NUMERICAL AND ANALYTICAL METHODS FOR SOLUTION OF PROBLEMS OF PLANE AND AXISYMMETRIC FLOWS OF IDEAL FLUID OVER A GAS BUBBLE

Nataliya M. Sherykhalina, Vladimir P. Zhitnikov

e-mail: n_sher@mail.ru, Web page: http://ugatu.rb.ru/~zhitnik

Key words: Axisymmetric Flow, Gas Bubble, Surface Tension, Laplace Equation, Analytic Function, Integral Transformations.

Abstract. Plane problems of hydrodynamics can be efficiently investigated by the theory of analytic functions. Well-known methods of the theory of analytic functions can’t be used directly in the case of axisymmetric flows. The integral transformation offered by G.N. Pology converts ordinary analytic functions to generalized analytic functions of special kind as, in particular, velocity potential and stream function of an axisymmetric flow, cross and arbitrary flows over rotation bodies. But nonadequacy of boundary conditions of plane and space problems (for example impenetrability conditions) and the different representation of singularities (source, dipole, stream separation point) allows to apply Pology’s transformation efficiently only for the rather simple problems (linear theory problems, non-separate flow over smooth bodies). This paper presents results of the investigation of nonlinear problem about flow over a gas bubble. The desired function is presented as a sum of a known function with given singularities and power series or a spline with coefficients found from a boundary condition. A flow domain is mapped onto a ring for the convenience of a function presentation.
1 INTRODUCTION

In plane problems of hydrodynamics the theory of a complex variable is a powerful method of investigation. In particular, within the framework of this theory the exact analytic solution for two special cases of flow past a gas bubble were found in [1, 2], an asymptotic solution for its small deformations was obtained in [3], an iterative method of calculation based on the expansion of the unknown functions in a power series in terms of as small parameter was processed in [4], and, finally, a detailed investigation of the past soft shells and bubbles was made in [5, 6]. In the last two studies the authors used the Levi-Chivita method after isolating the singularities of the unknown functions in advance.

In the case of an axisymmetric flow the methods of analytic functions cannot be used directly. In [7] generalized analytic functions of a special kind were introduced and the possibility of their integral representations in terms of the ordinary analytic functions was demonstrated. These generalized functions include the potential and the stream function of the axisymmetric flow. Consequently, the integral transformation [7] made it possible to reduce axisymmetric problems to plane ones, albeit with more complicated integral boundary conditions.

In this paper Pology’s integral transformation is used to solve the axisymmetric problem of flow past a bubble. Since, in the literature, this transformation is quite seldom used, the authors consider it appropriate first to present some information on integral transformations and on the behavior of analytic functions near critical and singular points.

2 POLOGY’S INTEGRAL TRANSFORMATION

In potential axisymmetric flow the potential $\phi$ and the stream function $\psi$ in the cylindrical coordinates $x, y$ must satisfy the following conditions

$$\frac{\partial \phi}{\partial x} = \frac{1}{y} \frac{\partial \psi}{\partial y}, \frac{\partial \phi}{\partial y} = -\frac{1}{y} \frac{\partial \psi}{\partial x} \tag{1}$$

In [7] the $p$ - analytic function $f(z) = u(x, y) + iv(x, y)$ formula satisfying the conditions $u_x = v_y / p$ and $u_y = -v_x / p$ was introduced and for characteristics $p = x^4$ the integral representation of $f(z)$ in terms of an analytic function (in the ordinary sense) $f(z) = u(x, y) + iv(x, y)$ was obtained. For a generalized analytic function $w = \phi + i\psi$ with the characteristic $p = y$ Pology’s transformation has the form

$$w(z) = \phi + i\psi = -\frac{1}{2} \int_{\Gamma} f(\zeta) \left(i + \zeta - \frac{z + \bar{z}}{2}\right) \frac{d\zeta}{g(\zeta)}, \quad g(\zeta) = \sqrt{(\zeta - z)(\zeta - \bar{z})} \tag{2}$$

where $\Gamma$ is the contour connecting the point $z = x - iy$ with the point $z = x + iy$ on the real $x$ axis the conditions $v(x, 0) = 0$, $\psi(x, 0) = 0$ are satisfied.

If at infinity as $z \to \infty$ the analytic function $f(z)$ satisfies the condition
\[ f(z) = O(|z|^{-\alpha}) \quad \alpha = const > 0 \], then the contour \( \Gamma \) may be deformed in such a way that it contains a point at infinity. In particular, from (2) we can obtain two integral transformations for the real functions \( \phi \) and \( \psi \).

\[
\phi = \text{Im} \int \frac{dx}{g(z)} f(\zeta), \quad \psi = -\text{Im} \int \frac{dx}{g(z)} \frac{d\zeta}{g(\zeta)}
\] (3)

For multifunction \( g(s) \) one of the single-valued branches should be chosen, for example, from the condition \( \text{Im} \, g(\zeta) > 0 \) on the interval \( (z, x + i\infty) \).

By means of the change of variables \( \zeta = z + i\eta \) and subsequent differentiation it is easy to verify directly, by means of (3), that the function \( \phi \) and \( \psi \) satisfy the condition (1). The partial derivatives are equal to

\[
\frac{\partial\phi}{\partial x} = \text{Im} \int \frac{dx}{g(z)} f'(\zeta), \quad \frac{\partial\psi}{\partial x} = -\text{Im} \int \frac{dx}{g(z)} \frac{d\zeta}{g(\zeta)}
\] (4)

where \( f'(z) \) denotes the derivatives of the function \( f(z) \) with respect to \( z \).

Let us consider the limiting value of the first integral (3) as \( z \to x + i0 \). In this case the function \( g(\zeta) \to \zeta - x \) and at the point \( \zeta = x \) the integrand will have a simple pole with the principal part \( u(x,0)/(\zeta - x) \). Let the contour of the axisymmetric body \( C \) be located to the right of the point \( z = x \). If in the upper half-plane of \( z \) \( (\text{Im} \, z > 0) \) outside the boundary \( C \) we consider two closed contours to the right and left of the straight line \( (x + i0, x + i\infty) \), the integral along which is equal to zero, then by deforming these contours we can obtain two forms for the same value of the velocity potential

\[
\phi(x,0) = \frac{\pi}{2} u(x,0), \quad \text{or} \quad \phi(x,0) = -\frac{\pi}{2} u(x,0) - \text{Im} \int f(\zeta) \frac{d\zeta}{\zeta - x}
\] (5)

Here, the integration is performed clockwise, i.e., the axisymmetric body remains on the right.

Analogous formulas can be found for a point \( x \) situated on the right of the axisymmetric body

\[
\phi(x,0) = -\frac{\pi}{2} u(x,0), \quad \text{or} \quad \phi(x,0) = \frac{\pi}{2} u(x,0) - \text{Im} \int f(\zeta) \frac{d\zeta}{\zeta - x}
\] (6)

From (5) and (6) it follows that the behavior of the function \( u(x) \) on the \( lx \) axis is the same as that of the potential \( \phi \).

The formulas (5) and (6) are also valid for a system of axisymmetric bodies situated on the same axis. The value of the potential at points \( x \) between the axisymmetric bodies can be found from the second of formulas (5) and (6).

Let the flow velocity at infinity be equal to \( V \). Then the velocity potential \( \Phi \) and the
stream function $\Psi$ for axisymmetric flow can be obtained by superposing the undisturbed and disturbed flows, i.e.,

$$\Phi = \phi + V_\infty x, \quad \Psi = \psi + \frac{V_\infty^2}{2}$$

(7)

Near the critical point $x_0$ the cone-shaped vertex of the body the potential is equal to $|\Phi(x,0) - \Phi(x_0,0)| O(x - x_0)^\gamma$, where the exponent $\gamma$ depends only on the angle of taper [13]. Consequently, we get

$$f(z) - f(x_0) = \pm \frac{2V_\infty}{\pi} (z - x_0) + O((z - x_0)^2)$$

where the upper sign corresponds to the right and the lower sign to the left critical points. For bodies with a smooth surface the exponent $\gamma = 2$, i.e., at these points the analyticity of the function $f(z)$ does not break down.

3 FORMULATION OF THE PROBLEM AND PARAMETRIC SOLUTION

Let a potential stream of ideal incompressible fluid flow around a bubble (or soft shell). On the surface of the bubble $\zeta$ the kinematic condition and the dynamic Laplace condition must be fulfilled

$$\Psi = \psi + 0.5V_\infty y^2 = 0$$

(8)

$$\frac{T \cos \vartheta}{y} + \frac{T}{R} = P_0 - P$$

(9)

Here, $V$ is the free-stream velocity, $T$ is the surface tension (the same in all directions and at any point on the surface), $y$ is the distance of the surface point from the $x$ axis, $1/R = -d\vartheta/ds$ is the meridional curvature of the surface, $s$ is the arc abscissa of the generator of the axisymmetric body, whose positive direction is clockwise, and $P_0$ and $P$ are the pressures inside and outside the bubble. If we introduce the dimensionless parameters

$$\mu = 2(P_0 - P_\infty)/(\rho V_\infty^2), \quad \text{We} = \rho V_\infty^2 L/(2T)$$

(10)

where $\rho$ is the density of the fluid, $P_\infty$ is the fluid pressure at infinity, $\text{We}$ is the Weber number, and $L$ is the length of the generator $\zeta$, and if we take advantage of the Bernoulli integral, then the condition (9) may be represented as follows

$$L\left(-\frac{d\theta}{ds} + \frac{\cos \vartheta}{y}\right) = \text{We} \left(\mu - 1 + \frac{\Phi_x^2 + \Phi_y^2}{V_\infty^2}\right)$$

(11)

As in the plane case [6], the solution of the problem may more easily be found in
parametric from by conformally mapping the region of flow in the axial plane \( z \) onto the exterior of the unit disk in the auxiliary plane \( \xi \). With allowance for the symmetry about the coordinate axes, the required analytic functions \( z(\xi) \) and \( f[\zeta(\xi)] \) can be represented in the series form

\[
z = L \left[ a_0 \zeta + \sum_{m=0}^{\infty} a_{2m+1} \zeta^{2m+1} \right], \quad f = V_0 L \sum_{m=0}^{\infty} \frac{b_{2m}}{\zeta^{2m}}
\]

with real coefficients \( a_0, a_{2m+1} \) and \( b_{2m} \).

At the critical points to the left and right of the bubble, in accordance with (5) and (6), the condition \( \frac{df}{dx} = \pm \frac{2V_0}{\pi} \) must be fulfilled. By substituting the function (12) and setting \( \zeta = \pm 1 \) we obtain the identical condition

\[
\sum_{m=0}^{\infty} 2mb_{2m} = \frac{2}{\pi} \left[ a_0 + \sum_{m=0}^{\infty} (2m+1)a_{2m+1} \right] = 0
\]

On the circle \( \zeta = \exp(i\sigma) \) the real and imaginary parts of the function (12) are equal, respectively, to

\[
x = L \left[ a_0 \cos \sigma + \sum_{m=0}^{\infty} a_{2m+1} \cos(2m+1)\sigma \right], \quad y = L \left[ a_0 \sin \sigma - \sum_{m=0}^{\infty} a_{2m+1} \sin(2m+1)\sigma \right]
\]

\[
u = -V_0 L \sum_{m=1}^{\infty} b_{2m} \cos 2m\sigma, \quad v = V_0 L \sum_{m=1}^{\infty} b_{2m} \sin 2m\sigma
\]

The length of the generator \( \zeta \) for the axisymmetric bubble is equal to

\[
L = \int_0^{\sigma} s'(\sigma) d\sigma, \quad s'(\sigma) = \sqrt{[x'(\sigma)]^2 + [y'(\sigma)]^2}
\]

In order to determine the coefficients \( a_{2m+1} \) and \( b_{2m} \) it is necessary to satisfy simultaneously the conditions (8) and (11), which should be expressed in terms of the variable \( \sigma \)

\[
\cos \vartheta = \frac{dx}{ds} = \frac{x'(\sigma)}{s'(\sigma)}, \quad -\frac{d\vartheta}{ds} = \frac{y'(\sigma)x'(\sigma) - x'(\sigma)y'(\sigma)}{[s'(\sigma)]^3}
\]

For points \( z \) on the boundary of the body the integrals in (3) and (4) can be reduced by deformation of the contour to integration along the boundary \( \zeta \) of the axisymmetric body [12] and, consequently, along an arc of the circle \( \zeta = e^{i\sigma} \)
\[
\psi = \text{Im} \int_0^\sigma f(\zeta) \zeta^{-x(\zeta)} g(\zeta) d\sigma,
\]

(17)

\[
\frac{\partial \phi}{\partial x} = -\text{Im} \int_0^\sigma f'(\zeta) \zeta^{-x(\sigma)} g(\zeta) \frac{\partial \phi}{\partial y} = -\frac{1}{y} \text{Im} \int_0^\sigma f'(\zeta) \zeta^{-x(\sigma)} g(\zeta) d\sigma
\]

(18)

\[z = z(e^{i\alpha}) = x(\sigma) + iy(\sigma), \quad \zeta = z(e^{i\beta}), \quad \text{Re} \zeta = x(\beta), \quad \text{Im} \zeta = y(\beta), \quad f'(\beta) = u'(\beta) + iv'(\beta)
\]

When \(\sigma = 0\) at the critical point the condition (11) takes the form

\[-\frac{2Lx'(0)}{[y'(0)]^2} = We(\mu - 1)
\]

(19)

4 NUMERICAL CALCULATIONS

For the numerical calculations the collocations method was used, i.e., the conditions (8) and (11) are satisfied at the discrete points \(f'(\beta) = u'(\beta) + iv'(\beta)\), and, in addition, the equalities (13), (14) and (19) are fulfilled. The system of \(2N + 3\) nonlinear equations was solved numerically with respect to the Weber number \(We\), the coefficient \(a_0\), the \(N\) first coefficients \(a_{2m+1}\), and the \(N\) coefficients \(b_{2m}\). All the other coefficients of the sum (12) are assumed to be equal to zero. The parameter \(\mu\) is assumed to be given. In going over to dimensionless variables the length of the generator \(L\) and the velocity \(V_0\) cancel out; therefore, without loss of generality, we can take \(L = 1, V = 1\).

The results of the numerical calculations are shown in figures 1-4, where the continuous curves correspond to the axisymmetric, and the broken curves to the cylindrical bubble.

Figure 1 shows the configuration of the bubble, when \(\mu = 5, 1, 0.681, 0\) (curves 1-4, respectively). In the third case the bubble surface is in contact at the critical points. In a real fluid this could lead to the formation of toroidal bubbles. Curves 5-7 correspond to the cylindrical bubble with \(\mu = 0.546, 1\) and 5. For the case of \(\mu = 1\) an exact analytic solution can be found [2].

The geometric dimensions of the bubble can be characterized by the distance between the critical points \(l\) and the diameter of the maximum cross-section \(d\). Figure 2 shows the dependence of the geometric dimensions of the axisymmetric and cylindrical bubbles on the parameter \(\mu\) when the volume is constant (curves 1-3 correspond to the functions \(\eta_1(\mu) = 1/2, \eta_2(\mu) = d/2, \) and \(\eta_3(\mu) = 2L/\pi\)).

Our calculations show that the change in the configuration of the axisymmetric bubble is similar to the change in the configuration of the plane one. As the parameter \(\mu\) decreases contraction in the direction of flow and expansion in the transverse direction are observed. When \(\mu < 0.6814\) (in the plane case \(\mu < 0.5463\)) the bubble boundaries self-intersect, which corresponds to the flow on a two-sheeted surface. In particular, when \(\mu < 0.5479\) (for the
plane bubble when \( \mu < 0.4412 \) the volume of the bubble calculated theoretically is equal to zero. For these values of \( \mu \) the curves in figure 2 have vertical asymptotes.

The results of the axisymmetric problem in comparison with the plane one show the following. The deformation of an axisymmetric bubble with pressure variation inside the bubble is similar to the plane one. The bubble is compressed in the stream direction and stretches in the cross one. The relation of the width to the length is some more for an axisymmetric bubble.

Practically, at low pressure a bubble break at its middle part is possible. Then a bubble transforms itself into other few bubbles or into toroidal bubble. The possibility of such
transformation depends on increment sign of the surface tension energy associated with variation of bubble surface area. When the bubble transforms itself into two equal ones, its volume $v$ is equal to $1/2$ of the initial bubble volume. Surface tension, velocity and pressure at the infinity don’t vary. Then the Weber number

$$\text{We} = \frac{\rho \sqrt{v}}{2T} = \frac{\lambda \sqrt{v}}{a}$$

(20)

becomes $2^{-1/3}$ of its previous value, and pressure inside the bubble and the surface area can be found from calculated relation $We(\mu)$ (figure 3, curve 1) from the equation

$$\text{We}(\mu) = 2^{-1/3} \text{We}''$$

(21).

Figure 3: Dependence of Weber number (1), bubble surface area of one (2) and two bubbles (3) on parameter $\mu$.

The obtained value $\mu'(\mu)$ is used for the determination of surface area $F$, resulting after bubble bisection. The curve 2 in figure 3 shows dependence $F' = F(\mu)/4\pi$ for one bubble. For surface area determination of two bubbles of half volume it is necessary to obtain the dependence $F'' = F''(\mu')$ and multiply it by $2/\sqrt{2^2} = \sqrt{2}$. The dependence $\sqrt{2}F''(\mu')$ is shown in figure 3 by the curve 3. Since the surface energy is equal to the surface area multiplied by $T$, the comparison of the curves 2 and 3 makes possible to know the sign of energy increment after bubble bisection. Evidently, that the energy is diminished for $\mu < \mu''=1.25$, so bisection becomes efficient. As the value of $\mu'$ found from (21) is equal to 2.5 for $\mu=1.25$, so the association of bubbles becomes efficient for $\mu>2.5$. The bisection and association leads to surface energy increase in interval $1.25<\mu<2.5$. So the bubbles are the most steady in this range of the pressure coefficient.

Dependence of radial bubble dimension $1/\text{We}$ number on the parameter $\mu$ for the flow over the bubble in tube are shown in figure 4, 5. An asymptotic evaluation of $\text{We}$ number for almost spherical bubble can be obtained by substituting $y=R_0 \cos \theta$ into (11): $\text{We} = 2/\mu$. For the flat bubble one have to substitute $y=\infty$ into (11), hence $\text{We} = 1/\mu$. Therefore the values of $\text{We}$
number, presented in figure 5 for the flat flow are doubled.

Figure 4: Dependence of radial bubble dimension on parameter $\mu$ for the flow over bubble in tube.

Figure 5: Dependence of $1/\text{We}$ber number on parameter $\mu$ for the flow over bubble in tube.

In this case the Weber number depends on both the pressure number $\mu$ and the ratio $h/L$ ($h$ is the radius of the tube or half width of the channel). The numerical investigation shows that the dependence of $\text{We}(\mu)$ for the fixed ratio $h/L$ is a two-valued function for $h/L<H=0.505$ and $\mu>\mu_0$ ($\mu_0$ depends on the ratio $h/L$), i.e. two kinds of flows exist for the value of $\mu$. But for $h/L>H$ the function $\text{We}(\mu)$ is single valued and the bubble has only one shape.

Figure 6 shows the bubble shapes for different Weber numbers and two values of the ratio...
The curves on the right of the y-axis are the bubbles for axisymmetric flows, whereas those on the left correspond to plane flow. Bubble shapes 1-6 in figure 5(a) correspond to $\mu = 14, 5.5, 5, 7, 8.5$ and 9, respectively, and curves 1-3 in figure 6(b) correspond to $\mu = 3, 1, 5, 0$. Figure 6(a) shows two kinds of bubble shapes (1) for the planar flow.

![Figure 6: Bubble shapes for different Weber’s number and two ratios of $h/L$:](image)

(a) $h/L=0.5$; (b) $h/L=0.557$.

REFERENCES


