NUMERICAL MODELING OF THE TIMPANI

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Abstract. We propose a time-domain fictitious domain method for the numerical simulation of a kettle drum (or timpani). Such an instrument is made of a circular elastic membrane stretched over an air cavity enclosed by a rigid shell and set into vibrations by the impact of a mallet. The mathematical model couples a 3D linear wave equation for the outside and inside fluids, a 2D wave equation with viscous damping terms on the membrane and a nonlinear differential equation for the mallet. The originality of the so-called fictitious domain approach is to be based on a formulation which does not distinguish explicitly the external fluid from the internal one. This formulation rests upon a velocity-pressure formulation of the acoustic wave equation in the air and a variational mixed formulation of the full problem. The boundary and fluid-structure interface conditions are taken into account in a weak way via a Lagrange multiplier which coincides with the pressure jump across the membrane and the shell. The numerical approximation is based on standard and mixed finite elements for the spacial discretization and centered finite differences for time discretization.
1 The continuous problem

Kettle drums (or timpani) are made of a circular elastic membrane $\Sigma$ stretched over an air cavity enclosed by a rigid shell $C$ and set into vibrations by the impact of a mallet. $\Gamma = \Sigma \cup C$ is the boundary of the instrument and $\vec{n}$ is the outside unit normal vector to $\Gamma$. It separates the interior domain $\Omega_i$ from the exterior one $\Omega_e$ (see figure 1). The motion of the membrane is coupled with both the external and internal sound pressure fields.

The unknowns of the model are:

- $w(x, t)$ the vertical displacement on the membrane ($x \in \Sigma$),
- $u(t)$ the vertical abscissa of the center of gravity of the mallet,
- $F(t)$ the force of interaction between the mallet and the membrane,
- $p_e(x, t)$ the exterior pressure ($x \in \Omega_e$),
- $v_e(x, t)$ the velocity in the exterior fluid ($x \in \Omega_e$),
- $p_i(x, t)$ the interior pressure ($x \in \Omega_i$),
- $v_e(x, t)$ the velocity in the interior fluid ($x \in \Omega_i$).

The equations of the model are taken from classical continuum mechanics together with
the small deformations and displacements assumptions which justify a linearization:

\[ \frac{m}{dt^2} \frac{du}{dt} = F(t) \quad \text{in } \mathbb{R}^+ \quad (1a) \]

\[ F(t) = K [(a - u(t) + W(t))^+ ]^p, \quad W(t) = \int_{\Sigma} g(x)w(x, t) \, d\sigma \quad \text{in } \mathbb{R}^+ \quad (1b) \]

\[ \sigma \frac{\partial^2 w}{\partial t^2} - \operatorname{div}(T \nabla \left( w + \eta \frac{\partial w}{\partial t} \right)) = -F(t)g - [p]_{\Sigma} \quad \text{in } \Sigma \times \mathbb{R}^+ \quad (1c) \]

\[ \mu_a \frac{\partial p_j}{\partial t} + \operatorname{div} v_j = 0, \quad j = e, i \quad \text{in } \Omega_j \times \mathbb{R}^+ \quad (1d) \]

\[ \rho_a \frac{\partial v_j}{\partial t} + \nabla p_j = 0, \quad j = e, i \quad \text{in } \Omega_j \times \mathbb{R}^+ \quad (1e) \]

with boundary conditions:

\[ w(x, t) = 0 \quad \text{on } \partial \Sigma \times \mathbb{R}^+ \quad (1f) \]

\[ v_j(x, t) \cdot \vec{n} = \frac{\partial w}{\partial t}(x, t) \quad \text{on } \Sigma \times \mathbb{R}^+ \quad j = e, i \quad (1g) \]

\[ v_j(x, t) \cdot \vec{n} = 0 \quad \text{on } C \times \mathbb{R}^+ \quad j = e, i \quad (1h) \]

and initial conditions:

\[ u_0 = u(0) = a \quad u_1 = \frac{du}{dt}(0) = -v_0 \quad (1i) \]

\[ w_0(x) = w(x, 0) = 0 \quad \text{on } \Sigma \quad (1j) \]

\[ w_1(x) = \frac{dw}{dt}(x, 0) = 0 \quad \text{on } \Sigma \quad (1k) \]

\[ p_j^0(x) = p_j(x, 0) = 0 \quad \text{in } \Omega_j \quad j = e, i \quad (1l) \]

\[ v_j^0(x) = v_j(x, 0) = 0 \quad \text{in } \Omega_j \quad j = e, i \quad (1m) \]

In the previous equations, note that

- (1a) and (1b) are the equations for the movement of the mallet whose behaviour is nonlinear (in practice \(2 < p < 3\)), \(m\) representing the mass of the mallet and \(a\) its radius at rest. The contact is supposed to be quasi-punctual and the position \(W(t)\) of the point of impact is calculated as a weighted mean value of \(w(x, t)\) around this point, thanks to the introduction of a regularized Dirac function \(g(x)\) (positive, with mass 1). The quantity \(a - u(t) + W(t)\) represents, when it is positive, the compression of the mallet.
• (1c) is the 2D wave equation (div and \(\nabla\) are surfacic operators) for the displacement of the membrane. \(T\) denotes the (possibly variable) tension of the membrane, \(\sigma\) its surfacic density. The velocity \(c = (T/\sigma)^{1/2}\) is about 100 m/s. \(\eta\) is an attenuation coefficient of viscoelastic nature. The right hand side is made of the various external forces acting on the membrane: the action of the mallet \(-F(t)g\) and the jump of pressure across \(\Sigma\), namely \([p]\Sigma = p_e - p_i\).

• (1d) and (1e) are the equations for the sound radiation. \(\rho_a\) is the air density. The sound velocity \(c_a = (\mu_a/\rho_a)^{1/2}\) in the air is about 330 m/s.

• (1f) expresses that the membrane is fixed along its boundary, (1g) is the standard fluid-structure interface condition (continuity of the normal displacement) and (1h) expresses the fact that the cavity \(C\) does not move.

• (1i) expresses that at time \(t = 0\) the mallet strikes the membrane with velocity \(v_0 > 0\) and the last four equations traduce the fact that the medium is initially at rest.

We denote by (1) the problem made of equations (1a) to (1m). The mathematical analysis of (1) is not too difficult. One can find for instance in [?] the following result:

**Theorem 1** If the initial data of the problem satisfy:

\[
\begin{aligned}
(w_0, w_1) & \in H_0^1(\Sigma) \times H_0^1(\Sigma) \\
\text{div}_\Gamma \left( T \nabla_\Gamma \left( w_0 + \eta w_1 \right) \right) & \in L^2(\Sigma) \\
(p_0^e, p_0^i) & \in L^2(\Omega_e) \times L^2(\Omega_i) \\
(v_0^e, v_0^i) & \in L^2(\Omega_e)^3 \times L^2(\Omega_i)^3
\end{aligned}
\]

problem (1) admits a unique strong solution with the regularity:

\[
\begin{aligned}
u & \in C^3(\mathbb{R}^+; \mathbb{R}), \quad F \in C^1(\mathbb{R}^+; \mathbb{R}) \\
 w & \in C^2(\mathbb{R}^+; L^2(\Sigma)) \cap C^1(\mathbb{R}^+, H_0^1(\Sigma)) \\
\text{div}_\Gamma \left( T \nabla_\Gamma \left( w + \eta \frac{\partial w}{\partial t} \right) \right) & \in C^0(\mathbb{R}^+, L^2(\Sigma)) \\
p_j & \in C^1(\mathbb{R}^+; L^2(\Omega_j)) \cap C^0(\mathbb{R}^+; H^1(\Omega_j)) \quad j = e, i \\
v_j & \in C^1(\mathbb{R}^+; L^2(\Omega_j)^3) \cap C^0(\mathbb{R}^+; H(\text{div}, \Omega_j)) \quad j = e, i
\end{aligned}
\]

which satisfies the energy identity:

\[
\frac{dE}{dt}(t) + \eta \int_\Sigma T(x) \left| \nabla_\Gamma \frac{\partial w}{\partial t}(x, t) \right|^2 d\sigma = 0
\]
where the total energy of the system is defined as:

\[
E(t) = \frac{1}{2} \left| \frac{du}{dt}(t) \right|^2 + K \left( \frac{(a - u(t) + W(t))^+}{p + 1} \right) + \frac{1}{2} \int_\Sigma \sigma(x) \left| \frac{\partial w}{\partial t}(x,t) \right|^2 d\sigma + \frac{1}{2} \int T(x) \left| \nabla_\Gamma w(x,t) \right|^2 d\sigma + \sum_{j=e,i} \int_{\Omega_j} \left( \mu_a |p_j(x,t)|^2 + \rho_a |v_j(x,t)|^2 \right) dx
\]  

(4)

2 The fictitious domain formulation

The originality of this approach consists in ignoring (in some sense) the presence of the boundary of the instrument by considering the pressure \( p \) and the velocity field \( v \) as defined in \( \Omega = \mathbb{R}^3 \): one does not distinguish any longer inside and outside pressures (namely \( p \) and \( p_e \)) or velocities (namely \( v_i \) and \( v_e \)). As a consequence of boundary or transmission conditions on \( C \) and \( \Sigma \), the normal velocity is continuous across \( \Gamma \). Therefore, the velocity field \( v \) can be sought in the space:

\[
v \in C^0(\mathbb{R}^3; H(\text{div};\Omega)),
\]

and satisfies in the sense of distributions:

\[
\mu_a \frac{\partial p}{\partial t} + \text{div} v = 0, \quad \text{in } \Omega.
\]  

(5)

As the pressure field is a priori discontinuous across \( \Gamma \) and \( \Sigma \), the equation for \( p \) in \( \Omega \) is:

\[
\rho_a \frac{\partial v}{\partial t} + \nabla p = \lambda \vec{n} \delta_\Gamma, \quad \text{in } \Omega,
\]  

(6)

where \( \delta_\Gamma \) is the Dirac distribution on \( \Gamma \) and:

\[
\lambda = [p]_\Gamma \in C^0(\mathbb{R}^3; H^1(\Gamma)),
\]  

(7)

which will be introduced as an additional unknown. Then, the equation for the membrane can be rewritten:

\[
\sigma \frac{\partial^2 w}{\partial t^2} - \text{div}_\Gamma \left( T \nabla_\Gamma \left( w + \eta \frac{\partial w}{\partial t} \right) \right) = -f(t)g - \lambda |\Sigma|, \quad \text{on } \Sigma,
\]  

(8)

while the transmission and boundary conditions on \( C \) and \( \Sigma \) can be rewritten as, if \( \chi_\Sigma \) denotes the characteristic function of \( \Sigma \):

\[
\chi_\Sigma \frac{\partial w}{\partial t} + v \cdot n = 0, \quad \text{on } \Gamma.
\]  

(9)
The “fictitious domain” variational formulation is obtained by multiplying (5) by a test function \( p^* \) in \( L^2(\Omega) \), (6) by a test function \( v^* \) in \( H(\text{div};\Omega) \), (8) by a test function \( w^* \) in \( H^1_0(\Sigma) \) and finally (9) by a test function \( \lambda^* \) in \( H^\frac{1}{2}(\Gamma) \). After integration over \( \Omega, \Sigma \) and \( \Gamma \), we obtain the following formulation (the details are left to the reader):

Find

\[
\begin{align*}
\forall w^* \in \mathcal{W}, & \quad \frac{d^2}{dt^2} m(w(t), w^*) + r(w(t), w^*) \\
& \quad + \eta \frac{d}{dt} r(w(t), w^*) + b_\Sigma(w^*, \lambda(t)) = -F(t) (g, w^*), \quad (10a) \\
\forall p^* \in \mathcal{P}, & \quad \frac{d}{dt} m_p(p(t), p^*) + d(v(t), p^*) = 0, \quad (10b) \\
\forall v^* \in \mathcal{V}, & \quad \frac{d}{dt} m_v(v(t), v^*) - d(v^*, p(t)) - b_\Gamma(v^*, \lambda(t)) = 0, \quad (10c) \\
\forall \lambda^* \in \mathcal{L}, & \quad \frac{d}{dt} b_\Sigma(w(t), \lambda^*) - b_\Gamma(v(t), \lambda^*) = 0, \quad (10d)
\end{align*}
\]

with initial conditions:

\[
\begin{align*}
\forall & \quad w(x, 0) = 0 \text{ on } \Sigma, \quad \frac{dw}{dt}(x, 0) = 0, \text{ on } \Sigma, \quad (10e) \\
\forall & \quad p(x, 0) = 0 \text{ in } \Omega, \quad v(x, 0) = 0, \text{ in } \Omega, \quad (10f)
\end{align*}
\]

where \( F(t) \) is determined by equations (1a) and (1b) and where we have defined the bilinear forms:

\[
\begin{align*}
\forall (w, w^*) \in L^2(\Sigma) \times L^2(\Sigma), & \quad m(w, w^*) = \int_\Sigma \sigma w(x) w^*(x) \, d\sigma, \quad (11a) \\
\forall (p, p^*) \in L^2(\Omega) \times L^2(\Omega), & \quad m_p(p, p^*) = \int_{\mathbb{R}^3} \mu_a p(x) p^*(x) \, dx, \quad (11b) \\
\forall (v, v^*) \in H(\text{div};\Omega) \times H(\text{div};\Omega), & \quad m_v(v, v^*) = \int_{\mathbb{R}^3} \rho_a v(x) \cdot v^*(x) \, dx, \quad (11c) \\
\forall (w, w^*) \in H^1_0(\Sigma) \times H^1_0(\Sigma), & \quad r(w, w^*) = \int_\Sigma T(x) \nabla_{\Gamma} w(x) \cdot \nabla_{\Gamma} w^*(x) \, d\sigma, \quad (12a) \\
\forall (v, p) \in H(\text{div};\Omega) \times L^2(\Omega), & \quad d(v, p) = \int_{\mathbb{R}^3} \text{div} \, v(x) p(x) \, dx, \quad (12b) \\
\forall (w, \lambda) \in L^2(\Sigma) \times H^\frac{1}{2}(\Gamma), & \quad b_\Sigma(w, \lambda) = \int_\Sigma w(x) \lambda(x) \, d\sigma, \quad (12c) \\
\forall (v, \lambda) \in H(\text{div};\Omega) \times H^\frac{1}{2}(\Gamma), & \quad b_\Gamma(v, \lambda) = <v \cdot \vec{n}|_{\Gamma}, \lambda>_\Gamma, \quad (12d)
\end{align*}
\]

where \(<.,.>_\Gamma\) denotes the duality product between \( H^{-\frac{1}{2}}(\Gamma) \) and \( H^{\frac{1}{2}}(\Gamma) \).
3 Numerical approximation

3.1 Space discretization

The space discretization is based on a regular mesh $T_a$ with cubes of side $h_a$ of the domain $\Omega$ for the unknowns $p$ and $v$, a triangular mesh $T_H$ of the boundary $\Gamma$ of stepsize $H$ for the unknown $\lambda$ and a triangular mesh $T_m$ of $\Sigma$ of stepsize $h_m$ for the unknown $w$. More precisely:

- $v_h$ is searched in the subspace $V_h$ of $V$ constructed with standard lowest order mixed finite elements of Raviart-Thomas [?] associated with the regular mesh $T_a$,
- $v_h$ is searched in the subspace $P_h$ of $P$ of piecewise constant functions associated to the same mesh $T_a$,
- $\lambda_h$ is searched in the subspace $L_h$ of $L$ of piecewise linear continuous functions associated with the mesh $T_H$,
- $w_h$ is searched in the subspace $W_h$ of $W$ of piecewise linear continuous functions associated with the mesh $T_m$.

One of the main advantages of the fictitious domain approach is thus to avoid the mesh of 3D structures (such as $\Omega_i$ or $\Omega_e$) and to work with structured data for the 3D unknowns. Moreover, it is easy to obtain explicit schemes for most of the equations, which promotes the speed of calculation.

The semi-discrete problem is written (we omit the initial conditions):

\[
\begin{align*}
\text{Find } & \quad w_h(t) : [0, T] \to W_h, \quad p_h(t) : [0, T] \to P_h \quad \text{such that} \\
& \quad v_h(t) : [0, T] \to V_h, \quad \lambda_h(t) : [0, T] \to L_h \\
& \forall w_h^* \in W_h, \quad \frac{d^2}{dt^2} m(w_h(t), w_h^*) + r(w_h(t), w_h^*) + \eta \frac{d}{dt} r(w_h(t), w_h^*) + b_\Sigma(w_h^*, \lambda_h(t)) = -F(t)(g_h, w_h^*), \\
& \forall p_h^* \in P_h, \quad \frac{d}{dt} m_p(p_h(t), p_h^*) + d(v_h(t), p_h^*) = 0, \\
& \forall v_h^* \in V_h, \quad \frac{d}{dt} m_v(v_h(t), v_h^*) - d(v_h^*, p_h(t)) - b_r(v_h^*, \lambda_h(t)) = 0, \\
& \forall \lambda_h^* \in L_h, \quad \frac{d}{dt} b_\Sigma(w_h(t), \lambda_h^*) - b_r(v_h(t), \lambda_h^*) = 0,
\end{align*}
\]
which leads to the following differential system, after having expanded the unknowns \((v_h, p_h, w_h, \lambda_h)\) in the standard finite element bases of the spaces \(V_h, P_h, W_h\) and \(L_h\) (\(V_h, P_h, W_h\) and \(L_h\) are the vectors of the corresponding degrees of freedom):

\[
\begin{align*}
\mathcal{M} \frac{d^2 W_h}{dt^2} + \mathcal{R} W_h + \eta \mathcal{R} \frac{dW_h}{dt} + B^T \Lambda_h &= -F(t) G_h, \quad (14a) \\
\mathcal{M}_p \frac{dP_h}{dt} + D^T V_h &= 0, \quad (14b) \\
\mathcal{M}_v \frac{dV_h}{dt} - D P_h - B^T \Lambda_h &= 0, \quad (14c) \\
B_v V_h - B_S \frac{dW_h}{dt} &= 0. \quad (14d)
\end{align*}
\]

The positive definite mass matrices \(\mathcal{M}, \mathcal{M}_v, \mathcal{M}_p\) are diagonal if the bilinear forms are computed approximately with an appropriate quadrature formula (mass lumping). The matrix \(D\) represents a discrete 3D-gradient, its transpose \(D^T\) a discrete 3D-divergence. The matrix \(\mathcal{R}\) represents a discrete 2D Laplace operator. The vector \(G_h\) represents a discrete approximation of the function \(g(x)\). The two rectangular matrices \(B_S\) and \(B_v\) couple volume unknowns and surface unknowns. They can be interpreted as discrete trace operators. Both are very sparse. Note that if \(B_S = B_v = 0\), one obtains two decoupled semi-discrete schemes respectively for the 2D membrane wave equation and for the 3D wave equation (written as a first order system) for the sound radiation. Thus, \(B_S\) and \(B_v\) take into account the fluid-structure interaction.

In order to close the system of equations, we have to add the following equations that take into account (1a) and (1b):

\[
\begin{align*}
m \frac{d^2 u}{dt^2} &= F(t), \quad \text{in } \mathbb{R}^+, \quad (15) \\
F(t) &= K \left[(a - \tilde{u}(t))^+\right]^p, \quad \tilde{u}(t) = u(t) - G_h^T W_h(t),
\end{align*}
\]

### 3.2 Time discretization

We use a constant time step \(\Delta t\). In order to get a centered finite difference scheme, \(W_h, P_h, \Lambda_h\) and \(F\) are approximated at times \(t^n = n\Delta t\), \(n \in \mathbb{N}\) while \(V_h\) is approximated at times \(t^{n+1/2} = (n + 1/2)\Delta t\). We use a standard explicit leap-frog scheme for the
discretization of equations (14a) to (14d):

\[
M \frac{W_h^{n+1} - 2W_h^n + W_h^{n-1}}{\Delta t^2} + R W_h^n + \eta R \frac{W_h^{n+1} - W_h^{n-1}}{2\Delta t} + B_\Sigma^T \Lambda_h^n = -G_h F^n, \quad (16a)
\]

\[
M_p \frac{P_h^{n+1} - P_h^n}{\Delta t} + D^T V_h^{n+\frac{1}{2}} = 0, \quad (16b)
\]

\[
M_v \frac{V_h^{n+\frac{1}{2}} - V_h^{n-\frac{1}{2}}}{\Delta t} - D P_h^n - B_v^T \Lambda_h^n = 0, \quad (16c)
\]

\[
B_x \frac{W_h^{n+1} - W_h^n}{\Delta t} - B_v V_h^{n+\frac{1}{2}} = 0, \quad (16d)
\]

while the nonlinear equation for the mallet (15) is discretized implicitly in a conservative form:

\[
\begin{align*}
\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} &= F^n, \quad F^n = -2K \frac{\Psi^{n+\frac{1}{2}} - \Psi^{n-\frac{1}{2}}}{(\tilde{u}^{n+1} - \tilde{u}^{n-1})}, \\
\Psi^{n+\frac{1}{2}} &= \Psi \left( \frac{\tilde{u}^{n+1} + \tilde{u}^n}{2} \right), \quad \Psi(\tilde{u}) = \frac{[(a - \tilde{u})^+]^{p+1}}{p+1}, \quad \tilde{u}^n = u^n - G_h^T W_h^n.
\end{align*}
\]

Note that the function \( \Psi(\tilde{u}) \) is such that:

\[
\Psi'(\tilde{u}) = -\varphi(\tilde{u}), \quad \varphi(\tilde{u}) = [(a - \tilde{u})^+]^p.
\]

so that it is not difficult to prove that this scheme is second order accurate in time (see [?]). The practical implementation deserves some explanation. Suppose that \( W_h^n, P_h^n, V_h^{n-\frac{1}{2}} \) are known (from previous time steps) and that the force term \( F^n \) is also known (we shall explain later how to compute it). Then \( W_h^{n+1}, V_h^{n+\frac{1}{2}} \) and \( \Lambda_h^n \) are computed by solving:

\[
\begin{pmatrix}
\frac{M}{\Delta t^2} + \frac{\eta}{2\Delta t} R & 0 & B_\Sigma^T \\
0 & M_v - B_v^T \Delta t & -B_\Sigma \\
-B_\Sigma & B_v \Delta t & 0
\end{pmatrix}
\begin{pmatrix}
W_h^{n+1} \\
V_h^{n+\frac{1}{2}} \\
\Lambda_h^n
\end{pmatrix}
= \begin{pmatrix}
h_1^n \\
h_2^n + F^n \\
h_3^n
\end{pmatrix}, \quad (18)
\]

where in the right hand side we have defined:

\[
\begin{pmatrix}
h_1^n \\
h_2^n \\
h_3^n
\end{pmatrix}
= \begin{pmatrix}
M \frac{2W_h^n - W_h^{n-1}}{\Delta t^2} - R \left( W_h^n - \eta \frac{W_h^{n-1}}{2\Delta t} \right) \\
M_v V_h^{n-\frac{1}{2}} - \Delta t D P_h^n \\
-B_\Sigma W_h^n
\end{pmatrix},
\]

\[
\begin{pmatrix}
h_1^n \\
h_2^n \\
h_3^n
\end{pmatrix}
= \begin{pmatrix}
M \frac{2W_h^n - W_h^{n-1}}{\Delta t^2} - R \left( W_h^n - \eta \frac{W_h^{n-1}}{2\Delta t} \right) \\
M_v V_h^{n-\frac{1}{2}} - \Delta t D P_h^n \\
-B_\Sigma W_h^n
\end{pmatrix},
\]

\[
\begin{pmatrix}
h_1^n \\
h_2^n \\
h_3^n
\end{pmatrix}

and $P^{n+1}_h$ is computed explicitly via:

$$P^{n+1}_h = P^n_h - \Delta t \, M_p^{-1} \, D^T \, V^{n+\frac{1}{2}}_h.$$  \hspace{1cm} (19)

By linearity, the solution of (18) is written as:

$$
\begin{pmatrix}
W^{n+1}_h \\
V^{n+\frac{1}{2}}_h \\
\Lambda^n_h
\end{pmatrix} =
\begin{pmatrix}
\hat{W}^{n+1}_h \\
\hat{V}^{n+\frac{1}{2}}_h \\
\hat{\Lambda}^n_h
\end{pmatrix} +
F^n 
\begin{pmatrix}
\bar{W}_h \\
\bar{V}_h \\
\bar{\Lambda}_h
\end{pmatrix},
$$

\hspace{1cm} (20)

where $\bar{W}_h$, $\bar{V}_h$ and $\bar{\Lambda}_h$ are computed by:

$$
\begin{pmatrix}
\frac{M}{\Delta t^2} + \frac{\eta}{2\Delta t} R & 0 & B^{rr}_\Sigma \\
0 & M_v & -B^c_v \Delta t \\
-\Sigma & B\Sigma & 0
\end{pmatrix}
\begin{pmatrix}
\bar{W}_h \\
\bar{V}_h \\
\bar{\Lambda}_h
\end{pmatrix} =
\begin{pmatrix}
-G_h \\
0 \\
0
\end{pmatrix},
$$

\hspace{1cm} (21)

and $\hat{W}^{n+1}_h$, $\hat{V}^{n+\frac{1}{2}}_h$ and $\hat{\Lambda}^n_h$ are computed by:

$$
\begin{pmatrix}
\frac{M}{\Delta t^2} + \frac{\eta}{2\Delta t} R & 0 & B^{rr}_\Sigma \\
0 & M_v & -B^c_v \Delta t \\
-\Sigma & B\Sigma & 0
\end{pmatrix}
\begin{pmatrix}
\hat{W}^{n+1}_h \\
\hat{V}^{n+\frac{1}{2}}_h \\
\hat{\Lambda}^n_h
\end{pmatrix} =
\begin{pmatrix}
h^n_1 \\
h^n_2 \\
h^n_3
\end{pmatrix}.
$$

\hspace{1cm} (22)

To complete our presentation, it remains to explain how we solve the systems (21) and (22) and to explain how we compute $F^n$ from (17).

**Resolution of (21) and (22).** Let us consider (21). We shall only explain the case $\eta = 0$. In practice, $\eta$, which is the attenuation coefficient, is very small and the general case is thus treated as a perturbation of the case $\eta = 0$ via a fixed point method (we omit the details, see [?]) for more information). When $\eta = 0$, if we eliminate $\bar{W}_h$ and $\bar{V}_h$, we get:

$$(B_\Sigma \, M^{-1} B^{rr}_\Sigma + B_v \, M^{-1} B^{rr}_v) \, \bar{\Lambda}_h = -\Delta t^2 B^{rr}_\Sigma \, M^{-1} \, G_h.$$  \hspace{1cm} (23)
Once $\overline{\Lambda}_h$ is known, $\overline{W}_h$ and $\overline{V}_h$ are computed explicitly by (remember that $\mathcal{M}$ and $\mathcal{M}_v$ are diagonal thanks to mass lumping):

$$\overline{W}_h = -\Delta t^2 \mathcal{M}^{-1} G_h - \Delta t^2 \mathcal{M}^{-1} B^T \Sigma \overline{\Lambda}_h,$$

$$\overline{V}_h = \mathcal{M}_v^{-1} B^T \overline{\Lambda}_h. \tag{24}$$

Thus, the only implicit step of the computation is the resolution of (23) for the determination of $\overline{\Lambda}_h$. Note that the matrix $B^T \Sigma \mathcal{M}^{-1} B + B^T \mathcal{M}_v^{-1} B^T$ is small (its dimension is the number of degrees of freedom for $\overline{\Lambda}_h$), positive, symmetric and sparse (in particular because $\mathcal{M}$ and $\mathcal{M}_v$ are diagonal after mass lumping). For system (21), we follow the same algorithm, which must be repeated at each time step, namely:

- $\hat{\Lambda}_{h}^{n+1}$ is computed by solving a linear system associated to the matrix $B^T \Sigma \mathcal{M}^{-1} B + B^T \mathcal{M}_v^{-1} B^T$,

- $\hat{V}_{h}^{n+\frac{1}{2}}$ and $\hat{W}_{h}^{n+1}$ are computed explicitly.

**Determination of $F^n$.** From the first equation of (17), we have:

$$F^n = K \Phi \left( \tilde{u}^{n+\frac{1}{2}}, \tilde{u}^{n-\frac{1}{2}} \right), \quad \tilde{u}^{n+\frac{1}{2}} = \frac{\tilde{u}^{n+1} + \tilde{u}^n}{2}, \tag{25}$$

where

$$\Phi(x, y) = -\frac{\Psi(x) - \Psi(y)}{(x - y)} \quad \text{if } x \neq y, \quad = \varphi(x) \quad \text{if } x = y.$$

Assuming that $u^n$ and $u^{n-1}$ are known, $\tilde{u}^{n-\frac{1}{2}}$ is known too and it suffices to find $\tilde{u}^{n+\frac{1}{2}}$ in order to compute $F^n$ and $u^{n-1}$. Defining $W_h^{n+\frac{1}{2}} = (W_h^{n+1} + W_h^n)/2$, we have

$$\tilde{u}^{n+\frac{1}{2}} = u^{n+\frac{1}{2}} - G^T h W_h^{n+\frac{1}{2}}.$$

Thanks to (20) we have

$$2 G^T h W_h^{n+\frac{1}{2}} = \frac{G^T h W_h^{n+1} + G^T h W_h^n}{2} + G^T h \overline{W}_h F^n.$$

Thus, the second equation of (17) leads to

$$2 \tilde{u}^{n+\frac{1}{2}} = -G^T h W_h^{n+1} + 2 u^n + \tilde{u}^n - u^{n-1} + \left( \frac{\Delta t^2}{m} - G^T h \overline{W}_h \right) F^n. \tag{26}$$
Eliminating \( F^n \) between (26) and (25), we obtain a nonlinear equation for \( \tilde{u}^{n+\frac{1}{2}} \)

\[
h^n(\tilde{u}^{n+\frac{1}{2}}) = 0,
\]

where

\[
\begin{align*}
\dot{h}^n(\tilde{u}) &= 2\tilde{u} + a^n + b \Phi(\tilde{u}, \tilde{u}^{n-\frac{1}{2}}), \\
a^n &= G_T^h \hat{W}^{n+1}_h - 2u^n + \tilde{u}_n + u^{n-1}, \\
b &= \left( \frac{\Delta t^2}{m} - G_T^h \hat{W}_h \right).
\end{align*}
\]

Note that since we can compute \( \hat{W}^{n+1}_h \) before \( F^n \) (see (22)), the function \( h^n(\tilde{u}) \) is known.

**Structure of the global algorithm.** Apart from the resolution of (21), the algorithm at each time step is the following:

(i) Apply one step of explicit scheme for the 3D wave equation on the uniform mesh \( T_a \) to determine \( P^{n+1}_h \) and \( \hat{V}^{n+\frac{1}{2}}_h \).

(ii) Apply one step of explicit scheme for the 2D wave equation on the non uniform mesh \( T_m \) to determine \( \hat{W}^{n+1}_h \).

(iii) Solve the system (23) associated to the non uniform mesh \( T_H \) to determine \( \hat{\Lambda}^{n+1}_h \).

(iv) Solve the nonlinear equation (26) to determine \( F^n \).

(v) Apply the correction (20) in order to determine \( V^{n+\frac{1}{2}}_h, W^{n+1}_h \) and \( \Lambda^{n+1}_h \).

The reader will easily convince himself that the computation differs from the decoupled explicit resolution of the 3D wave equation in the air and the 2D membrane wave equation only by the two additional (but cheap) steps (iii) (resolution of a “small” sparse positive symmetric system) and (iv) (resolution of a nonlinear equation in \( \mathbb{R} \)). This makes the fictitious domain method very efficient in practice.

### 3.3 Theoretical issues

**Existence of the solution.** From the algorithm described in the previous section, it is clear that the existence (and uniqueness) of the solution to our discrete scheme is a consequence of the two following properties:

- The matrix \( B_v M^{-1} B_t^\tau + B_r M^{-1} B_r^\tau \) is positive definite. This will be true as soon as \( B_r M^{-1} B_r^\tau \) is itself positive definite (contrary to \( B_v M^{-1} B_v^\tau \) which is not, see \([？]\)).

This is a particular consequence of the well-known uniform discrete inf-sup condition that is written:

\[
\exists k > 0, \forall h > 0, \inf_{\lambda_h \in L_h} \sup_{\tilde{v}_h \in V_h, \tilde{v}_h \neq 0} \frac{b_t(\tilde{v}_h, \lambda_h)}{\|\tilde{v}_h\|_{\text{div}, \Omega} \|\lambda_h\|_{H^{1/2}(\Gamma)}} \geq k. \tag{28}
\]
Such a condition imposes some compatibility between the surface mesh of stepsize $H$ for the Lagrange multiplier $\Lambda_h$ and the volume mesh of stepsize $h_a$ for the volume unknowns $v$ and $p$. More precisely, it is shown in [?] that assuming that the meshes $T_H$ are uniformly regular, i.e.

$$\exists 0 < \nu \leq 1 \text{ such that } \forall K \in T_H, \text{diam}(K) \geq \nu H.$$ 

then, there exists a constant $\alpha > 0$ such that if $H \geq \alpha h_a$, the uniform inf-sup condition (28) holds. This result does not give a numerical value for $\alpha$ however, in practice, $\alpha \simeq 1.1$ is sufficient.

- For each $n \geq 1$, the nonlinear equation (27) is uniquely solvable. In fact, it is easy to prove that, exploiting the fact that the function $\Psi(\tilde{u})$ is decreasing, convex and identically zero for large values of $\tilde{u}$, that the function $h^n(\tilde{u})$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$ provided that $b$ is positive, which is the case since

$$\begin{cases}
-G^T_h \nabla h = \Delta t^2 G^T_h \mathcal{M}^{-1} G_h + \Delta t^2 G^T_h \mathcal{M}^{-1} B^T_v \lambda_h, \\
G^T_h \mathcal{M}^{-1} B^T_v \lambda_h = \Delta t^2 G^T_h \mathcal{M}^{-1} (B^T_S \mathcal{M}^{-1} B^T_v + B^T_v \mathcal{M}^{-1} B^T_v)^{-1} B^T_S \mathcal{M}^{-1} G_h.
\end{cases}$$

**Stability analysis.** It appears that the stability of our numerical scheme is guaranteed provided that the stability conditions for the discrete 2D wave membrane equation, namely:

$$\frac{\Delta t^2}{4} \| R_h \| \leq \gamma < 1, \quad \| R_h \| = \sup_{w_h \in \mathcal{W}_h} \frac{r(w^*_h, w^*_h)}{m(w^*_h, w^*_h)},$$

and for the discrete 3D wave equation, namely:

$$\frac{\Delta t}{2} \| D_h \| \leq \gamma < 1, \quad \| D_h \| = \sup_{v^*_h \in \mathcal{V}_h} \sup_{p^*_h \in \mathcal{P}_h} \frac{d(v^*_h, p^*_h)}{|p^*_h|_{\mu_a} |v^*_h|_{\rho_a}},$$

(where $|\cdot|_{\mu_a} = m_p(\cdot, \cdot)^{1/2}$ and $|\cdot|_{\rho_a} = m_v(\cdot, \cdot)^{1/2}$) are simultaneously satisfied. In particular, the stability of the scheme does not depend on the mesh $T_H$ for the Lagrange multiplier $\lambda$. The proof of this stability result is a consequence of a discrete equivalent of the energy identity (3). More precisely, it can be shown [?] that:

$$\frac{E^{n+\frac{1}{2}}_h - E^{n-\frac{1}{2}}_h}{\Delta t} + \eta r \left( \frac{w^{n+1}_h - w^{n-1}_h}{2\Delta t}, \frac{w^{n+1}_h - w^{n-1}_h}{2\Delta t} \right) = 0,$$
where $E_{h}^{n+\frac{1}{2}}$ denotes the discrete energy:

$$
E_{h}^{n+\frac{1}{2}} = \frac{1}{2} m \left( \frac{w_{h}^{n+1} - w_{h}^{n}}{\Delta t}, \frac{w_{h}^{n+1} - w_{h}^{n}}{\Delta t} \right) + \frac{1}{2} \mu \left( v_{h}^{n+\frac{1}{2}}, v_{h}^{n+\frac{1}{2}} \right) + \frac{1}{2} \left( \frac{\tilde{u}^{n+\frac{1}{2}}}{\Delta t} \right)^{2} + K \Psi \left( \tilde{u}^{n+\frac{1}{2}} \right)
$$

Indeed, (31) proves that the discrete energy $E_{h}^{n+\frac{1}{2}}$ is decaying and from conditions (29) and (30) we deduce uniform bounds (i.e. independent on $H$, $h_{a}$, $h_{m}$ and $\Delta t$) for the $L^{2}$ norms of $w_{h}^{n}, p_{h}^{n}$ and $v_{h}^{n+\frac{1}{2}}$. Finally, the use of the uniform discrete inf-sup condition (28) permits to obtain analogous estimates for the $H^{1/2}$ norm of $\lambda_{h}^{n}$. These are the basic estimates that lead to the convergence of the method. We refer the reader to [?] for more details. As the mesh $\mathcal{T}_{a}$ is uniform, it is easy to see, using discrete Fourier techniques, that condition (30) is equivalent to:

$$
c_{a} \frac{\Delta t}{h_{a}} \leq \frac{\sqrt{3}}{3}.
$$

If we consider a uniformly regular family (in the sense of [?]) of meshes $\mathcal{T}_{m}$ for $\Sigma$, it is easy to prove the existence of a dimensionless constant $0 < K < 1$ (which depends on uniform bounds about the structure of the meshes) such that a sufficient condition for (29) is of the form:

$$
c \frac{\Delta t}{h_{m}} \leq K.
$$

4 Numerical experiments

The numerical experiment we present below corresponds to the following data:

- The density of the air is $\rho_{a} = 1.21$ kg m$^{-3}$. The sound velocity is $c_{a} = 344$ m s$^{-1}$.

- The radius of the membrane is 31 cm. Its tension, assumed uniform, is $T = 3324$ N m$^{-1}$, its density $\sigma = 0.262$ kg m$^{-2}$, which corresponds to the membrane velocity $c = 115$ m s$^{-1}$. The attenuation parameter $\eta$ is equal to $0.610^{-6}$ s.

- The rigid shell is assimilated to the half of an ellipsod (with symmetry of revolution) whose height is 50 cm. This shape is close to the one of the real timpani.

- The radius of the mallet at rest is 2.5 cm. Its mass $m$ is 28 g. The coefficient $K$ is equal to $1.6 \times 10^{8}$ N m$^{-p}$ and the exponent $p$ equal to 2.54.
• The point of impact (center of the support of the function $g$) is located at 21 cm from the center of the membrane. The velocity of impact is $v_0 = 1.4 \text{ m s}^{-1}$.

The corresponding results will be compared with experimental measures. We refer the reader to [?] for more information about the experimental techniques that have been used.

The computational domain will be chosen as a cubic box enclosing the instrument. Each side of this box is one meter long. This domain is artificially bounded with higher order absorbing boundary conditions with appropriate edge conditions. Such conditions can be incorporated (see [?]) in the variational formulation of the problem.

4.1 Choice of the discretization parameters

The discretization parameters are chosen according to various criteria:

(i) The time step $\Delta t$: the frequency content of the signals that are measured during the experiment is mainly contained in the interval $[0, 600 \text{ Hz}]$. We choose:

$$\Delta t = \frac{1}{24000},$$

which corresponds to 40 time steps per minimum period. This time step may appear as unnecessarily small. In fact, we have to compute the sound for approximately 3 seconds, that is to say 1800 periods for the highest frequency. This is quite large and we have to control the numerical dispersion in order to avoid large errors.

(ii) The space step $h_a$: it is chosen in adequation with the sound velocity $c_a$ in order to minimize the numerical dispersion. According to a standard dispersion analysis, we take:

$$h_a = \sqrt{3} c_a \Delta t \simeq 0.025 \text{ m}.$$

This ensures that the error committed on the phase velocity of plane waves is less than 3 per cent for all frequencies between 0 and 600 Hz.

(iii) The space step $H$: it must be sufficiently large with respect to $h_a$. In practice, we choose it according to the heuristic criterion:

$$H = 1.1 h_a.$$

(iv) The space step $h_m$: it is chosen in adequation with the membrane velocity $c$, once again in order to minimize the numerical dispersion. To estimate this dispersion (which is not known on a general mesh), we consider the particular case of a regular mesh made of squares of side $h_m$, each of them being splitted into two square
triangles. The analysis of the corresponding dispersion curves shows that the error committed on the phase velocity of plane waves is less than 2 per cent for all frequencies between 0 and 600 Hz if we take:

\[ h_m \simeq \sqrt{2} c \Delta t, \]

which corresponds approximately to (we take into account the ratio \(c_a/c \simeq 3\))

\[ h_m = \frac{H}{4}. \]

In practice, we begin by constructing the mesh \( \mathcal{T}_H \) for \( \Gamma \) and then construct the mesh \( \mathcal{T}_m \) for \( \Sigma \) as a refinement of \( \mathcal{T}_H \) according to Figure 2.

Figure 2: Mesh refinement on \( \Gamma \) from \( \mathcal{T}_H \) to \( \mathcal{T}_m \)

### 4.2 Numerical results

We first observe qualitatively the evolution of the solution during the ten first milliseconds of simulation, which corresponds approximately to the period of contact between the mallet and the membrane. We represent, at equally spaced instants, the deformation of the membrane (Figure 3), the evolution of the pressure field in the air (in a plane of symmetry of the instrument which passes by the point of impact mallet-membrane - Figure 4) and the evolution of the pressure jump across \( \Gamma \) (Figure 5).

In Figure 6, we superpose:

- the evolution of the energy of the mallet:

\[
E_M(t) = \frac{1}{2} \left\| \frac{du}{dt}(t) \right\|^2 + K \left[ (a - u(t) + W(t))^+ \right]^{p+1},
\]

- the evolution of the energy of the mallet:

\[
E_m(t) = \frac{1}{2} \int_{\Sigma} \sigma(x) \left\| \frac{\partial w}{\partial t}(x, t) \right\|^2 d\sigma + \frac{1}{2} \int_{\Sigma} T(x) |\nabla_\Gamma w(x, t)|^2 d\sigma,
\]
- the evolution of the energy associated to the sound radiation:

\[
E_a(t) = \frac{1}{2} \sum_{j=\varepsilon,i} \int_{\Omega_j} \left( \mu_a |p_j(x,t)|^2 + \rho_a |v_j(x,t)|^2 \right) dx,
\]

- the evolution of the total energy of the system:

\[
E(t) = E_M(t) + E_m(t) + E_a(t).
\]

This result clearly illustrates the very slow decay in time of the total energy, the transfer of energy from the mallet to the membrane during the period of impact (7 ms), and finally the oscillatory exchanges of energy between the air and the membrane after the impact.

With a microphone located at 10 cms above the membrane, we register the pressure field at a given point as a function of time. In the same way, we compute with our code the
variations of the pressure field at this point. We compare in Figure 7 the two corresponding curves for short and long times. This shows a very good agreement between experimental and numerical results. To complete this comparison, we compute the Fourier transform of the two (numerical and experimental) signals registered between 0 and 3 seconds. The two spectra are compared in Figure 8. Comparing the peaks of the two curves shows that the numerical simulation enables us to recover the main frequencies of the signal, which is essential for the sound perception (the low frequency content of the experimental result has to be attributed to noise generated by the measurement process).
Figure 7: Experimental results (continuous line) versus numerical results (dashed line) - Short times (left) and Long times (right)

Figure 8: Experimental spectrum (left) versus numerical spectrum (right)

REFERENCES


