simply to reduce the complexity of the algebra.

$a > 0$ leads to an exponential growth without the presence of the diffusion. The source term may be positive locally in regions where there is a turbulent growth. Positive source terms do not restrict the local time step, they are integrated explicitly and will eventually be balanced by the negative source terms or the diffusion when the turbulence stops to grow. Positive source terms do no possess a threat to turn q negative since q is growing, therefore only negative source terms $a < 0$ are considered which may push q towards zero.

The diffusion term is integrated explicitly, as in the CFD calculation. Without the source term explicit Euler forward integration gives the following stability bound on the time step:

$$\frac{q_i^{n+1}-q_i^n}{\Delta t_v} = \frac{\nu}{\Delta x^2}(q_{i+1}^n - 2q_i^n + q_{i-1}^n) \Rightarrow \Delta t_v \le \frac{\Delta x^2}{2\nu} \tag{2}$$

which is easily verified by a von Neumann analysis. With the negative source term present and explicit integration leads to

$$\frac{q_i^{n+1}-q_i^n}{\Delta t} = \frac{\nu}{\Delta x^2}(q_{i+1}^n - 2q_i^n + q_{i-1}^n) + aq_i^n \Rightarrow \Delta t \le \frac{1}{\frac{2\nu}{\Delta x^2}-\frac{a}{2}} = \frac{\Delta t_v}{1-\frac{a\Delta t_v}{2}} \tag{3}$$

for a stable time step provided Δt_v is chosen as its maximum. The source term restricts the time step by an additional term in the denominator. To guarantee positivity in the integration (3) the following constraint must be made to the time step:

$$Positivity\colon \Delta t \le \frac{1}{\frac{2\nu}{\Delta x^2}-a} = \frac{\Delta t_v}{1-a\Delta t_v} \tag{4}$$

When the source term is dominating the positivity requirement restricts the time step by a factor of two.

Jongen & Marx[10] propose to treat negative source terms point implicitly leading to

$$\begin{aligned} &\frac{q_i^{n+1}-q_i^n}{\Delta t} = \frac{\nu}{\Delta x^2}(q_{i+1}^n - 2q_i^n + q_{i-1}^n) + aq_i^{n+1} \Rightarrow \\ &\left(\frac{1}{\Delta t}-a\right)(q_i^{n+1}-q_i^n) = \frac{q_i^{n+1}-q_i^n}{\Delta t^*} = \frac{\nu}{\Delta x^2}(q_{i+1}^n - 2q_i^n + q_{i-1}^n) + aq_i^n \end{aligned} \tag{5}$$

which increases the stability limit for the time step Δt:

$$\Delta t \le \frac{1}{\frac{2\nu}{\Delta x^2}+\frac{a}{2}} \tag{6}$$

However, the time step used in the time integration is the efficient time step $\Delta t^* = \Delta t/(1-a\Delta t)$ which is restricted by the same stability and positivity criteria as the explicit scheme, (3) and (4). In either case of explicit or point implicit time integration, the time step has to be restricted due to the source term. In the case of this scalar, one-stage integration problem explicit and point-implicit time integration become identical.

It is not attractive to restrict the time step due to a positivity criterion more restrictive than the stability criterion since it will decrease the rate of convergence. Ideally, the positivity requirement should be used initially only when the residual is large and may cause the solution to turn negative. When the residual becomes smaller the stability limit should determine the size of the time step. Below such an approach is presented.

If the residual is assumed to be negative and large and the residual is dominated by the source term the model problem may be approximated as:

$$\frac{dq_i}{dt} = R(q) \approx aq_i \tag{7}$$

Hence the size of the source term may be estimated as

$$a \approx \frac{R(q)}{q} \tag{8}$$

According to the expression for the time step (3) in the model problem a positive time step Δt_p may be obtained as

$$\Delta t_p \le \frac{\Delta t}{1 - \Delta t \; min\left(\frac{R(q)}{q}, 0\right)} \tag{9}$$

which ensures that the time step is determined by the stability requirement only as the residual becomes small compared to the solution. Δt is then determined from reasons of stability only, the additional term in the denominator in (9) guarantees positivity. For a point implicit approach, the time step may be determined from the explicit terms (advection and diffusion) which for the example above would lead to

$$\Delta t_p \le \frac{\Delta t_v}{1 - \Delta t_v \; min\left(\frac{R(q)}{q}, \frac{a}{2}\right)} \tag{10}$$

where Δt_v is given in (2). Note that, compared to (3), the time step is reduced only when the residual is negative, i.e. when the solution is decreasing. When it increases the time step is determined from stability reasons only and the expressions (9) and (10) are identical. Note that the expressions (9) and (10) are not identical for negative residuals. (9) is more restrictive for large negative residuals. The expressions (9) and (10) are chosen because of their similar appearance, other choices of time steps are possible that guarantees positivity.

For a general *m*-stage Runge-Kutta scheme the condition (9) may be written

$$\Delta t_p = \frac{\Delta t}{1 - \frac{\Delta t}{CFL_v} \; min\left(2\frac{R(q)}{q}, 0\right)} \tag{11}$$

for an explicit integration of the viscous and source term where Δt is a stable time step based on the spectral radius of the viscous and source term

$$\Delta t = \frac{CFL_v}{\frac{4\nu}{\Delta x^2} - a} \tag{12}$$

and where CFL_v is the absolute value of the intersection of the stability region with the negative real axis in the complex plane.

For a point implicit approach the equation (10) for a general *m*-stage Runge-Kutta scheme may be written

$$\Delta t_p = \frac{\Delta t_v}{1 - \frac{\Delta t_v}{CFL_v} min\left(2\frac{R(q)}{q}, a\right)} \tag{13}$$

where the time step is based on the spectral radius of the viscous term only:

$$\Delta t_v = \frac{CFL_v}{\frac{4\nu}{\Delta x^2}} \tag{14}$$

The equations (11) and (13) are valid for explicit and point implicit time integration respectively and guarantee a positive and stable time integration for an *m*-stage Runge-Kutta scheme applied to the model problem (1) provided the Runge-Kutta coefficients $0 \le \alpha_{1 \le i \le m} \le 1$.

For vanishing source terms ($a \to 0$) Shu[13] gives the necessary CFL-conditions for arbitrary *m*-stage Runge-Kutta to be TVD.

2.2 The *k*-ε equations

The positive time steps for the model problem above may easily be extended to two-equation turbulence models As an example the *k*-ε equations are used, but any two-equation turbulence model may be used. The equations for the Chien *k*-ε[2] is:

$$\begin{aligned} \frac{D}{Dt}\rho k &= \nabla \cdot \left(\mu + \frac{\mu_T}{\sigma_k}\right)\nabla k + P - \rho\varepsilon - 2\frac{\mu k}{y^2} \\ \frac{D}{Dt}\rho\varepsilon &= \nabla \cdot \left(\mu + \frac{\mu_T}{\sigma_\varepsilon}\right)\nabla\varepsilon + (C_{\varepsilon 1}P - C_{\varepsilon 2} f_2 \rho\varepsilon)\frac{\varepsilon}{k} - 2\frac{\mu\varepsilon}{y^2}e^{-\frac{1}{2}y^+} \end{aligned} \tag{15}$$

where

$$P = S^2\mu_T - \frac{2}{3}\rho k D, \mu_T = C_\mu f_\mu \rho\frac{k^2}{\varepsilon} \tag{16}$$

The negative source terms are treated point implicitly and must be accounted for so that a stable time step is obtained. The complete Jacobian of the negative source S^- terms may be used

$$S^- = \begin{pmatrix} -\rho\varepsilon - 2\frac{\mu k}{y^2} \\ (-C_{\varepsilon 2} f_2 \rho\varepsilon)\frac{\varepsilon}{k} - 2\frac{\mu\varepsilon}{y^2} e^{-\frac{1}{2}y^+} \end{pmatrix}, \frac{\partial}{\partial q}(S^-) = \begin{bmatrix} -\frac{2\mu}{\rho y^2} & -1 \\ C_{\varepsilon 2} f_2 \frac{\varepsilon^2}{k^2} & -2C_{\varepsilon 2} f_2 \frac{\varepsilon}{k} - 2\frac{\mu\varepsilon}{y^2} e^{-\frac{1}{2}y^+} \end{bmatrix} \quad (17)$$

or an estimate of the spectral radius ρ^- of it may be used. In the computations presented in this paper ρ^- is used and approximated as

$$\rho^- = min\left(-2\frac{\varepsilon}{k} - 2\frac{\mu}{y^2\rho}, -1.5C_{\varepsilon 2} f_2 \frac{\varepsilon}{k} - 2\frac{\mu}{y^2\rho} e^{-y^+/2}\right) \quad (18)$$

Using the spectral radius or the complete Jacobian in the determination of the local time step has most often a small influence on the rate of convergence.

The time step used in the computations is

$$\Delta t^* = \frac{\Delta t}{1 - \frac{\Delta t}{CFL} min\left(2\frac{R(\rho k)}{\rho k}, 2\frac{R(\rho\varepsilon)}{\rho\varepsilon}, \rho^-\right)} \quad (19)$$

where Δt is determined from a stability analysis of the convective and diffusive terms of the mean flow, Rizzi *et al*[14]. Hence Δt is used to update the mean flow, Δt^* is used for the turbulence. A common time step instead of two separate time steps was chosen for the two turbulent equations since they are closely coupled.

3 MULTIGRID

To accelerate the convergence of the explicit Runge-Kutta time integration with local time steps, FAS multigrid is used. The multigrid is applied to both the mean flow as well as to the turbulence. Below the positive update in the time integration is extended to the update of the corrections in multigrid. A higher order restriction operator is also presented.

3.1 Positivity

The dependent variables are updated in the Runge-Kutta time integration but also in the multigrid procedure where a prolongated correction is added to the variables, see e.g. Hackbush[7]. Positivity of the turbulence is then not only of concern in the time integration but also when the multigrid corrections are added.

The corrections are added as:

$$q^{n+1} = q^n + \Delta q \quad (20)$$

where Δq is the correction. There is nothing in the update of the corrections (20) that prevents the turbulent variables to turn negative.

By considering the corrections as residuals the approach from the time integration may be extended. The following modified expression for adding the corrections ensures positivity:

$$q^{n+1} = q^n + \frac{\Delta q}{1 - \beta min\left(\frac{\Delta q}{q^n}, 0\right)} \tag{21}$$

Note that the correction is only reduced when it is negative and that the reduction becomes small as soon as the correction becomes small compared to the solution. An additional parameter $\beta > 1$ has been introduced to keep the turbulence sufficiently far away from zero. In the computations presented below it will be shown that the value of β has very small influence on the rate of convergence provided it is chosen big enough to prevent initial divergence and blow up. β should for most cases be chosen $\beta \geq 10$ for reasons of robustness, in the computations $\beta = 100$ is usually used.

For the *k*-ε equations the following expression is used:

$$q^{n+1} = q^n + \frac{\Delta q}{1 - \beta min\left(\frac{\Delta k}{k^n}, \frac{\Delta \varepsilon}{\varepsilon^n}, 0\right)} \tag{22}$$

where q is either k or ε and Δq the correction Δk or $\Delta\varepsilon$ respectively. Note that k and ε are equally restricted not to destroy their close coupling.

3.2 A higher order restriction operator

In multigrid the residuals of a fine grid is restricted to coarser grids to form a forcing function on the right hand side, Hackbush[7]. A semi-discrete equation discretized by a finite-volume method is denoted

$$V_l \frac{dq_l}{dt} = R(q_l) + d_l \tag{23}$$

where V_l is the volume of a cell and where subscript $l \geq 1$ denotes the grid level, $l = 1$ being the finest. d_l is the forcing function restricted from finer levels, $d_1 = 0$. For more details see e.g. Rizzi *et al.*[14].

A restriction operator commonly used is the transpose of the simplest prolongation operator, the injection. These restriction and prolongation operators are illustrated in Figure 1 in two space dimensions. Note that the restriction is for the residuals only, to restrict the unknowns a factor of 1/4 multiply fine grid unknowns. This is a result of the finite-volume discretization.

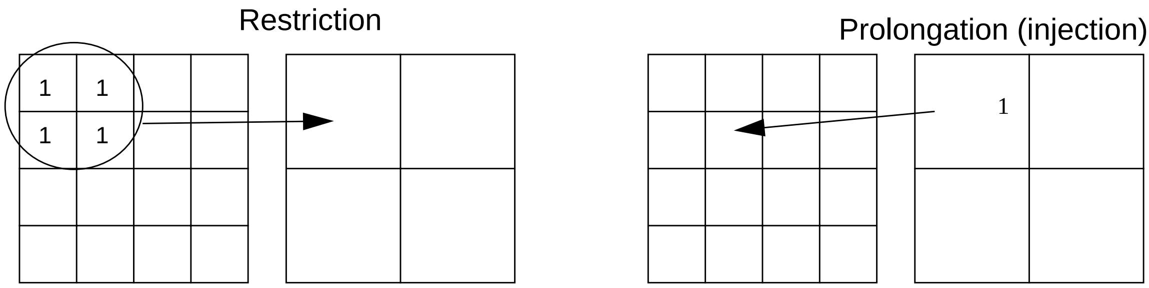

Figure 1: Prolongation operator of lowest order (injection) in two dimensions, restriction as its transpose.

The injection prolongation is usually not accurate enough so a linear prolongation operator may be used instead. The linear prolongation may be used to form a higher order restriction by taking the transpose of the prolongation, Figure 2.

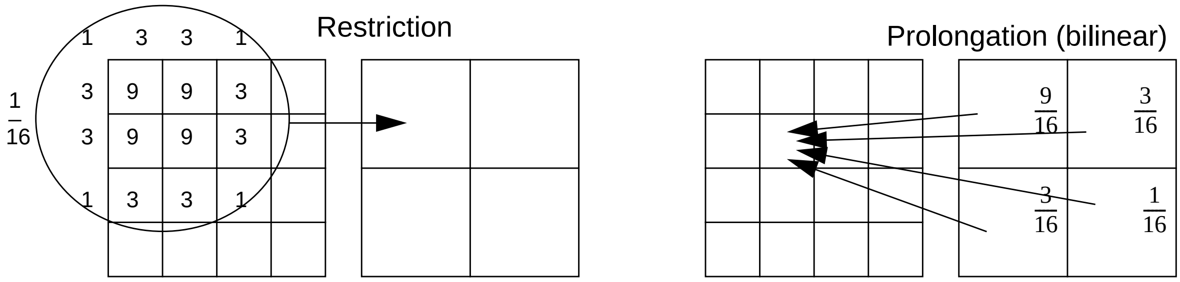

Figure 2: Linear prolongation in two dimensions and restriction as its transpose.

The higher order restriction needs values outside the boundaries, it needs boundary conditions on the residual. At solid walls the boundary condition

$$R_0 = -R_1 \tag{24}$$

is used for all residuals. At external boundaries the same boundary condition is used although the boundary condition on these boundaries have negligible impact on the rate of convergence. For more details about the multigrid operators as well as analysis for model problem, see Eliasson[3], Eliasson *et al.*[4].

Note that the higher order restriction should not be used for the variables. Although it should be possible to apply any consistent operator in theory, in practice the large variations and non-linearities of the turbulent variables often cause problem for this wider operator and hence the lower order, compact operator is used for the variables.

It will be demonstrated that in some cases the lower order operators prevent convergence.

4 SOLUTION TECHNIQUE

The equations are integrated to steady state using an explicit 5-stage Runge-Kutta scheme with point-implicit treatment of the turbulent negative source terms described above. The Runge-Kutta coefficients are:

$$\alpha_1 = 0.0814,\ \alpha_2 = 0.1906,\ \alpha_3 = 0.342,\ \alpha_4 = 0.574,\ \alpha_5 = 1 \tag{25}$$

The viscous terms and the source terms are calculated in the first stage only. This scheme provides optimum damping for both central and upwind schemes[3], the CFL number in the computations is CFL=1.5.

The mean flow convective terms as well as the turbulent convective terms are discretized using a second order accurate symmetric TVD scheme[11]. Second order accuracy is obtained by using a van Leer limiter for the mean flow applied to the characteristics and a minmod limiter for the turbulence.

An entropy fix is used to prevent unphysical solution, the eigenvalues of the mean flow and turbulence are not allowed to be less than 5% of the spectral radius $|\bar{u}| + c$.

On coarser grids the mean flow and the turbulent convective terms are reduced to first order accuracy. The entropy fix is increased to 20% and the turbulent production is set to zero, i.e. it is assumed to be constant from the fine grid.

5 NUMERICAL RESULTS

5.1 Model equation

To investigate the time discretization and the different time step restrictions a model equation is used[15] The model problem varies in time only and is basically the k-ε equations without convection and near-wall damping:

$$\frac{dq}{dt} = R(q) = V(q) + S(q) = -\alpha\begin{bmatrix} k-1 \\ \varepsilon - 1 \end{bmatrix} + \begin{bmatrix} P_k - \varepsilon \\ (C_{\varepsilon 1}P_k - C_{\varepsilon 2}\varepsilon)\dfrac{\varepsilon}{k} \end{bmatrix} \tag{26}$$

where $q = [k, \varepsilon]^T$ and $V(q)$ represents schematically the viscous terms where α is positive and related to $\mu / \Delta x^2$. $S(q)$ is the source term, $P_k = C_m k^2 / \varepsilon$ is the turbulent production and $C_{\varepsilon 1} = 1.45, C_{\varepsilon 2} = 1.9$.

The spectral radius to the negative terms of $S(q)$ may be found from the Jacobian

$$J(q) = \frac{\partial S^-}{\partial q} = \frac{\partial}{\partial q}\begin{bmatrix} -\varepsilon \\ -C_{\varepsilon 2}\dfrac{\varepsilon^2}{k} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ C_{\varepsilon 2}\dfrac{\varepsilon^2}{k^2} & -2C_{\varepsilon 2}\dfrac{\varepsilon}{k} \end{bmatrix} \tag{27}$$

to be

$$\rho^{-} = -(C_{\varepsilon 2} + \sqrt{C_{\varepsilon 2}(C_{\varepsilon 2} - 1)})\frac{\varepsilon}{k} \qquad (28)$$

One stage explicit Euler time integration is made with three different limitations to the time step. In the first approach a strictly positive time step is chosen:

$$\Delta t = \frac{CFL_v}{\alpha - 2\rho^{-}} \qquad (29)$$

The second time step used is a time step based on stability requirements only

$$\Delta t = \frac{CFL_v}{\alpha - \rho^{-}} \qquad (30)$$

and may therefore initially turn k or ε negative. An ‘ad hoc’ approach is then used by setting the negative value to 1% of the initial value.

In the third approach the new way of restricting the time step is used in accordance with Equation (11)

$$\Delta t = \frac{CFL_v}{\alpha - \rho^{-} - min(R(k)/k, R(\varepsilon)/\varepsilon, 0)} \qquad (31)$$

or, equivalently, with the notation in Equation (11)

$$\Delta t_p = \frac{\Delta t}{1 - \frac{\Delta t}{CFL_v} min(R(k)/k, R(\varepsilon)/\varepsilon, 0)} \qquad (32)$$

where Δt is defined in Equation (30).

In Figures 3-4 two computations with different values of the parameters are visualized. The results are compared against an exact solution in time. Since a steady state problem is used the focus is the number of iterations required to reach the asymptotic steady state values. The computed results are therefore plotted using the same time step although different time steps (29)-(31) were used in the computations. As can be seen, for both cases the new proposed time integration reaches the asymptotic value fastest.

Other ‘ad hoc’ resetting of negative values can result in both worse and better behaviour than demonstrated below. The example demonstrates the danger with that procedure and the increased robustness with the new proposed time step.

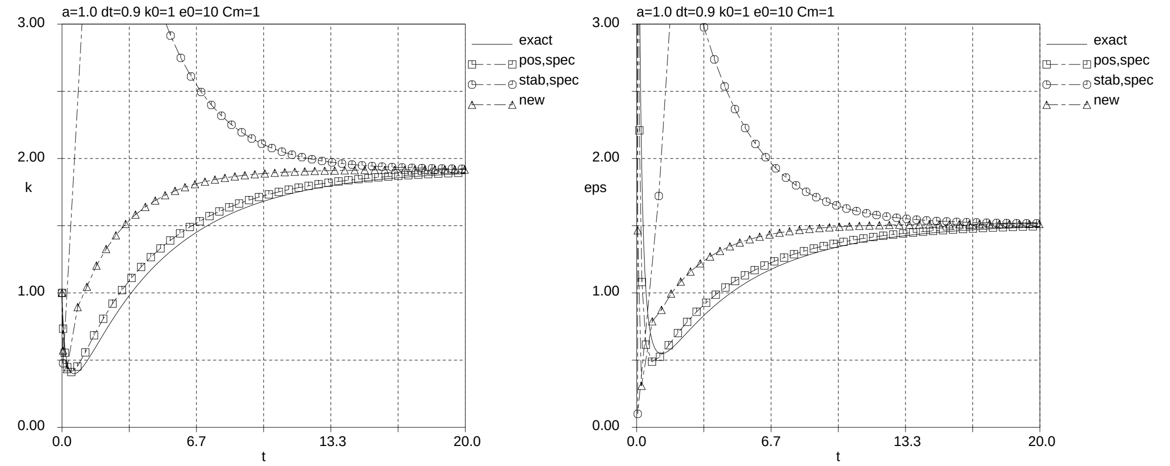

Figure 3: Model example with $k_0 = 1, \varepsilon_0 = 10, \alpha = 1$. Time evolution of k (left) and ε (right). ___ exact solution, □ positive time step, O stable time step with 'ad hoc' resetting, Δ new proposed time step.

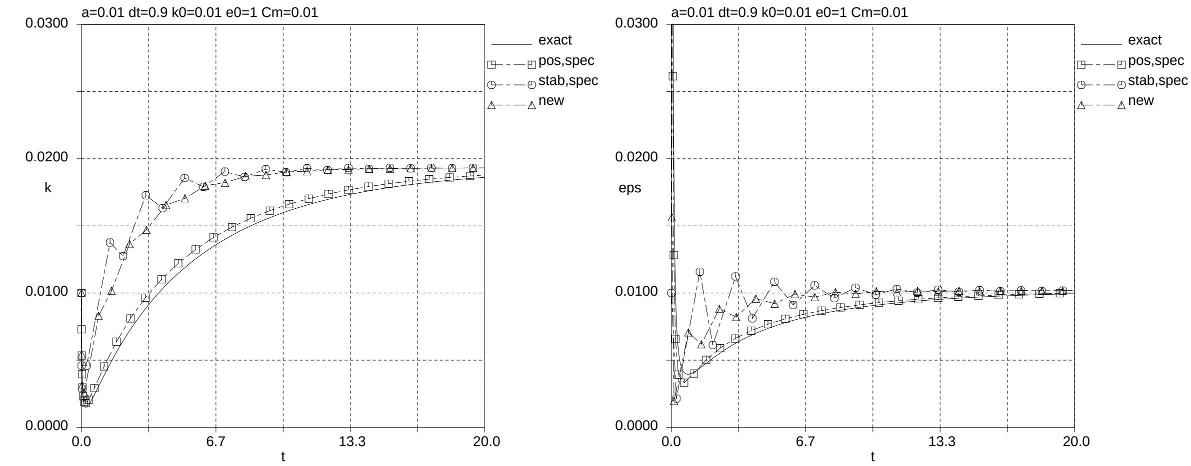

Figure 4: Model example with $k_0 = 0.01, \varepsilon_0 = 10, \alpha = 0.01$. Time evolution of k (left) and ε (right). ___ exact solution, □ positive time step, O stable time step with 'ad hoc' resetting, Δ new proposed time step.

5.2 RAE2822 airfoil, Case10.

A demanding test case is the transonic flow over a RAE2822 airfoil. This case has been subject for several earlier investigations in Euroval[5] and ECARP[6], the particular case investigated here is denoted *Case 10* and involves shock boundary layer separation. In this paper only aspects concerning robustness and rate of convergence are considered although the case is very interesting concerning the prediction of shock location, leading edge suction peak, skin friction etc.

The flow conditions are

$$M = 0.754, \alpha = 2.57°, Re = 6.2\times10^6. \quad (33)$$

The trailing edge of the airfoil is sharp and hence a C-type of grid was used with a size of 257×65 nodes, 65 nodes from the wall to the outer boundary located about 10 chords away. A close up of the grid can be seen Figure 5. The distance to the second layer of nodes from the wall varies from 0.3×10^{-5} at the leading edge to 1×10^{-5} at the trailing edge. Transition is specified at 3% of the chord on both upper and lower side.

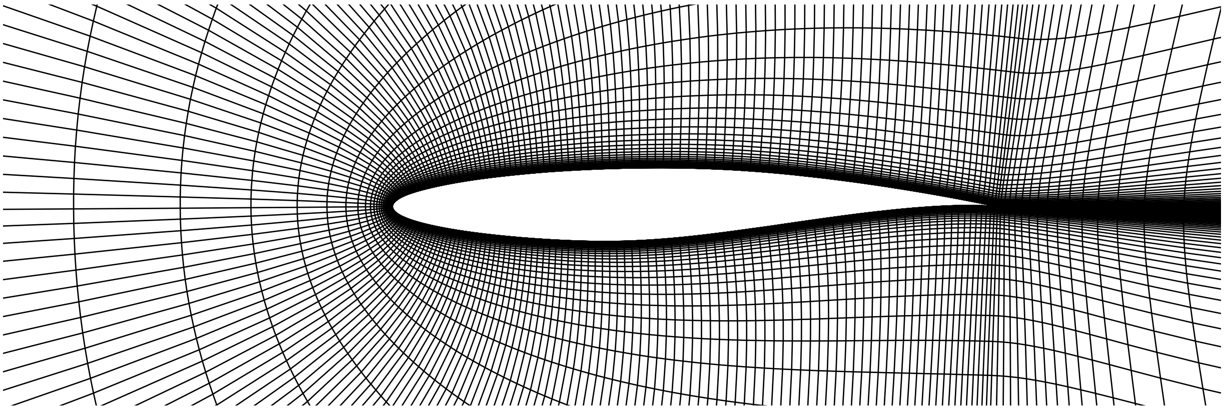

Figure 5: C-type of grid around the RAE2822 airfoil used in the calculation with 257×65 nodes.

Unless otherwise stated 3 levels of multigrid with V-cycles is used in the computations as well as a higher order prolongation operator, higher order restriction operator, the restriction damping factor $\beta = 100$ from Equation (22), everything initialized to free stream values and a CFL number of $CFL = 1.5$. No smoothing of the multigrid corrections is applied. The rate of convergence is presented for the density and turbulent kinetic energy residuals.

Without the new proposed time step (19) and correction relaxation (22) the computations starting from free stream values diverge immediately. The option to the new approach would be to drastically reduce the CFL number initially, finding a better initial solution, leave out multigrid initially etc. All of these option imply additional time and cost to find a converged solution which makes the proposed improvements superior. It is difficult though, to quantify the benefits from it since the options involve different procedures in getting a solution.

The rate of converge using 1-4 multigrid levels can be seen in Figure 6.

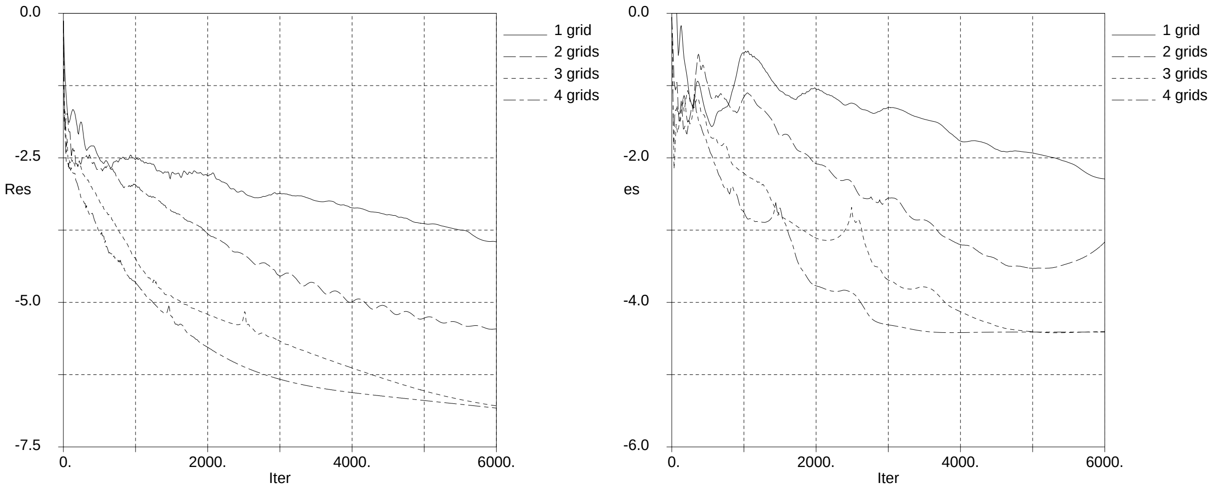

Figure 6: Rate of convergence using 1-4 multigrid levels. Left: density residual, right: k residual.

As can be seen the speed up is considerable. The residual of the turbulent kinetic energy drops to a constant level from which there is no further decrease.

The ambition has not been to obtain the best possible convergence but to study delta effects on the rate of convergence. The rate of convergence can be further improved by optimizing CFL numbers, using residual smoothing, increasing numerical diffusion etc. Although the density residual is reduced more than 7 orders of magnitude it is not required for engineering accuracy, e.g. the integrated forces converge to its steady state values within 0.1% in about 1500 iterations using 4 grid levels, see Figure 7:

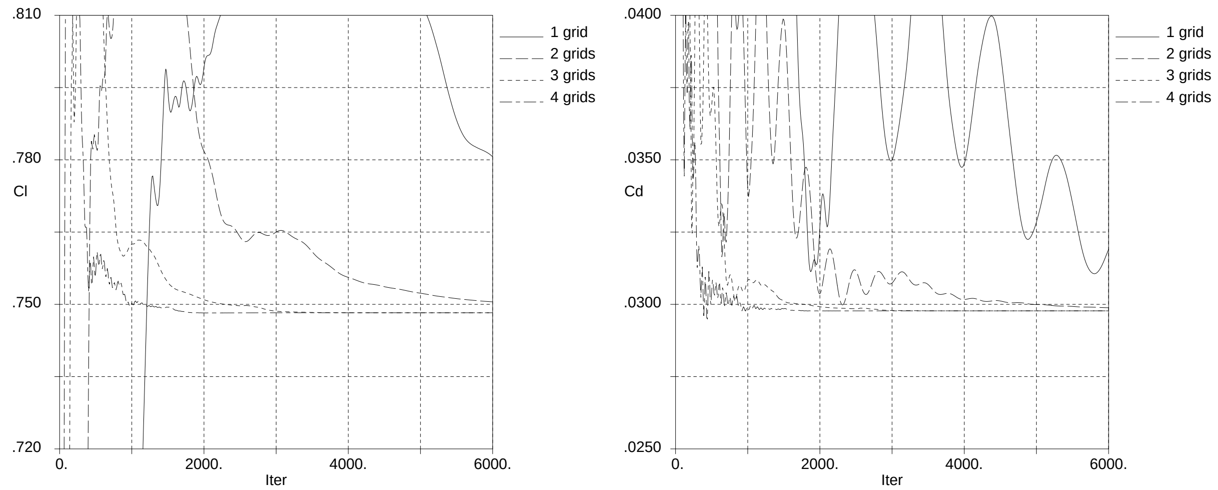

Figure 7: Rate of convergence using 1-4 multigrid levels. Left: lift coefficient Cl, right: drag coefficient, Cd

The choice of the parameter β in the update of the multigrid corrections in Equation (22) is investigated in Figure 8 using 4 grid levels. The rate of convergence is practically unchanged for different β provided β is chosen large enough. Divergence is obtained for $\beta = 2$ which is obviously a too low value for this case.

In Figure 9 different restriction operators are used. With a lower order restriction operator the residuals show a somewhat oscillatory behaviour using 3 grid levels although convergence is obtained. With 4 grid levels there is no longer convergence using 4 grid levels.

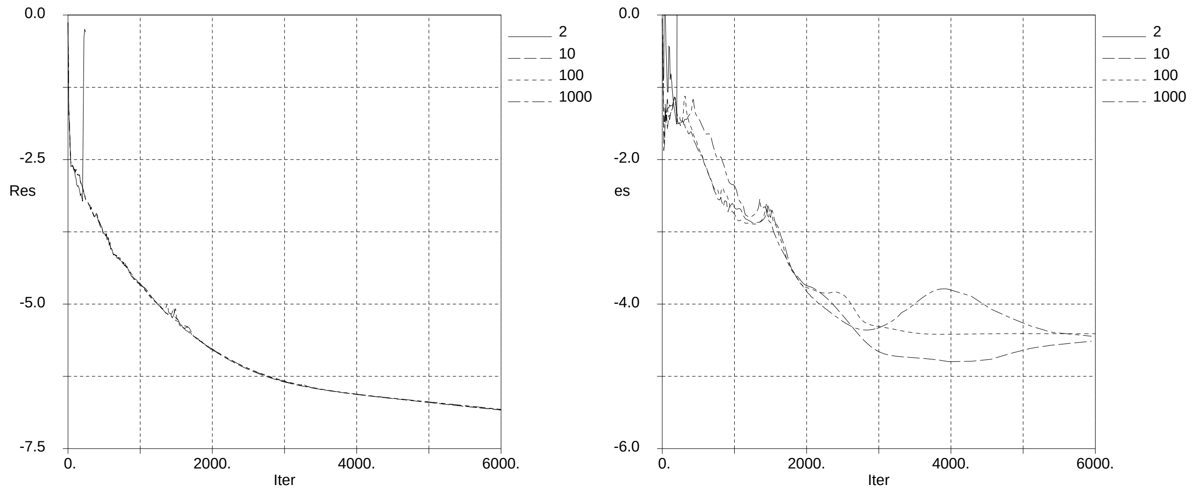

Figure 8: Rate of convergence using 4 grids and variation of $\beta = 2, 10, 100, 1000$. Left: density residual, right: k residual.

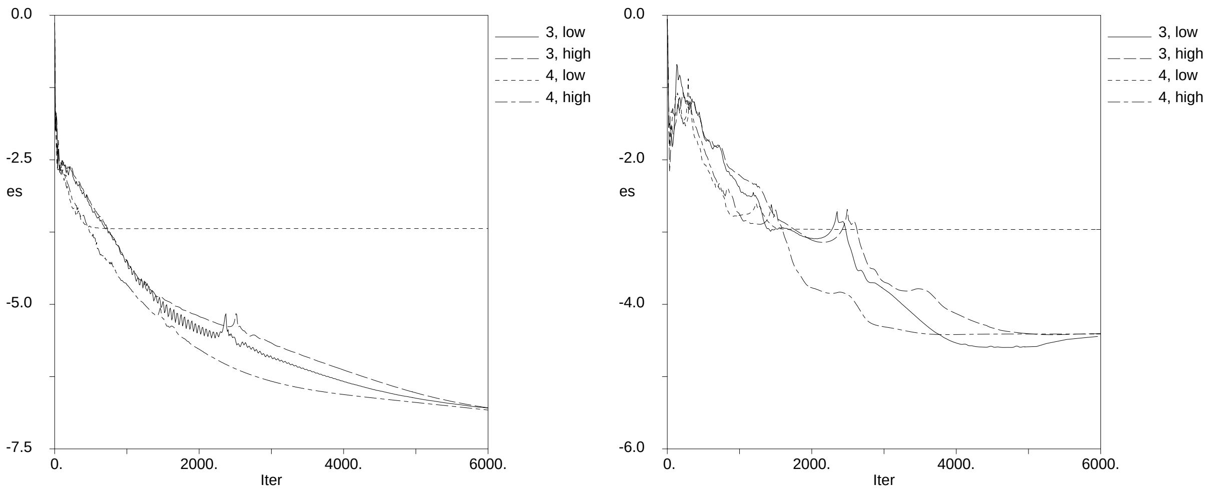

Figure 9: Rate of convergence using 3 and 4 multigrid levels with lower and higher order restriction. Left: density residual, right: k residual.

In general multigrid is often less robust applied to the turbulent equations due to their stiffness. In many industrial codes multigrid is therefore only applied to the mean flow equations. In Figure 10 the difference between using multigrid and single grid for the turbulence is plotted. As can be seen the rate of convergence for the density is not affected much whereas the kinetic turbulent kinetic energy converges much slower. This also indicates that the evolution of the mean flow and the turbulence is fairly decoupled.

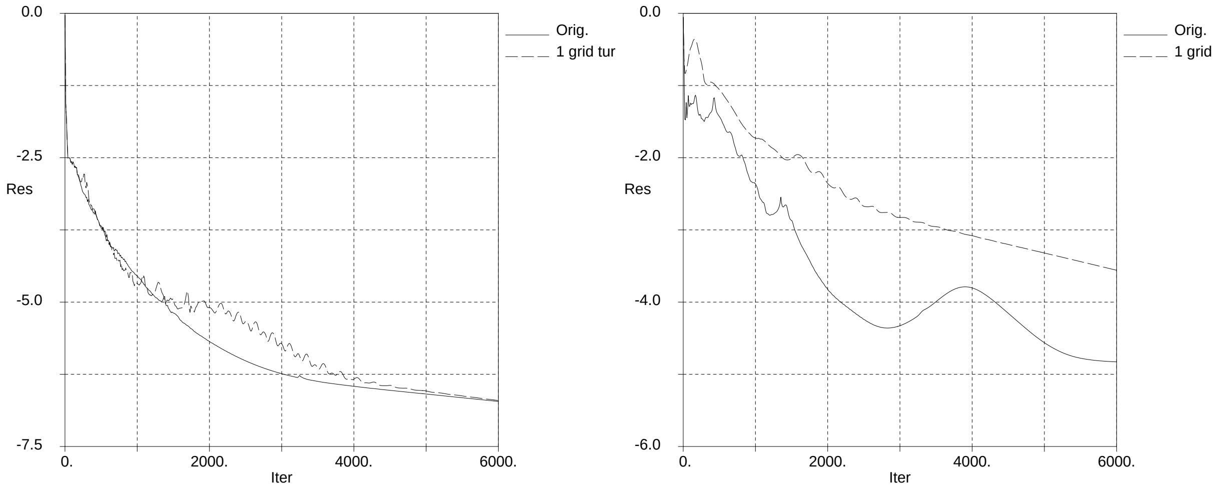

Figure 10: Rate of convergence with 4 multigrid levels. Comparison with multigrid applied to all equations (solid line) and multigrid applied only to the mean flow equations (dashed line). $\beta = 1000$. Left: density residual, right: k residual.

5.3 Multigrid applied to 'bad grids'

In practical application for complex geometry in three dimensions it not possible to avoid 'bad grids' with large stretchings, grid singularity, high cell skewness etc. These distortions often lead to a degradation in the quality of the solution and in the rate of convergence. In this section the influence of different multigrid transfer operators on grids with large and sudden stretchings to the rate of convergence is carried out.

To simulate a complex 3D grid with large and sudden stretchings, a less complex problem was chosen. Every third point has been removed in the 2D grid of the RAE2822 original grid of 257×65 points. Points are removed in the I-direction, in the J-direction and in both directions. The grids can be seen in Figure 11. The grid sizes and the sizes of the coarser grids in multigrid can be seen in Table 1. Note that for the grid with points removed in both directions only 3 multigrid levels are possible and hence used. Also note that removing points in one directions implies that the grid is made coarser in only one direction between grid levels 3 and 4.

The flow conditions and the numerical conditions are the same as in the previous Section 5.2.

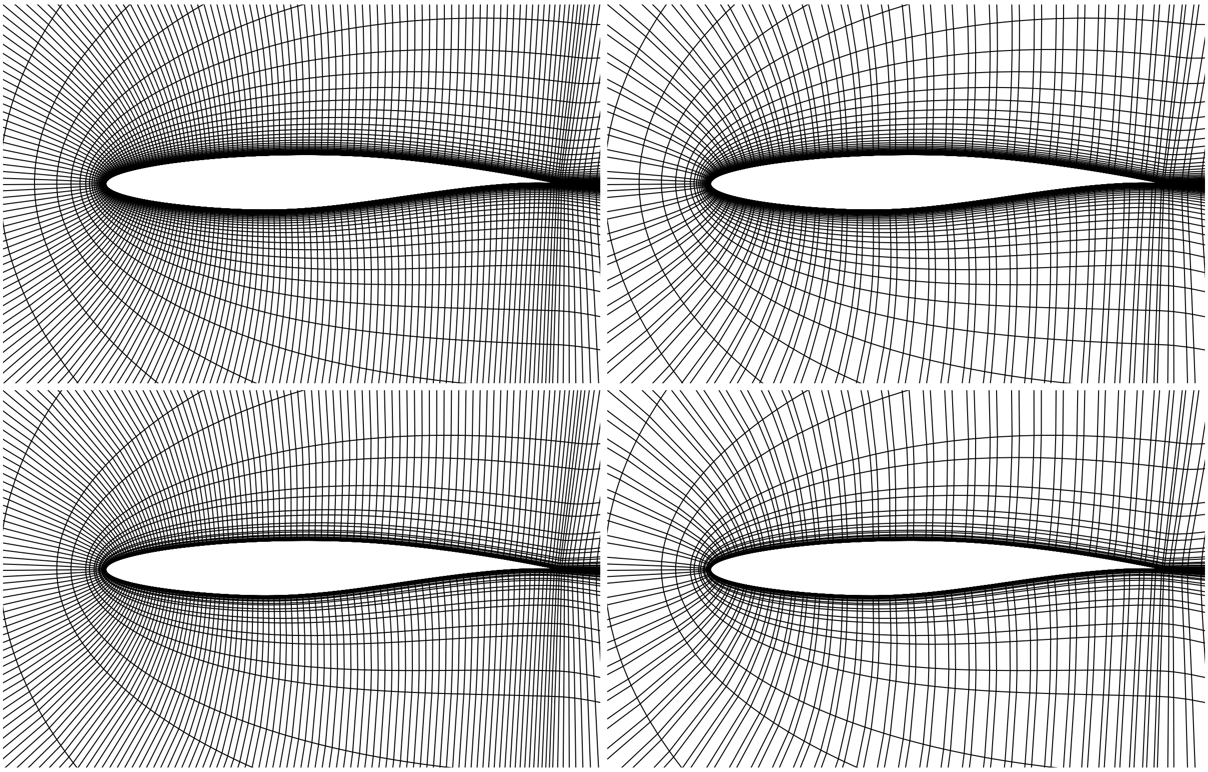

Figure 11: Upper left: original grid. Upper right: grid with every third point removed in I. Lower left: grid with every third point removed in J. Lower right: grid with every third point removed in I and J.

Grid	Size
1	257×65
2	129×33
3	65×17
4	33×9

Original

Grid	Size
1	173×65
2	87×33
3	44×17
4	44×9

Removal in I

Grid	Size
1	257×45
2	129×23
3	65×12
4	33×12

Removal in J

Grid	Size
1	173×45
2	87×23
3	44×12

Removal in I,J

Table 1. Size of the original grids and the grids with every third point removed. Sizes of the coarser grids used in multigrid are displayed as well (Grid 2). J is the direction normal to the airfoil to the outer boundary

The rate of convergence using the lower and higher order multigrid operators in Figures 1 and 2 for the four grids in Figure 11 can be seen in Figure 12-15.

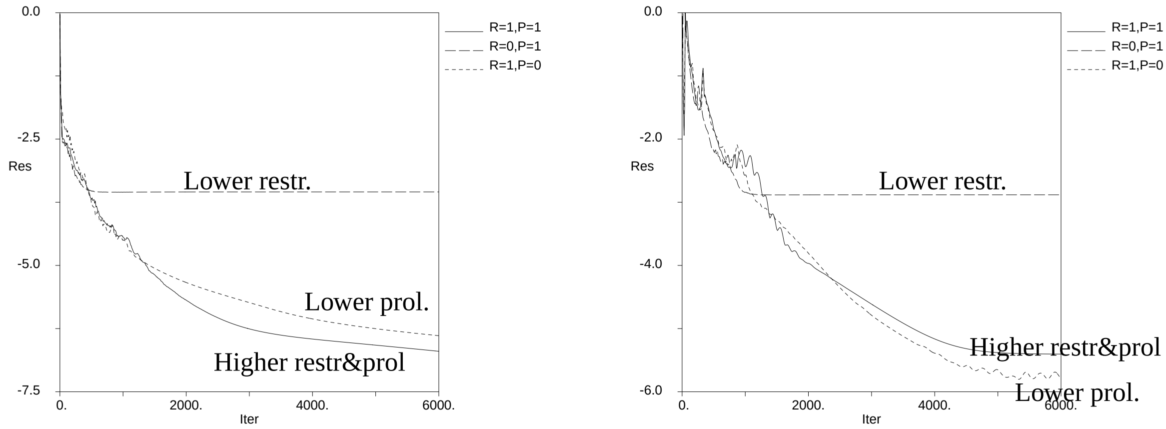

Figure 12: Convergence rate, original grid with 4 multigrid levels. Left: density residual, right: k residual.

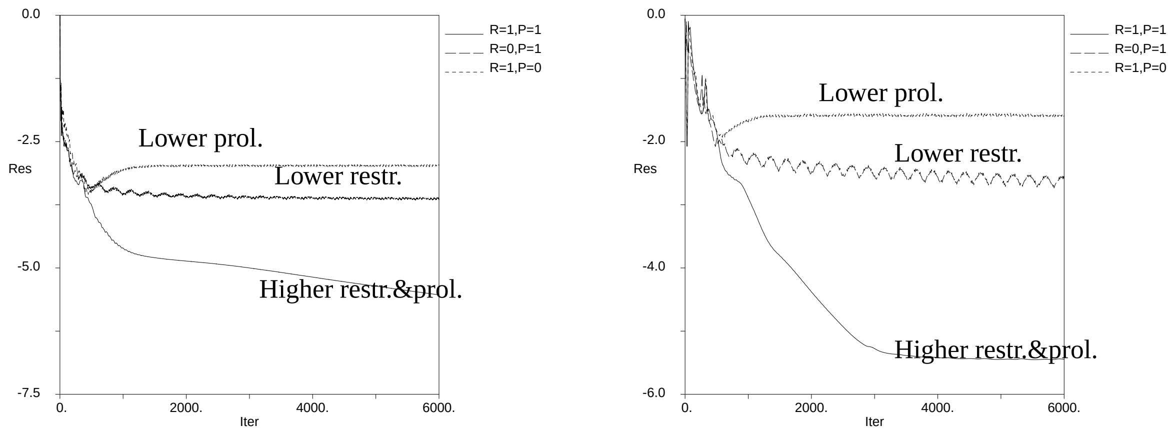

Figure 13: Convergence rate with points removed in I, 4 multigrid levels. Left: density residual, right: k residual.

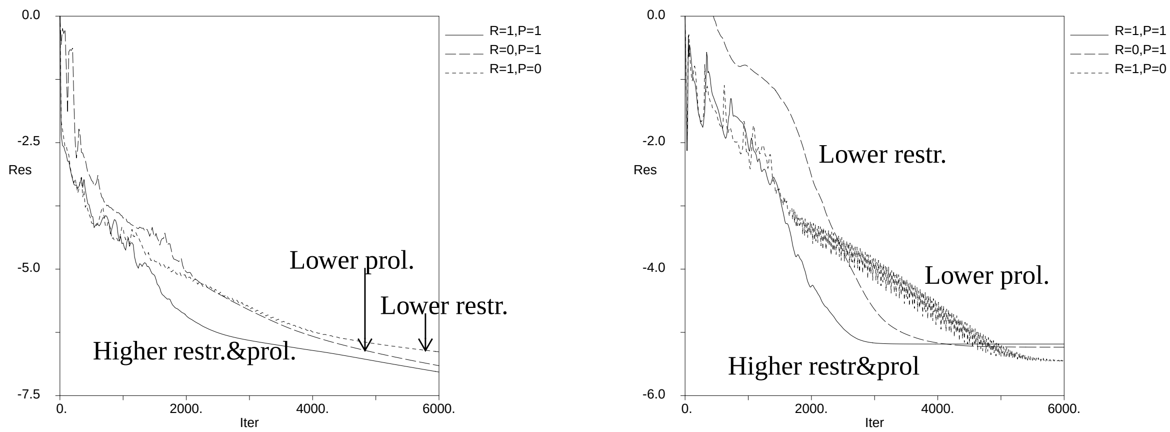

Figure 14: Convergence rate with points removed in J, 4 multigrid levels. Left: density residual, right: k residual.

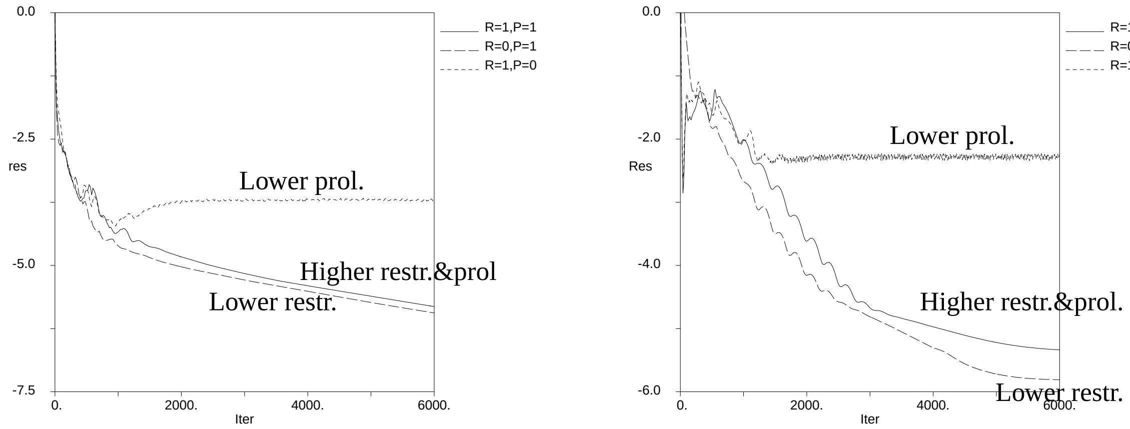

Figure 15: Convergence rate with points removed in I and J, 3 multigrid levels. Left: density residual, right: k residual.

It is evident that the combination of higher order prolongation and restriction operators give the best convergence as well as convergence curves with least oscillations. Lower order restriction or prolongation may prevent convergence in some cases. The benefit from the higher order operators is most evident when removing points in the I-direction and hence increasing the local cell aspect ratio. The combination of lower order restriction and prolongation operators give divergence and can usually not be used for problems with more than two multigrid levels.

6 SUMMARY AND CONCLUSIONS

A new way of keeping the turbulent dependent variables positive in the time integration is proposed and evaluated. The local time step in the steady state time integration is restricted by the local residual in such a way that the solution cannot turn negative. The restriction is only initial and the asymptotic rate of convergence is not effected. A similar procedure is used in the update of the variables from the multigrid corrections. The benefits of the proposed method is demonstrated on a model example and on the RAE2822 airfoil at transonic speed.

A higher order restriction operator is demonstrated to give improved rate of convergence in some cases. Especially when the grid has large local variations and the cell aspect ratio is high it is shown that only with the combination of linear prolongation and higher order restriction convergence is obtained.

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