NUMERICAL SIMULATION AND ADAPTIVE METHODS FOR TRANSONIC FLOW

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Abstract. We present an efficient numerical method for the numerical solution of the system of the compressible Navier – Stokes equation. This method was originally determined for the numerical solution of the nonlinear convection – diffusion scalar equation, where the proof of the convergence of the numerical solution to the exact one was performed. The convective terms are discretized by the barycentric finite volumes and the viscous ones by the nonconforming linear finite elements. To increase the quality of the solution, the anisotropic mesh adaptation technique was employed. We present a short theoretical background and the motivation of the method. A special technique was determined for the capture of viscous effects as boundary layers and wakes. Several numerical results are presented.
1 INTRODUCTION

Many processes in science and technology are described by convection–diffusion equations with convection dominating over diffusion. In [1] we derived an efficient numerical scheme for the numerical solution of a nonlinear convection–diffusion scalar equation. This scheme is based on a combination of the barycentric finite volume and nonconforming finite element methods. Here we have applied this scheme for the system of the Navier–Stokes equation describing the motion of a viscous compressible gas. In what follows we describe the method combining barycentric finite volumes with nonconforming piecewise linear finite elements, applied to the solution of high-speed flow past an isolated profiles and a cascade of profiles modeling the flow in steam and gas turbines or compressors. The goal was to construct sufficiently efficient, robust and reliable method for the solution of complex flow fields with shock waves, boundary layers and their interaction. This scheme is not based on the operator splitting method as in [5] and both (viscous and inviscid) terms are discretized on the same time level.

Although this scheme gives promising results, they can not be achieved without a suitable grids. The simulation of shock waves and thin boundary layers characteristic for transonic flow with a high Reynolds number requires a special mesh adaptation. Therefore the second part of this contribution is devoted to the anisotropic mesh adaptation. We slightly extend the results presented in [5], namely the theoretical background of the method.

2 FORMULATION OF THE PROBLEM

We consider gas flow in a space-time cylinder $Q_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain representing the region occupied by the fluid and $T > 0$. The complete system of viscous compressible flow consisting of the continuity equation, Navier–Stokes equations and energy equation can be written in the form

$$
\frac{\partial w}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(w)}{\partial x_s} = \sum_{s=1}^{2} \frac{\partial R_s(w, \nabla w)}{\partial x_s} \quad \text{in} \ Q_T.
$$

Here

$$w = (w_1, w_2, w_3, w_4)^T = (\rho, \rho v_1, \rho v_2, e)^T,$$

$$w = w(x, t), \quad x \in \Omega, \ t \in (0, T),$$

$$f_i(w) = (\rho v_i, \rho v_i v_1 + \delta_{i1} p, \rho v_i v_2 + \delta_{i2} p, (e + p) v_i)^T,$$

$$R_i(w, \nabla w) = \left(0, \tau_{i1}, \tau_{i2}, \tau_{i1} v_1 + \tau_{i2} v_2 + \frac{\gamma}{Re Pr} \frac{\partial \theta}{\partial x_i}\right)^T,$$

$$\tau_{ij} = \frac{1}{Re} \left[ \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} \text{div} \mathbf{v} \delta_{ij} \right], \quad i, j = 1, 2.$$
From thermodynamics we have
\[ p = (\gamma - 1) \left( e - \rho |\mathbf{v}|^2 / 2 \right), \quad e = \rho(\theta + |\mathbf{v}|^2 / 2). \] (3)

We use the standard notation for dimensionless quantities: \( t \) – time, \( x_1, x_2 \) – Cartesian coordinates in \( \mathbb{R}^2 \), \( \rho \) – density, \( \mathbf{v} = (v_1, v_2) \) – velocity vector with components \( v_i \) in the directions \( x_i \), \( i = 1, 2 \), \( p \) – pressure, \( \theta \) – absolute temperature, \( e \) – total energy, \( \tau_{ij} \) – components of the viscous part of the stress tensor, \( \delta_{ij} \) – Kronecker delta, \( \gamma > 1 \) – Poisson adiabatic constant, \( Re \) – Reynolds number, \( Pr \) – Prandtl number. We assume that \( Pr = 0.72 \). We neglect outer volume force. The functions \( f_i \), called inviscid (Euler) fluxes, are defined in the set \( D = \{ (w_1, \ldots, w_4) \in \mathbb{R}^4; w_1 > 0 \} \). The viscous terms \( R_i \) are defined in \( D \times \mathbb{R}^8 \). (Due to physical reasons it is also suitable to require \( p > 0 \).

System (1), (3) is equipped with the initial condition
\[ w(x, 0) = w^0(x), \quad x \in \Omega \] (4)
(which means that at time \( t = 0 \) we prescribe, e.g., \( \rho, v_1, v_2 \) and \( \theta \)) and boundary conditions. In the simulation of flow past a cascade of profiles the region occupied by the fluid is represented by a plane infinitely connected domain \( \tilde{\Omega} \), bounded in one space direction (say \( x_1 \)) and unbounded but periodic in the other direction (\( x_2 \)). Assuming also the periodicity of the flow field, we can choose the computational domain \( \Omega \) in the form of one period of the original domain \( \tilde{\Omega} \). The boundary \( \partial \Omega \) is formed by disjoint parts \( \Gamma_I, \Gamma_O, \Gamma_W, \Gamma^+ \) and \( \Gamma^- \). On \( \Gamma_I, \Gamma_O \) and \( \Gamma_W \), representing the inlet, outlet and impermeable profile, respectively, we prescribe the conditions
\[
\begin{align*}
(i) & \quad \rho = \rho^*, \quad v_i = v_i^*, \quad i = 1, 2, \quad \theta = \theta^* \quad \text{on } \Gamma_I, \\
(ii) & \quad v_i = 0, \quad i = 1, 2, \quad \frac{\partial \theta}{\partial n} = 0 \quad \text{on } \Gamma_W, \\
(iii) & \quad \sum_{i=1}^{2} \tau_{ij} n_i = 0, \quad j = 1, 2, \quad \frac{\partial \theta}{\partial n} = 0 \quad \text{on } \Gamma_O.
\end{align*}
\] (5)

Here \( \partial / \partial n \) denotes the derivative in the direction of unit outer normal \( \mathbf{n} = (n_1, n_2)^T \) to \( \partial \Omega \); \( w^0, \rho^*, v_i^*, \theta^* \) are given functions.

Moreover, the arcs \( \Gamma^- \) and \( \Gamma^+ \) are piecewise linear artificial cuts such that
\[ \Gamma^+ = \{ (x_1, x_2 + \tau); (x_1, x_2) \in \Gamma^- \}, \] (6)
where \( \tau > 0 \) is the width of one period of the cascade in the direction \( x_2 \). On \( \Gamma^\pm \) we consider the periodicity condition
\[ w(x_1, x_2 + \tau, t) = w(x_1, x_2, t), \quad (x_1, x_2) \in \Gamma^- . \] (7)
The same condition is imposed on the first-order derivatives of the function \( w \).

Let us note that equations (1) and (3) are of hyperbolic-parabolic type and that nothing is known about the existence and uniqueness of the solution of problem (1), (3)–(5) and (7).
3 NUMERICAL METHOD

We carry out the discretization of the system (1), (3) with the use of the combination of finite volume and finite element methods. Let \( \Omega_h \) be a polygonal approximation of the domain \( \Omega \). By \( T_h = \{ T \}_{T \in T_h} \), we will denote a triangulation of \( \Omega_h \) with standard properties for finite element method. By \( S_h \) we denote the set of all sides of all triangles \( T \in T_h \). We introduce a numbering of triangles \( T \in T_h \) and their sides \( S \in S_h \) in such a way that

\[
T_h = \{ T_i; \ i \in I \}, \quad S_h = \{ S_j; \ j \in J \},
\]

where \( I \) and \( J \) are suitable index sets. By \( Q_j \) we denote the centre of a side \( S_j \in S_h \) and put \( P_h = \{ Q_j; \ j \in J \} \). Moreover, we set

\[
J^o = \{ i \in J; \ Q_i \in \Omega_h \} \tag{9}
\]

Now let us construct the barycentric mesh \( D_h = \{ D_i; i \in J \} \) over the basic mesh \( T_h \). The barycentric finite volume \( D_i \) is a closed polygon defined in the following way: We join the barycentre of every triangle \( T \in T_h \) with its vertices. Then around the side \( S_i, \ i \in J^o \), we obtain a closed quadrilateral containing \( S_i \). If \( S_j \subset \partial \Omega_h \) is a side with vertices \( P_1, P_2 \) of a triangle \( T \in T_h \) adjacent to \( \partial \Omega_h \), then we denote by \( D_j \) the triangle with the sides \( S_j \) and segments connecting the barycentre of \( T \) with \( P_1 \) and \( P_2 \), see Fig. 1 and Fig. 2. It is obvious that

\[
\overline{D_i} = \bigcup_{i \in J} D_i \tag{10}
\]

If \( D_i \neq D_j \) and the set \( \partial D_i \cap \partial D_j \) contains more than one point, we call \( D_i \) and \( D_j \) neighbours and set \( \Gamma_{ij} = \partial D_i \cap \partial D_j \) (= a common side of \( D_i \) and \( D_j \)). Further, we define the set \( s(i) = \{ j \in J; \ D_j \text{ is a neighbour of } D_i \} \). If \( Q_i \in \partial \Omega_h \), then we set \( S(i) = s(i) \cup \{-1\} \) and \( \Gamma_{i,-1} = S_i \subset \partial \Omega_h \), otherwise (for \( i \in J^o \)) we put \( S(i) = s(i) \).

In the sequel we use the following notation: \( |T| \) = area of \( T \in \mathcal{T}_h \), \( |D_i| \) = area of \( D_i \in \mathcal{D}_h \) (i.e., \( i \in J \)), \( l_{ij} \) = length of the segment \( \Gamma_{ij} \), \( n_{ij} = (n_{ij1}, n_{ij2}) = \text{unit outer normal to } \partial D_i \text{ on } \Gamma_{ij} \) (i.e., \( n_{ij} \) points from \( D_i \) to \( D_j \)). Moreover, let us consider a partition \( 0 = t_0 < t_1 < \ldots \) of the interval \((0, T)\) and set \( \tau_k = t_{k+1} - t_k \) for \( k = 0, 1, \ldots \). Obviously, we have

\[
\partial D_i = \bigcup_{j \in S(i)} \Gamma_{ij} \tag{11}
\]

Let us define the following space over the grid \( \mathcal{T}_h \):

\[
X_h = \left\{ v_h \in L^2(\Omega_h); v_h|_T \text{ is linear } \forall T \in \mathcal{T}_h, v_h \text{ is continuous at } Q_j \forall j \in J \right\} \tag{12}
\]
Figure 1: Barycentric finite volumes, \(D_i, D_j \in D_h\), \(Q_i, Q_j \in P_h\), \(S_i, S_j \in S_h\), \(S_j \subset \partial \Omega_h\)

Figure 2: Triangular mesh and the associated barycentric finite volume mesh

We use nonconforming piecewise linear finite elements. This means that the components of the state vector are approximated by functions from the finite dimensional space \(X_h\) defined in (12). Further, we set \(X_h = [X_h]^4\) and

a) \(V_h = \{ \varphi_h = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in X_h; \varphi_i(Q_j) = 0 \text{ for midpoints } Q_j \text{ lying on the part of } \partial \Omega_h \text{ where } w_i \text{ satisfies the Dirichlet condition and } \varphi_h \text{ satisfies periodicity condition (7)} \}\),

b) \(W_h = \{ w_h \in X_h; \text{ its components satisfy the Dirichlet boundary conditions following from (5) and periodicity condition (7)} \}\).

Multiplying (1) considered on time level \(t_k\) by any \(\varphi_h \in V_h\), integrating over \(\Omega_h\), using Green’s theorem, taking into account the boundary conditions (5) and the periodicity
followingscheme for the calculation of an approximate solution

\[
\int_{\Omega_h} \frac{\partial w}{\partial t} \varphi_h \, dx + \int_{\Omega_h} \sum_{s=1}^{2} \frac{\partial f_s(w)}{\partial x_s} \varphi_h \, dx + \int_{\Omega_h} \sum_{s=1}^{2} R_s(w, \nabla w) \frac{\partial \varphi_h}{\partial x_s} \, dx = 0. \tag{13}
\]

We approximate the convective terms with fluxes \( f_s \) by a form \( b_h \) with the aid of the finite volume approach. Let \( u \in H^1(\Omega_h), v_h \in X_h \). Then we write

\[
\int_{\Omega} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} v \, dx \approx \sum_{i \in J} \int_{\partial D_i} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} v(Q_i) \, dx = \sum_{i \in J} v(Q_i) \int_{\partial D_i} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} \, dx
\]

\[
= \sum_{i \in J} v(Q_i) \sum_{s=1}^{2} f_s(u) n_s \, dS = \sum_{i \in J} v(Q_i) \sum_{j \in S(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} f_s(u) n_s \, dS
\]

\[
\approx \sum_{i \in J} v(Q_i) \sum_{j \in S(i)} H(u(Q_i), u(Q_j), n_{ij}) \ell_{ij}.
\]

The function \( H \) defined on \( \mathcal{R}^2 \times \mathcal{S} \), where \( \mathcal{S} = \{ n \in \mathcal{R}^2; |n| = 1 \} \), is called a numerical flux. The form

\[
b_h(u, v) = \sum_{i \in J} v(Q_i) \sum_{j \in S(i)} H(u(Q_i), u(Q_j), n_{ij}) \ell_{ij} \tag{14}
\]

obtained above has sense for all \( u, v \in X_h \). Now approximating the time derivative by a forward difference, (14) and evaluating the integrals with the aid of the quadrature formula using midpoints of sides as integration points, i.e.,

\[
\int_{T} F \, dx \approx \frac{1}{3} |T| \sum_{i=1}^{3} F(Q_T^i), \tag{15}
\]

for \( F \in C(T) \) and a triangle \( T \) with midpoints of sides \( Q_T^i, \ i = 1, 2, 3 \), we arrive at the following scheme for the calculation of an approximate solution \( w_h^{k+1} \) on the \((k + 1)\)-st time level:

a) \( w_h^{k+1} \in W_h \), \( \tag{16} \)

b) \( \int_{\Omega_h} w_h^{k+1} \varphi_h \, dx = \int_{\Omega_h} w_h^{k} \varphi_h \, dx - \tau_k \{ b_h(w_h^{k}, \varphi_h) + a_h(w_h^{k}, \varphi_h) \} \quad \forall \varphi_h \in V_h. \)

Here

\[
(w_h, \varphi)_h = \frac{1}{3} \sum_{T \in T_h} |T| \sum_{i=1}^{3} w_h(Q_T^i) \varphi_h(Q_T^i), \quad w_h, \varphi \in X_h \tag{17}
\]
and \( a_h(w_h^k, \varphi_h) \) approximates the viscous terms of the form

\[
\int_{\Omega_h} \sum_{s=1}^{2} R_s(w_h^k, \nabla w_h^k) \frac{\partial \varphi_h}{\partial x_s} \, dx.
\]

Namely,

\[
a_h(w_h, \varphi_h) = a_h^1(w_h, \varphi_h) + \ldots + a_h^n(w_h, \varphi_h), \tag{18}
\]

\[
a_h^1 \equiv 0,
\]

\[
a_h^2(w_h, \varphi_h) = \sum_{T \in \mathcal{T}_h} |T| \left\{ 2\mu \frac{\partial v_{h,1}}{\partial x_1} \left| \frac{\partial \varphi_{h,2}}{\partial x_2} \right| T + \lambda(\text{div} \, \mathbf{v}_h) \left| \frac{\partial \varphi_{h,2}}{\partial x_2} \right| T \right\},
\]

\[
a_h^3(w_h, \varphi_h) = \sum_{T \in \mathcal{T}_h} |T| \left\{ \mu \left( \frac{\partial v_{h,2}}{\partial x_1} \left| \frac{\partial \varphi_{h,3}}{\partial x_1} \right| T + \frac{\partial v_{h,1}}{\partial x_2} \left| \frac{\partial \varphi_{h,3}}{\partial x_1} \right| T \right) + 2\mu \left| \frac{\partial \varphi_{h,3}}{\partial x_2} \right| T \right\},
\]

\[
a_h^4(w_h, \varphi_h) = \sum_{T \in \mathcal{T}_h} \left\{ \frac{1}{3} |T| \left( \tau_{h,11} \sum_{i=1}^{3} v_{h,1}(Q_i^T) + \tau_{h,12} \sum_{i=1}^{3} v_{h,2}(Q_i^T) \right) \frac{\partial \varphi_{h,4}}{\partial x_1} \right| T + \frac{1}{3} |T| \left( \tau_{h,21} \sum_{i=1}^{3} v_{h,1}(Q_i^T) + \tau_{h,22} \sum_{i=1}^{3} v_{h,2}(Q_i^T) \right) \frac{\partial \varphi_{h,4}}{\partial x_2} \right| T + \frac{k |T| \sum_{j=1}^{2} \frac{\partial \theta_h}{\partial x_j} \left| \frac{\partial \varphi_{h,4}}{\partial x_1} \right| T \right\},
\]

\[
\tau_{h,rs}|T| = \frac{1}{2} \left( \frac{\partial v_{h,r}}{\partial x_s} + \frac{\partial v_{h,s}}{\partial x_r} \right) \bigg|_T = \text{const.}
\]

By \( v_{h,s} \) and \( \theta_h \) we denote the functions from the space \( X_h \) approximating the velocity components and temperature. Moreover, \( b_h \) representing the approximation of convective terms reads

\[
b_h(w_h, \varphi_h) = \sum_{i \in J} \varphi_h(Q_i) \sum_{j \in S(i)} \sum_{a=1}^{\beta_{ij}} H(w_h(Q_i), w_h(Q_j), n_{ij}^a) \ell_{ij}^a, \quad w_h, \varphi_h \in X_h. \tag{19}
\]

As \( H \) we use here the well-known Osher-Solomon numerical flux (cf. [7], [9]).

From (16) we see that the used scheme is fully explicit. The reason is its simple algorithmization. However, its application is conditioned by the use of a suitable stability condition. Namely, the following condition has been used in practical computations:
\[ \max \left\{ \max_{i \in J} \frac{\tau_k}{|\partial D_i|} \left( \max_{j \in S(i), \alpha = 1, \ldots, \beta(i,j)} \rho(\mathcal{P}(w_k, n_{ij})) \right) \right\}, \]

\[ \max_{T \in T_h} \frac{3}{4 \sigma_T} \frac{h_T}{|T|} \max(\mu, k) \leq \text{CFL} \approx 0.85, \]

where \( \mathcal{P}(w, n) = \sum_{s=1}^{2} (D f_s(w)/Dw) n_s \), \( \rho(\mathcal{P}) \) is spectral radius of the matrix \( \mathcal{P} \), \( h \) is the length of the maximal side in \( T_h \) and \( \sigma = \min_{T \in T_h} \sigma_T \), \( \sigma_T \) is radius of the largest circle inscribed into \( T \). Condition (20) is obtained on the basis of linearization and in analogy with a scalar problem.

The numerical analyses of this method was performed for the scalar problem: Find \( u : Q_T \rightarrow \mathbb{R} \), \( u = u(x,t), \) \( x \in \Omega, \ t \in [0,T] \), such that

\[ \frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} - \nu \Delta u = g \text{ in } Q_T, \]

where \( \nu > 0 \) is a given constant and \( f_s : \mathbb{R} \rightarrow \mathbb{R} \), \( s = 1, 2 \), \( g : Q_T \rightarrow \mathbb{R} \), \( u^0 : \Omega \rightarrow \mathbb{R} \) are given functions. The homogenous Dirichlet boundary condition is given. The proof of the convergence of the combined barycentric finite volume and nonconforming finite element method for the problem (21) was done in [1].

4 THEORETICAL BACKGROUND OF AMA

We slightly extend the results presented in [5], namely the theoretical background of the anisotropic mesh adaptation (AMA). Let \( \Omega \subset \mathbb{R}^2 \) and \( T_h = \{T_i\}_{T_i \in T_h} \) be a triangulation of \( \Omega \) with standard properties from the finite element method with \( h \) denoting the step of mesh.

Definition 1 Let \( u \) be an exact solution of a given problem considered in the computational domain \( \Omega \) and \( u_h \) be its piecewise linear approximation. The interpolation error is given as in [2] by

\[ E_I(x,y) = |u(x,y) - u_h(x,y)|, \ (x,y) \in \Omega. \]

Our aim is to construct a triangulation \( T_h \) of \( \Omega \) so that the interpolation error \( E_I(x,y) \) is smaller than a given tolerance for each \( (x,y) \in \Omega \) and on the other hand the number of elements of triangulation is not too high.

Definition 2 Let \( u \in C^2(\overline{\Omega}) \) and \( T_h = \{T_i\}_{T_i \in T_h} \) be a triangulation of \( \Omega \). Let \( (x_i, y_i) \) be the center of gravity of \( T_i \in T_h \). We say that \( T_h \) is a \( \omega \)-triangulation if the uniquely defined function \( u_h \), \( u_h \in L^2(\Omega) \), \( u_h|_T \) is linear for each \( T \in T_h \) such that

\[ u(x_i, y_i) = u_h(x_i, y_i) \quad \forall T_i \in T_h, \]

\[ \nabla u(x_i, y_i) = \nabla u_h(x_i, y_i) \quad \forall T_i \in T_h, \]
satisfies the condition
\[ \max_{(x,y)\in\Omega_h} |u(x,y) - u_h(x,y)| \leq \omega. \] (24)

**Remark 1** In general, \( u_h \notin C(\Omega) \), i.e. it represents the discontinuous finite elements.

Without proof we present the two following Lemmas:

**Lemma 1** Let \( \Omega \in \mathbb{R}^2 \) be a computational domain and \( h > 0 \). Then there exists a triangulation \( T_h \) of \( \Omega \) with the step of mesh \( h \).

**Lemma 2** Let \( u \in C^2(\Omega) \) and \( \omega > 0 \). Then there exists at least one \( \omega \)-triangulation.

**Definition 3** Let \( u \in C^2(\Omega) \). We say that \( T_{h,\text{opt}} \) is the \( \omega \)-optimal triangulation if \( T_{h,\text{opt}} \) is \( \omega \)-triangulation and
\[ \text{Card}(T_{h,\text{opt}}) = \min_{T_h, \omega-\text{optimal}} \text{Card}(T_h), \] (25)
where \( \text{Card}(T_h) \) denotes the number of elements of \( T_h \).

**Theorem 1** Let \( u \in C^2(\Omega) \). Then the \( \omega \)-optimal triangulation always exists.

Proof. It follows from Theorem 2, that the set of \( \omega \)-triangulations is not empty. As \( \text{Card}(T_h) \) gives only integer values, Definition 3 implies the existence of \( \omega \)-optimal triangulation.

**Remark 2** The motivation of Definition 3 is the following. We want to receive a solution whose interpolation error is smaller than a given tolerance using the smallest possible number of elements of triangulation (\( \approx \) CPU time). But Definition 3 is not suitable for the practical use.

Let \( u \in C^2(\Omega) \) and its Hessian matrix \( H \) given by
\[ H(x,y) = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix} \] (x,y), \quad (x,y) \in \Omega \] (26)
be symmetric and positively definite. Let \( T_h \) be an \( \omega \)-triangulation and \( u_h \in L^2(\Omega) \) a function from Definition 2 defined by (23). Let \( (x_i, y_i) \) be the center of gravity of \( T_i \in T_h \) and \( (x,y) \in T_i \). Then, using the Taylor series expansion at \( (x_i, y_i) \in \Omega \), we have
\[ u(x,y) = u(x_i, y_i) + \frac{\partial u(x_i, y_i)}{\partial x}(x-x_i) + \frac{\partial u(x_i, y_i)}{\partial y}(y-y_i) + \] \[ + \frac{1}{2} \left[ \frac{\partial^2 u(x_i, y_i)}{\partial x^2} (x-x_i)^2 + 2 \frac{\partial^2 u(x_i, y_i)}{\partial x \partial y} (x-x_i)(y-y_i) + \frac{\partial^2 u(x_i, y_i)}{\partial y^2} (y-y_i)^2 \right] + \] \[ + \ o(|x-x_i, y-y_i|^2). \]
As
\[ u_h|_{T_i}(x, y) = u(x_i, y_i) + \frac{\partial u(x_i, y_i)}{\partial x}(x - x_i) + \frac{\partial u(x_i, y_i)}{\partial y}(y - y_i), \quad (x, y) \in T_i, \quad (27) \]
we have (omitting the terms of order \( o(|x - x_i, y - y_i|^2) \))
\[ e_1, |T_i(x, y) = |u(x, y) - u_h|_{T_i}(x, y)| \approx \]
\[ \approx \frac{1}{2} \left| \frac{\partial^2 u(x_i, y_i)}{\partial x^2}(x - x_i)^2 + 2 \frac{\partial^2 u(x_i, y_i)}{\partial x \partial y}(x - x_i)(y - y_i) + \frac{\partial^2 u(x_i, y_i)}{\partial y^2}(y - y_i)^2 \right|, \quad (28) \]
which is valid for \((x, y) \in T_i\). As \( T_h \) is the \( \omega \)-triangulation we have
\[ \left| \frac{\partial^2 u(x_i, y_i)}{\partial x^2}(x - x_i)^2 + 2 \frac{\partial^2 u(x_i, y_i)}{\partial x \partial y}(x - x_i)(y - y_i) + \frac{\partial^2 u(x_i, y_i)}{\partial y^2}(y - y_i)^2 \right| \leq 2\omega, \quad (29) \]
which can be rewritten as
\[ \left( x - x_i, y - y_i \right) \left( \begin{array}{cc} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} \end{array} \right)_{(x_i, y_i)} \left( x - x_i, y - y_i \right) \leq 2\omega \quad (30) \]
as the Hessian matrix from (30) is positive definite. Under this assumption the points \((x, y)\) satisfying the relation (30) form an ellipse \( \epsilon_i(\omega) \subset \mathbb{R}^2 \). From this it follows that the interpolation error \( E_I(x, y) \) given by (22) is less or equal to \( \omega \) for all \((x, y) \in \epsilon_i(\omega)\). Furthermore, if \( T \) is such a triangle that \( T \subset \epsilon(\omega) \), then \( E_I(x, y) \leq \omega \) for all \((x, y) \in T\).

**Definition 4** Let \( u \in C^2(\Omega) \) be a function with symmetric and positively definite Hessian matrix, \( T_h \) an \( \omega \)-triangulation, \( T_i \subset T_h \) and \( \epsilon_i(\omega) \) an ellipse given by (30). We say that \( T_i^0 \) is an optimal triangle to \( T_i \) if
\[ T_i^0 \subset \epsilon_i(\omega) \quad (31) \]
and
\[ |T_i^0| \geq |T| \quad \text{for all } T \in \epsilon_i(\omega), \quad (32) \]

**Lemma 3** Let \( u \in C^2(\Omega) \) be a function with symmetric and positively definite Hessian matrix, \( T_h \) an \( \omega \)-triangulation and \( T_i \subset T_h \). Then there exist four different optimal triangles with the identical center of gravity, whose position is in the center of \( \epsilon_i(\omega) \).

Proof. See [6].

**Definition 5** Let \( M \) be a \( 2 \times 2 \) symmetric and positively definite matrix and \( u \in \mathbb{R}^2 \). We define the norm of the vector \( u \) corresponding to the matrix \( M \) as
\[ \|u\|_M \equiv \left( uMu^T \right)^{\frac{1}{2}}. \quad (33) \]

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If $M = I$ (= identical matrix) then $\|u\|_M = |u|$, where $|u|$ is the Euclidean norm of $u$.

**Lemma 4** Let $u \in C^2(\Omega)$ be a function with symmetric and positively definite Hessian matrix, $T_h$ a triangulation of $\Omega$ and $T_i \in T_h$ with center of gravity $(x_i, y_i)$. Let $V_i^j$, $j = 1, 2, 3$ be vertexes of $T_i$. Then $T_i$ is the optimal triangle if and only if

$$\|V_i^1 - V_i^2\|_{H(\{u(x_i, y_i)\})} = \|V_i^2 - V_i^3\|_{H(\{u(x_i, y_i)\})} = \|V_i^3 - V_i^1\|_{H(\{u(x_i, y_i)\})} = 3 \cdot 2\omega. \quad (34)$$

Proof. See [6].

**Remark 3** Definition 5 and Lemma 4 give to us a more suitable instigation for the practical construction of the optimal triangulation. In Section 5.1, we define an optimal triangulation as a triangulation, whose edges satisfy a relation similar to (34). As each nonboundary edge of triangulation is a common side of two triangles, there are two Hessians matrixes, which can be used in (34). So that we pass from the center of gravities associated Hessians matrixes to the vertex associated Hessians matrixes.

5 Anisotropic Mesh Adaptation Method

5.1 Optimal triangulation

Let $M = I (= a unit matrix)$, then the equation

$$(x, y)^T M (x, y) = 2\omega, \quad (x, y) \in R^2 \quad (35)$$

defines a circle with the radius $\sqrt{2\omega}$. We consider $\omega$ as a scaling parameter and we chose $\omega = 1/2$ in the following, so that (35) defines a unit circle.

**Definition 6** Let $T$ be a triangulation of the computational domain $\Omega$. Let $M_j$ be a symmetric positive definite $2 \times 2$ matrix defined for each node $P_j$, $j \in J (= an index set)$ of the triangulation $T$. Then for each edge of the triangulation $S_k$, $k \in K (= an index set)$, we put $M_k := \frac{1}{2}(M_{k_1} + M_{k_2})$ where $P_{k_1}, P_{k_2}$ are the initial and final points forming $S_k$, and $M_{k_1}, M_{k_2}$ are the corresponding matrices. Now, we say that the mesh $T$ is optimal, if the norm of $S_k$ corresponding to $M_k$ is equal to $\sqrt{3}$ for all $k \in K$ (compare with (34)), i.e.

$$\|S_k\|_{M_k} = \sqrt{3} \quad \forall k \in K. \quad (36)$$

**Remark 4** The matrix $M$ can be interpreted as an error matrix and the norm $\|u\|_M$ as an error of solution over the edge $u$ (see [3], [8]). It means that the error (considered as $\|\cdot\|_M$) for each edge of $T$ is the same, i.e., it is uniformly distributed over the triangulation.

It is easy to show [6], that the optimal triangulation exists only for special cases. So that we have to define a parameter $Q_T$, which measures how the triangulation $T$ is close to the optimal one.
Definition 7 Let $T$ be a triangulation of $\Omega$ and $M_j$ be a symmetric positive definite $2 \times 2$ matrix defined for each node $P_j$, $j \in J$ of $T$.

Then the quality parameter of a triangle $T_n \in \mathcal{T}$ is given by
\[
q_{T_n} = \sum_{l=1}^{3} \left( \|u_{nl}\|_{M_{nl}} - \sqrt{3} \right)^2,
\]
(37)
where $\|u_{nl}\|_{M_{nl}}$, $l = 1, 2, 3$, are the norms of the sides of the triangle $T_n$, and the quality parameter $Q_T$ of the triangulation $\mathcal{T}$ by
\[
Q_T = \frac{1}{\text{nelem}} \sum_{T_n \in \mathcal{T}} q_{T_n},
\]
(38)
where $q_{T_n}$ is the quality parameter of triangle $T_n$ and $\text{nelem}$ is a number of triangles in $\mathcal{T}$.

Remark 5 It is clear that $Q_T \geq 0$ always and if the mesh $\mathcal{T}$ is optimal then $Q_T = 0$. We will construct a mesh for which the quality parameter $Q_T$ is minimal.

5.2 Optimization of the mesh

Let $\Omega$ denote a computational domain with the boundary $\partial \Omega_h$. Let us suppose that we have a triangulation $\mathcal{T}_{\text{int}}$ of $\Omega$ with matrices $M_j$, $j \in J$ ($M_j$ symmetric and positive definite). We want to find a new triangulation $\mathcal{T}_{\text{new}}$ such that $Q_{\mathcal{T}_{\text{new}}}$ is minimal from all possible triangulation (for given $M_j$, $j \in J$). To obtain $\mathcal{T}_{\text{new}}$, an iterative process is used. Each iteration consists of a certain number of the following local operations (see [5], [6]):

- Adding a node ($A$) in the center of an edge,
- Removing an edge ($R$),
- Swapping ($S$) the diagonal of the quadrilateral formed by any pair of adjacent elements,
- Moving a node ($M$) into a new more suitable position.

5.3 Generation of the matrix $M$

There is still an open question how to define matrixes $M_i$, $i \in J$. This is the crucial point of AMA because the quality of triangulation strongly depends on $M_i$, $i \in J$.

Let $H_i$, $i \in J$, be a Hessian of a function $u \in C^2(\Omega)$, which is a solution of a given problem. We decompose $H_i$ in the following way:
\[
H_i = \left( \begin{array}{cc}
\frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\
\frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2}
\end{array} \right)_{P_i} = R_i \begin{pmatrix}
\Lambda_{i1} & 0 \\
0 & \Lambda_{i2}
\end{pmatrix} R_i^{-1} \quad i \in J.
\]
(39)
The matrices $H_i$ are not suitable as the Riemann metric, because they are not positive definite. Therefore, we put

$$H_i = R_i \begin{pmatrix} |\Lambda_{i1}| & 0 \\ 0 & |\Lambda_{i2}| \end{pmatrix} R_i^{-1} \quad i \in J. \quad (40)$$

The matrices $H_i$, $i \in J$ are symmetric and positive definite. It follows from (39) and (40), that if $u$ is a polynomial of degree less than 2 then $H_i = 0$, $i \in J$. For this case we require to have an uniform triangulation (i.e. $M_i = cI$ for all $i \in J$). This and the numerical experiments lead us to put

$$M_i = c[I + \alpha(||H_i||)H_i] \quad i \in J, \quad (41)$$

where $c > 0$ is a constant and $\alpha(||H_i||)$ is a function of the norms of the matrix $H_i$. We put $||H|| = \max_{i,j=1,2} |h_{ij}|$, where $h_{ij}$, $i, j = 1, 2$ are the elements of $H$. The first term in the square bracket of (41) guarantees, that the matrixes $M_i$ are regular for all functions $u$. The determination of $c$ and $\alpha$ can be found in [6].

### 5.4 Special aspects for viscous flow

The simulation of viscous flow requires also a special type of mesh adaptation near impermeable walls and at the vicinity of wakes. To ensure the aspect ration of the adapted triangulation for viscous flow, the size of a triangle adjacent to an impermeable wall in the direction perpendicular to the wall should be

$$h_Y \approx \frac{1}{\sqrt{Re}}, \quad (42)$$

where $Re$ is the Reynolds number and $h_Y$ is the size of the triangle in the direction perpendicular to the wall, see Fig. 3. On the other hand the size of this triangle in the direction parallel with the wall $h_X$ is restricted only by the minimal angle condition required e.q. for FEM. Therefore we put

$$h_X = c_B^\alpha h_Y \tan \alpha_0, \quad (43)$$

where $\alpha_0$ is the given minimal angle value and $0 < c_B^\alpha < 1$ is an ensured constant. To keep the regularity of the triangulation we put

$$h_Y = \frac{1}{\sqrt{Re}} + c_B^\beta r_y, \quad (44)$$

where $r_y$ is the distance from the impermeable wall and $c_B^\beta$ is a suitable constant. Then the desired matrix $M_B$ has the form

$$M_B = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \frac{1}{h_X} & 0 \\ 0 & \frac{1}{h_Y} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (45)$$
where $\theta$ is the angle between direction parallel with the wall and the axis $x$ and $h_X$ and $h_Y$ are given by (43) and (44), respectively.

We use a similar approach for nodes in vicinity of a wake. The relations (43) and (44) rest the same, but instead of the constants $c_\alpha^B$ and $c_\alpha^B$ we put $c_\alpha^W$ and $c_\alpha^W$, respectively.

Now we have in each node of $\mathcal{T}$ two matrix $M$ (given by (41) and $M^B$ (given by (45). Therefore we use the *intersection of metric* [4] which is illustrated in Fig. 4 to guaranty the validity of both effects:

- bound of the interpolation error (matrix $M$)
- preserving the aspect ration (matrix $M^B$)

\[ h_X \]

**Figure 3:** The size $h_X$ and $h_Y$ of the triangle.

\[ h_Y \]

\[ M \]
\[ M^B \]

**Figure 4:** Intersection of two metrics.

The new metric $(M, M^B)$ is given by ellipse which is the subset of $M$ and $M^B$ and has the maximal possible areas. We apply the transformation of coordinates to the system in
which $M$ and $M^B$ are diagonal, i.e. let $P$ be a matrix such that

$$\tilde{M} = P^T M P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \tilde{M}^B = P^T M^B P = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix},$$  \hspace{1cm} (46)

where $P^T$ denotes the matrix transpose to $P$. Then putting

$$(M, M^B) = (P^{-1})^T \begin{pmatrix} \max(\lambda_1, \mu_1) & 0 \\ 0 & \max(\lambda_2, \mu_2) \end{pmatrix} P^{-1}$$  \hspace{1cm} (47)

we obtain the desired matrix. Here $P^{-1}$ means the inverse matrix to $P$.

6 NUMERICAL RESULTS

We applied our computational method and anisotropic mesh adaptation technique for several problems of CFD. The steady state solution was achieved by a time marching method. All triangulation were generated by AMA technique (http://www.ms.mff.cuni.cz/~dolejsi/angener/angener.htm). We present the transonic flow past a NACA0012 profile. The inlet Mach number was 2.0 and the Reynolds number $Re = 10^5$ and $Re = 10^6$. Fig. 5 and Fig. 6 show the color Mach number distribution with the corresponding barycentric finite volume mesh for the Reynolds number $10^5$ and $10^6$, respectively. We observe the thinner boundary layers for $Re = 10^6$ than for $Re = 10^5$ at the map of Mach number distribution as well as at the graph of triangulation.

The second example concerns the flow past a cascade of profiles modeling the flow in steam turbines. Fig. 7 shows the detail of one period of the color map of the Mach number near the trailing edge and Fig. 8 shows the corresponding barycentric finite volume mesh.

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REFERENCES


Figure 5: NACA0012 – color map of Mach number for $Re = 10^5$ and the corresponding triangulation
Figure 6: NACA0012 – color map of Mach number for $Re = 10^6$ and the corresponding triangulation
Figure 7: Cascade of profiles – color map of Mach number
Figure 8: Cascade of profiles – corresponding triangulation