NONLINEAR VISCOELASTIC CONSTITUTIVE MODELLING OF A CONTINUUM

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Abstract. This paper discusses a new continuum formulation for a non-linear viscoelastic solid. The physical problem is idealised using the generalised Maxwell model. The model is based upon the multiplicative decomposition of the deformation gradient into elastic and viscous components. The isochoric component is further split into long term and viscous contributions. Nonlinear rate equations for the internal variables will be proposed. Enforcing the condition of isochoric deformation of the viscous contribution ensures a non-linear relationship for the non-equilibrium stresses. In recognition of the need for future computer implementation of the constitutive equations, the internal variables are updated using incremental kinematics. This leads to a simple update of the non-equilibrium stresses which resemble the radial return algorithms used in Von Mises plasticity. The incremental update of the viscous component requires the solution of six non-linear equations at each increment. The constitutive equations will be derived from typical strain energy functions, with a dimensionless proportionality factor relating the long term and viscous expressions. Dynamic applications of viscoelastic material behaviour are demonstrated to illustrate the accuracy of the formulations.
1. INTRODUCTION

The proceeding work extends the well-established linear rheological models to the finite strain regime. The deformation gradient, the main quantity used in finite deformation analysis, is split into the product of the elastic and permanent viscous contributions as proposed by Lubliner. The final rate of relaxation of the viscous deformation is only dependent on the choice of linear evolution equations. However, in finite strain problems the permanent strains satisfy non-linear constraints and their evolution cannot be described by a linear equation. Simo, Holzapfel & Reitter, Simo and Holzapfel have all proposed linear differential equations to simulate the stress relaxation in the viscous component of the linear rheological mechanical model. The assumption of splitting the elastic and non-equilibrium stresses into volumetric and deviatoric responses respectively by the multiplicative procedure is discussed in the work by Ogden, Simo et al. and Simo & Taylor.

The work proposed herein is based on the model introduced by Lubliner. In common with this work the deformation gradient is decomposed into elastic and permanent components. The Generalised Maxwell model consisting of springs and a number of viscous dashpots allows the long term and inelastic contributions to be treated separately. The strains in the each viscous dashpot constitute an internal variable and are required in order to fully describe the state of the system. In contrast with the model by Lubliner, non-linear forms of the rate equation governing the evolution of the internal variables will be proposed. In preparation for future computer implementation of the constitutive equations, the volumetric and deviatoric stresses are updated in incremental form. The resulting viscoelastic formulation can be seen as a particular case of the general large strain elasto-viscoplastic formulation proposed by Simo in which the elastic region has been collapsed to the zero.

The paper discusses stresses in the Lagrangian configuration and is therefore suitable for material exhibiting anisotropy. Unfortunately, at each increment the solution of a cubic equation is required in order to update the right Cauchy viscous tensor. The stresses for both long term and viscous components are derived from well-known strain energy functional.

Finally, some dynamic, large strain, finite element applications are shown to illustrate the validity of the new constitutive equations. In particular, the simple problem of the bending of a cantilever beam will allow a direct comparison between displacement patterns for both hyperelastic and viscoelastic cases.

2. MOTIVATION: GENERALISED MAXWELL MODEL

A mechanical model consisting of spring and dashpot elements in parallel is commonly used to depict viscoelastic material behaviour. The system, known as the generalised Maxwell model (see Figure 1) enables the long term and viscous contributions to be treated separately.
Clearly, the total internal force of the system consists of both steady state and non-equilibrium components. Consequently, the total internal force of the system can be written as,

\[ f = f_\infty + \sum f_\alpha \]  

(1)

In order to derive equations relating force and displacement, it is first necessary to re-write the above equation as a function of the total internal energy. The deviatoric component is further split into long term and viscous contributions as,

\[ \psi(x) = \psi_\infty(x) + \sum \psi_\alpha(x, x_\alpha) \]  

(2)

The energy for the long term and each viscous component can be written in terms of the total system displacement and spring constants as,

\[ \psi_\infty = \frac{1}{2} K_\infty x^2 \]  

(3.a)

\[ \psi_\alpha = \frac{1}{2} K_\alpha (x - x_\alpha)^2 \]  

(3.b)

To fully define the state of the mechanical model, the displacements in the viscous elements are required. The resulting total internal force of the system is obtained by differentiating Equation (2) with respect to the total displacement of the system as,

\[ f = \frac{d\psi}{dx} = K_\infty x + \sum K_\alpha (x - x_\alpha) \]  

(4)

The force evolution in each viscous dashpot can be determined in two separate descriptions as,
\[ c_\alpha \dot{x}_\alpha = f_\alpha \]  
\[ f_\alpha = K_\alpha (x - x_\alpha) \]  

(5.a)  
(5.b)

Where \( c_\alpha \) represents the linear viscosity of dashpot \( \alpha \) and is related to the corresponding spring stiffness via the retardation time parameter \( \tau_\alpha \) as,

\[ c_\alpha = \tau_\alpha K_\alpha \]  

(6)

Combining Equations (6) and (5.a, b) enables the derivation of a direct relationship between the evolution of the internal variables and the net displacement in the corresponding spring element as,

\[ \dot{x}_\alpha = \frac{1}{\tau_\alpha} (x - x_\alpha) \]  

(7)

Since the evolution of the internal variables are non-linear bounded, they cannot be described by the simple linear evolution law given in Equation (7). To ensure that each viscous element obeys non-linear constraints, Equation (5.a) is expressed in terms of the relaxation of non-equilibrium forces \( f_\alpha \) at constant total strain \( x \) as,

\[ \left. \frac{df_\alpha}{dt} \right|_{x=\text{constant}} = -\frac{1}{\tau_\alpha} f_\alpha \]  

(8)

It can be shown that by a combination of Equations (8), (5.a) and (6), the linear evolution of each viscous element described in Equation (7) is satisfied. In the non-linear context, a generalisation of Equation (8) will lead to the correct evolution equations for the non-linear internal variables.

3. NONLINEAR VISCOELASTIC CONSTITUTIVE EQUATIONS

The motivation behind the development of accurate non-linear viscoelastic constitutive equations concerns deriving suitable models for the behaviour of human body parts under impact conditions. It is hoped that the proceeding formulations will accurately simulate the behaviour of biological soft tissues such as muscle and flesh during high-speed impact problems.

Biological soft tissues exhibit distinctive anisotropic behaviour. Consequently, the equivalent internal forces are calculated in the material or Lagrangian configuration via the Second Piola-Kirchoff stress tensor. In cases where behaviour in the current configuration is required, the Cauchy stress tensor is obtained via the push forward operation.

3.1 Multiplicative decomposition of the deformation gradient

As is commonly practised in deriving large strain elasto-plastic constitutive equations, the deformation gradient is decomposed into isochoric and volumetric contributions. For viscoelastic material behaviour, the isochoric component is further split into volume preserving elastic and viscous components. The deformation gradient \( \mathbf{F} \) and its
corresponding isochoric tensor $\hat{F}$ are directly related via the Jacobian $J$ as,

$$\mathbf{F} = J^{\frac{1}{2}}\hat{\mathbf{F}}, \quad J = \det \mathbf{F}$$  \hspace{1cm} (9)

For most metals and high-polymeric materials it is sufficient to consider the volumetric deformation as purely elastic and the viscous effects are subsequently constrained to the isochoric contribution of the deformation. In order to develop general equations for the material formulations, it is first necessary to use the multiplicative decomposition procedure to separate the deformation gradient into multiple elastic and viscous components as,

$$\hat{\mathbf{F}} = \hat{\mathbf{F}}_e \hat{\mathbf{F}}_v$$  \hspace{1cm} (10)

As in the generalised Maxwell model the number of viscous elements $\alpha$ is arbitrary. The deformation of each viscous dashpot at the current state is given by the tensor $\hat{\mathbf{F}}_v$. By construction each viscous deformation tensor satisfies $\det \hat{\mathbf{F}}_v = 1$.

Further expressions required such as strain measures for total, elastic and viscous contributions are given in terms of the deformation components described in Equation (10) as,

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}$$  \hspace{1cm} (11.a)

$$\hat{\mathbf{C}} = \hat{\mathbf{F}}^T \hat{\mathbf{F}} = I^{\frac{1}{2}} \mathbf{C}; \quad I^{\frac{1}{2}} = \det \mathbf{C} = J^2$$  \hspace{1cm} (11.b)

$$\hat{\mathbf{C}}_e = \hat{\mathbf{F}}_e^T \hat{\mathbf{F}}_e$$  \hspace{1cm} (11.c)

$$\hat{\mathbf{C}}_v = \hat{\mathbf{F}}_v^T \hat{\mathbf{F}}_v$$  \hspace{1cm} (11.d)

The above Lagrangian strain tensors are based at the reference configuration except for the tensor $\mathbf{C}_v$, which is based at the unloaded state. Consequently, the variables $J$, $\hat{\mathbf{C}}$ and $\mathbf{C}_v$ are used to describe the stress state at the current configuration. Note also that the condition $\det \hat{\mathbf{F}}_v = 1$ also implies $\det \mathbf{C}_v = 1$.

### 3.2 Strain energy function

The generalised energy functional for a three-dimensional model is simply an extension of the one-dimensional case as described by Equation (2). The total strain energy of the system is similarly decomposed into volumetric and deviatoric components, with the deviatoric contribution further split into steady state and non-equilibrium components. Since the volumetric component is independent of the Maxwell model, it is described in terms of the Jacobian $J$. The long term contribution is given in terms of the isochoric right Cauchy-Green tensor, with the inelastic right Cauchy-Green tensor required to define the viscous effects. Subsequently, the total strain energy functional can be written as,

$$\psi(\mathbf{C}, \mathbf{C}_v) = U(J) + \psi_\infty(\hat{\mathbf{C}}) + \sum_\alpha \psi_\alpha(\hat{\mathbf{C}}_e, \mathbf{C}_v)$$  \hspace{1cm} (12)

where typical expressions for the volumetric energy and long term functions are given by the simple compressible neo-Hookean material as,
where $\kappa$ and $\mu$ are the bulk and shear modulus material parameters respectively. For the simple neo-Hookean model used above, the viscous contribution is directly related to the long term expression via the positive non-dimensional proportionality factors $\beta_{\alpha}$ as,

$$
\psi_{\infty} (\hat{\hat{C}}) = \frac{1}{2} \mu \left( tr \hat{\hat{C}} - 3 \right) = \frac{1}{2} \mu \left( I_{3C}^{1/3} C : I - 3 \right)
$$

(14)

By noting that the invariants for $\hat{\hat{C}}_{\infty}$ coincide with the invariants of $C_{\infty}^{-1}$, enables Equation (15) to be re-written in the form,

$$
\psi_{\alpha} (\hat{\hat{C}}, C_{\infty}) = \frac{1}{2} \beta_{\alpha} \mu \left( \hat{\hat{C}} : C_{\infty}^{-1} - 3 \right)
$$

(16)

### 3.3 Second Piola-Kirchoff stress tensor

The total Second Piola-Kirchoff stress tensor is obtained by differentiating Equation (12) with respect to the right Cauchy-Green strain tensor. As with the strain energy functional, the stress tensor is decomposed into volumetric and deviatoric components as,

$$
S = 2 \frac{\partial \psi}{\partial C} = S_{\text{vol}} + S_{\text{dev}}; \quad S_{\text{vol}} = \frac{\partial U}{\partial C}
$$

(17)

It can also be shown that the trace of the true deviatoric component of $S$ with the right Cauchy-Green strain tensor equals zero\textsuperscript{11}. In order to derive an expression for $S$, it is first necessary to derive the following relationships\textsuperscript{11},

$$
\frac{\partial I_{3C}}{\partial C} = I_{3C} C^{-1}
$$

(18)

The volumetric component of $S$ is given by\textsuperscript{11},

$$
S_{\text{vol}} = p J C^{-1};
$$

(19)

$p$ is found by differentiating Equation (13) with respect to the Jacobian $J$ as,

$$
p = \frac{dU}{dJ} = \kappa (J - 1)
$$

(20)

In comparison with expression (1), the deviatoric component $S_{\text{dev}}$ consists of an arbitrary number of elements $\alpha$. Consequently the total Second Piola-Kirchoff stress tensor can be written as,

$$
S_{\text{dev}} = S_{\infty} + \sum_{\alpha} S_{\alpha}
$$

(21)
where the long term and viscous contributions of the deviatoric second Piola-Kirchoff stress tensor are defined by variables at the current configuration as,

$$ S_\infty = 2 \frac{\partial \psi_\infty}{\partial \hat{C}} $$  \hspace{1cm} (22) 

$$ S_\alpha = 2 \frac{\partial \psi_\alpha}{\partial \hat{C}} (\hat{C}, C_{v_\alpha}) $$  \hspace{1cm} (23) 

The Lagrangian stress state for the long term contribution is found by differentiating Equation (14) with the aid of Expression (22) to yield,

$$ S_\infty = \mu I^{1/3} \{ I - \frac{1}{3} I_{1C} C^{-1} \}; \quad I_{1C} = tr C $$  \hspace{1cm} (24) 

Similarly, the non-equilibrium component of the deviatoric second Piola-Kirchoff stress tensor is obtained using Equations (16) and (22) as,

$$ S_\alpha = \beta_\alpha \mu I^{1/3} \{ C_{v_\alpha}^{-1} - \frac{1}{3} (C; C_{v_\alpha}^{-1}) C^{-1} \} $$  \hspace{1cm} (25) 

Note as the strain in the viscous elements $C_{v_\alpha}$ approaches $\hat{C} = I^{1/3} C$, the total strain of the system, the stresses from the viscous elements reduce to zero. The identical manner in which the spring force $f_\alpha$ in the generalised Maxwell model tends to zero as the strain in the viscous dashpots $x_\alpha$ verges towards the total strain $x$.

### 3.4 Evolution equation

Clearly, viscous behaviour is a rate dependent phenomenon, which implies that the final expressions must be a function of time. In particular, the rate of deformation of each viscous element $C_{v_\alpha}$ is required in order to fully describe the stress state of the system. To ensure that the viscous elements behaviour in a non-linear fashion, each dashpot element must satisfy the condition $\text{det} C_{v_\alpha} = 1$. The non-equilibrium stresses are updated using the approach described in Equation (8), where the non-equilibrium forces are found at a constant strain of the system. To achieve such an expression for viscous stresses, simply replace the non-equilibrium forces with the non-equilibrium stresses and the total strain of the system is now given in terms of the right Cauchy-Green tensor to give,

$$ \frac{dS_\alpha}{dt} \bigg|_{C=\text{constant}} = -\frac{1}{\tau_\alpha} S_\alpha $$  \hspace{1cm} (26) 

where $\tau_\alpha$ is once again the retardation time parameter which determines the rate of dissipation of the viscous stresses. Since $S_\alpha$ is a function of both $C$ and $C_{v_\alpha}$, Equation (26) can be rewritten using the chain rule of differentiation as,
\[
\frac{\partial S_\alpha}{\partial C_{v_\alpha}} : \dot{C}_{v_\alpha} = \frac{1}{\tau_\alpha} S_\alpha
\]  

which can be further manipulated to give the following evolution equation for the state variable \( C_{v_\alpha} \) as,

\[
\dot{C}_{v_\alpha} = -\frac{1}{\tau_\alpha} N^{-1}_{\alpha} : S_\alpha; \quad N_\alpha = 2 \frac{\partial^2 \psi_\alpha \left( \dot{C} ; C_{v_\alpha} \right)}{\partial C \partial C_{v_\alpha}} \quad (28\text{.a,b})
\]

### 3.5 Incremental evolution equation

To determine the new state variable \( C_{v_\alpha} \) at the current configuration would require the integration of Equation (28.a, b). Clearly, this would further complicate the problem, and in a computational context the left hand side can be rewritten in incremental form as,

\[
\left. \frac{dS_\alpha}{dt} \right|_{C=\text{constant}} = \frac{1}{\Delta t} \left[ S_\alpha \left( C_{n+1}, C_{v_\alpha}^{n+1} \right) - S_\alpha \left( C_{n+1}, C_{v_\alpha}^{n} \right) \right] \quad (29)
\]

where \( \Delta t \) denotes the time increment between steps \( n \) and \( n+1 \). The term \( S_\alpha \left( C_{n+1}, C_{v_\alpha}^{n} \right) \) indicates the viscous stress state at the current configuration if the deformation for step \( n \) to \( n+1 \) occurred instantaneously. This in elastoplasticity is described as the trial stress state, which may or may not be compatible with the yield surface inequality. Clearly the assumption there is not relaxation of the viscous stresses during the step \( n \) to \( n+1 \) is invalid. The final stresses are found by substituting Equation (28) back into Equation (25) as,

\[
S_\alpha \left( C_{n+1}, C_{v_\alpha}^{n+1} \right) - S_\alpha \left( C_{n+1}, C_{v_\alpha}^{n} \right) = -\frac{\Delta t}{\tau_\alpha} S_\alpha \left( C_{n+1}, C_{v_\alpha}^{n+1} \right) \quad (30)
\]

Although it is appreciated that there are more accurate integration techniques available, the requirement for ease of future computational implementation of the constitutive equations must also be considered. By integrating Equation (25) using a backward Euler rule, which is only first order accurate in time, enables the final expressions to be presented in a particularly simple form. The viscous stress contribution at the current configuration is determined by rearranging Equation (29) as,

\[
S^{(n+1)}_\alpha = \frac{\tau_\alpha}{\tau_\alpha + \Delta t} S^{(n+1)}_\alpha; \quad S^{(n+1)}_\alpha = S_\alpha \left( C_{n+1}, C_{v_\alpha}^{n} \right) \quad (31)
\]

Note that the final stresses are simply proportional to instantaneous stresses. Obviously, this expression is much simpler to implement in a computational context than the integration of Equation (25). Once the new stresses are found, it is then necessary to update the new state variable \( C_{v_\alpha} \). This is achieved by solving for \( C_{v_\alpha} \) from the non-linear equations,

\[
S_\alpha \left( C_{n+1}, C_{v_\alpha}^{n+1} \right) = S^{n+1}_\alpha \quad (32)
\]
Hence Equation (25) can be re-arranged as the subject of the new internal variables in terms of the stresses as,

\[
(C_{n+1}^{-1})^{-1} = \frac{1}{\beta \mu I_3 C^2} S_{n+1}^{(n+1)} + \eta C_{n+1}^{-1}; \quad \eta = \frac{1}{3} C : C_{n+1}^{-1}
\]

(33.a,b)

where \( \eta \) is an additional unknown which can be solved using the condition \( \det C_{n+1} = 1 \).

This produces a non-linear equation in \( \eta \) as,

\[
\det[A + \eta C_{n+1}^{-1}] = 1; \quad A = \frac{1}{\beta \mu I_3 C^2} S_{n+1}^{(n+1)}
\]

(34.a,b)

which can be solved either directly or iteratively depending on the number of real roots within the equation.

4. Applications

The Lagrangian viscoelastic constitutive equations were implemented into an explicit large strain dynamic finite element code. A simple example of the use of the new formulations is described below. In particular, the bending of a vertical cantilever beam subjected to an initial velocity allows for a direct comparison between hyperelastic and viscoelastic material behaviour.

4.1 Bending of a vertical cantilever

Consider the problem of a vertical cantilever subjected to an initial horizontal velocity. The beam is modelled using 1269 tetrahedral elements, which are integrated using a simple average nodal pressure method developed by Bonet et al. The original mesh, initial boundary conditions and material properties for both hyperelastic and viscoelastic cases are shown in Figure 2. In addition, the problem time is 45s.

![Figure 2, Original Mesh, Boundary Conditions and Material Properties](image)
For hyperelastic material behaviour there is zero reduction in total kinetic energy of the body during the duration of the problem. Consequently, the beam sways from side to side with constant amplitude. By comparison, for viscoelastic material behaviour the viscous effects allow for dissipation of the total kinetic energy resulting in the beam returning to its original position. This type of behaviour is illustrated in Figure 3, which shows how the horizontal velocity of the top centre node varies with time for both hyperelastic and viscoelastic cases. The rate at which the beam loses energy depends directly on the retardation time and free energy parameters.

![Figure 3, X-Velocity Distribution for both Hyperelastic and Viscoelastic Cases](image)

As a further illustration of the energy dissipation of the viscoelastic beam, Figures 4 (a) and (b) depicts the deformed beams and xx-stress plots for both cases at various times. Clearly, the magnitude of sway of the viscoelastic beam reduces at each complete cycle until the beam returns to its original position. In contrast, the hyperelastic beam continues to move from side to side with a constant displacement from the initial position.

It’s also worth noting how the instantaneously deformation is affected by the viscous elements. The maximum sway at the first time interval is considerable reduced for the viscoelastic beam. By adding a further spring into the system, the total model behaves in a stiffer manner resulting in less instantaneous deformation of the vertical cantilever beam. Clearly, the addition of further viscous elements would result in an even stiffer model.

The choice of material parameters directly affects the rate at which the beam dissipates energy. However, care must be taken when deciding on suitable material parameters for a particular viscoelastic example. Problems can arise if inappropriate values for the retardation time and free energy parameters are chosen. Subsequent problems include the beam behaving in a hyperelastic fashion, with the viscous contributions unable to take effect.
Figure 4(b), XX-Stress Plots for Viscoelastic Beam
5. Concluding Remarks

The paper has presented the constitutive equations for a non-linear viscoelastic continuum in the Lagrangian configuration. Expressions for energy from known displacements were derived from the well-known generalised linear Maxwell model. This allowed for both the steady state and non-equilibrium components to be treated separately. The linear evolution equations were extended to the non-linear regime by determining the viscous stresses at a constant strain of the system. The multiplicative split of the deformation gradient into elastic and inelastic contributions allowed for the examination of three-dimensional bodies. Steady state and viscous stress tensors were derived by differentiating the compressible neo-Hookean strain energy functional with respect to the right Cauchy-Green tensor. A further non-dimensional proportionality factor was required in order to relate long term and viscous stresses. The total second Piola-Kirchoff stress tensor was further split into volumetric and true deviatoric components. The viscous right Cauchy-Green satisfied a non-linear law ensuring the strain in the viscous elements followed a non-linear relationship. The final relaxed stresses were calculated in incremental form using a simple first order accurate in time integration rule. This enabled the final stresses to be found in a particularly simple form. Finally, the new internal state variables were found using the non-linear compatible equation.

A simple dynamic example was subsequently presented, namely the bending of a vertical cantilever subjected to an initial horizontal velocity. The beam was modelled using hyperelastic and the new viscoelastic constitutive equations. The former beam demonstrated a distinctive cyclic horizontal velocity distribution for the duration of the problem, thus indicating that energy is conserved throughout. In contrast, the later beam dissipated energy resulting in the beam returning to its original position. The rate of dissipation depends directly on the viscous material parameters. Alternating such parameters would result in a variety of stress histories for the vertical cantilever beam.

REFERENCES


