NEW MIXED FORMULATIONS FOR THIN PLATE MODELS WITH PHYSICAL BOUNDARY CONDITIONS

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Abstract. This paper deals with the Kirchhoff-Love and the Reissner-Mindlin models for bending thin plates, with physical boundary conditions. We study new mixed formulations for these problems, whose main variable is the bending moment, which is a second order symmetric tensor. Then a simple postprocessing allows us to calculate the deflection of the plate and, in the Reissner-Mindlin case, the rotation vector. We prove that the problems are well-posed and we propose finite element approximation methods which are both conforming and of low degree. We establish that the methods are unconditionally convergent and, whenever the continuous solution is sufficiently smooth, the convergence rates are optimal. Moreover, the discretization of the Reissner-Mindlin model avoids the numerical locking phenomenon, which usually appears as the plate’s thickness tends towards zero.
1 INTRODUCTION

We are interested here in the analysis, from the theoretical and the numerical point of view, of two models for bending thin plates, satisfying natural boundary conditions cf. [1]. More precisely, we study the Kirchhoff-Love and the Reissner-Mindlin models, for which we propose new mixed formulations based on the calculus of the bending moment. This is a second order symmetric tensor and represents, in practice, the quantity of physical interest. We equally show how to recover the deflection of the plate by solving a laplacian problem, and - in the Reissner-Mindlin case - the additional unknown which is the rotation vector. We establish the convergence of the approximation methods proposed, which employ conforming low-order finite elements. We also state that no locking occurs when discretizing the Reissner-Mindlin model, that is the convergence of the method is uniform with respect to the plate’s thickness.

2 CONTINUOUS PROBLEMS

Let us briefly present the two plate models which we studied. The boundary $\Gamma$ of the medium surface of the plate $\Omega \subset \mathbb{R}^2$ is decomposed into three disjoint parts, onto which the plate is supposed to be clamped (on $\Gamma_0$), simply supported (on $\Gamma_1$), respectively free (on $\Gamma_2$). In view of the finite element approximation, we suppose that $\Omega$ is a polygonal and connected bounded domain, with no crack. The mechanical framework considered here is linear elasticity, and for the sake of simplicity the constitutive material is taken homogeneous and isotropic. Before describing our approach, let us first make some notations. In this paper, we employ the summation convention of Einstein and we denote by the letter $c$ any positive constant independent upon the discretization and the plate’s thickness. We agree to write the vectors in bold letters and the tensors in underlined letters. Finally, we note $n = (n_i)_{1 \leq i \leq 2}$ the unit outward normal vector along $\Gamma$ and $t = (t_i)_{1 \leq i \leq 2}$ the unit tangent vector to $\Gamma$ such that $t_1 = n_2, t_2 = -n_1$.

2.1 The Kirchhoff-Love model

We begin with the Kirchhoff-Love equations, which write as below:

$$\begin{cases}
\Delta^2 u = f & \text{in } \Omega \\
u = \partial_n u = 0 & \text{on } \Gamma_0 \\
u = \sigma_{ij} n_i n_j = 0 & \text{on } \Gamma_1 \\
\sigma_{ij} n_i n_j = \partial_r (\sigma_{ij} t_i n_j) + \partial_j \sigma_{ij} n_i = 0 & \text{on } \Gamma_2 \\
\sigma_{ij} = (1 - \nu) \partial_{ij} u + \nu \Delta u \delta_{ij} & \text{in } \Omega,
\end{cases}$$

(1)

where $\nu$ is the Poisson’s coefficient. A transverse loading is applied, of force density denoted (after scaling) by $f$; we shall take in the sequel $f \in L^2(\Omega)$. The unknowns of the problem are the deflection $u$ and the bending moment $\sigma = (\sigma_{ij})_{1 \leq i, j \leq 2}$.
2.2 The Reissner-Mindlin model

Let us now introduce the Reissner-Mindlin model corresponding to a bending thin plate of thickness $2\varepsilon$, which is written as below:

$$\begin{cases}
    -\frac{1}{\varepsilon} \text{div}\, \mathbf{g}^\varepsilon + \frac{1}{\varepsilon^2} (\mathbf{r}^\varepsilon - \nabla u^\varepsilon) = 0 & \text{in } \Omega \\
    1 - \nu \text{div}(\mathbf{r}^\varepsilon - \nabla u^\varepsilon) = f & \text{in } \Omega \\
    u^\varepsilon = 0, \quad \mathbf{r}^\varepsilon = 0 & \text{on } \Gamma_0 \\
    u^\varepsilon = \mathbf{r}^\varepsilon \cdot \mathbf{t} = \sigma^\varepsilon \mathbf{n} \cdot \mathbf{n} = 0 & \text{on } \Gamma_1 \\
    \partial_i (\mathbf{g}^\varepsilon \cdot \mathbf{n}) + \text{div}\, \mathbf{g}^\varepsilon \cdot \mathbf{n} = 0, \quad \mathbf{r}^\varepsilon \cdot \mathbf{t} = \partial_i u^\varepsilon & \text{on } \Gamma_2 \\
    \sigma^\varepsilon_{ij} = (1 - \nu) \varepsilon_{ij} (\mathbf{r}^\varepsilon) + \nu (\text{div} \mathbf{r}^\varepsilon) \delta_{ij} & \text{in } \Omega.
\end{cases}
$$

The unknowns are now the transverse displacement $u^\varepsilon$, the rotation vector $\mathbf{r}^\varepsilon$ and the bending moment $\sigma^\varepsilon$ and clearly, they are depending upon the thickness. Let us notice here that (2) is a singular problem with respect to the small parameter $\varepsilon$, and its finite element approximation usually suffers from locking as the thickness becomes very small (phenomenon known as "shear locking").

When $\varepsilon$ goes towards 0, this model tends towards the Kirchhoff-Love model (1) cf.[1], in the following sense:

$$\begin{align*}
    u^\varepsilon & \to H^1(\Omega) u, & \mathbf{r}^\varepsilon & \to \nabla u, \\
    \mathbf{g}^\varepsilon & \to \mathbf{g}, & \text{div}\, \mathbf{g}^\varepsilon & \to \text{div}\, \mathbf{g}.
\end{align*}$$

2.3 Mathematical framework

One of the first questions which arise in the analysis of problems (1) and (2) is to give a mathematical framework in which they are well-posed. To do that, we consider the following spaces:

$$\begin{align*}
    V &= \{ v \in H^1(\Omega) ; \ v = 0 \text{ on } \Gamma_0 \cup \Gamma_1 \} \\
    X &= \{ \tau = (\tau_{ij})_{1 \leq i,j \leq 2} ; \ \tau_{ij} \in L^2(\Omega) , \ D(\tau) \in L^2(\Omega) \} \\
    X^\varepsilon &= \{ \tau \in X ; \ \varepsilon (\text{div}\, \tau) \in L^2(\Omega) \},
\end{align*}$$

endowed with the natural norms:

$$\begin{align*}
    \| v \|_V &= |v|_{1,\Omega} \\
    \| \tau \|_X &= (\| \tau \|^2_{0,\Omega} + \| D(\tau) \|^2_{0,\Omega})^{1/2} \\
    \| \tau \|_{X^\varepsilon} &= (\| \tau \|^2_{X} + \varepsilon^2 \| \text{div}\, \tau \|^2_{0,\Omega})^{1/2},
\end{align*}$$

where $D(\tau) = \text{div}(\text{div}\, \tau) = \partial_{ij} \tau_{ij}$. It is obvious that $V, X, X^\varepsilon$ are Hilbert spaces with respect to these norms. The fact that $D(\Omega)^4$ is a dense subspace of $X$ allows us to define
rigorously, by continuity, the trace operators for the bending tensor. Thus, we can show that:

\[ \gamma_0 : X \rightarrow H^{-1/2}(\Gamma), \quad \gamma_0(\tau) = \tau \cdot n = \tau_{ij} n_i n_j \]

\[ \gamma_1 : X \rightarrow H^{-3/2}(\Gamma), \quad \gamma_1(\tau) = \partial_t(\tau \cdot t) + \text{div} \tau \cdot n = \partial_t(\tau_{ij} n_j) + \partial_j \tau_{ij} n_i \]

are linear continuous operators and moreover, we can establish the following Green’s formula, which holds for any \( v \in H^2(\Omega) \) and any \( \tau \in X \):

\[
\int_{\Omega} D(\tau) v \, d\Omega = \int_{\Omega} \tau_{ij} \partial_v v \, d\Omega - \langle \gamma_0(\tau), \partial_v v \rangle_{-\frac{1}{2}, \frac{1}{2}, \Gamma} + \langle \gamma_1(\tau), v \rangle_{-\frac{3}{2}, \frac{3}{2}, \Gamma}.
\] (3)

3 VARIATIONAL FORMULATIONS IN THE BENDING MOMENT

In order to give variational formulations of the above plate models, let us introduce the spaces:

\[
M = \left\{ v \in H^{3/2}(\Gamma) : v = 0 \text{ on } \Gamma_0 \cup \Gamma_1 \right\}
\]

\[
N = \left\{ v \in H^{1/2}(\Gamma) : v = 0 \text{ on } \Gamma_0 \right\},
\]

as well as the subsets of \( X \), respectively of \( X^\varepsilon \):

\[ X^g = \{ \tau \in X ; D(\tau) = g \}, \quad X^{\varepsilon, g} = \{ \tau \in X^\varepsilon ; D(\tau) = g \}, \]

for any \( g \in L^2(\Omega) \).

Let us first obtain the formulation of the Kirchhoff-Love problem. For that, we define the following continuous bilinear forms on \( X \times X \), respectively on \( X \times (M \times N) \):

\[
a(\sigma, \tau) = \frac{1}{1 - \nu} \int_{\Omega} \sigma : \tau \, d\Omega - \frac{\nu}{1 - \nu} \int_{\Omega} (\text{tr} \sigma)(\text{tr} \tau) \, d\Omega \]

(4)

\[
b(\tau, (\mu, \lambda)) = \langle \gamma_1(\tau), \mu \rangle_{-\frac{1}{2}, \frac{1}{2}, \Gamma} - \langle \gamma_0(\tau), \lambda \rangle_{-\frac{3}{2}, \frac{3}{2}, \Gamma}.
\]

By using the Green’s formula (3), we get the following mixed variational formulation of (1):

\[
\left\{ \begin{array}{l}
\text{find } \sigma \in X^f, \ (u_0, u_1) \in M \times N \text{ such that } \\
\forall \tau \in X^0, \quad a(\sigma, \tau) + b(\tau, (u_0, u_1)) = 0 \\
\forall (v_0, v_1) \in M \times N, \quad b(\sigma, (v_0, v_1)) = 0.
\end{array} \right.
\] (5)

It is equally useful to introduce the boundary value problem:

\[
\left\{ \begin{array}{l}
\Delta \phi = f \text{ in } \Omega \\
\phi = 0 \text{ on } \Gamma_0 \cup \Gamma_1 \\
\partial_n \phi = 0 \text{ on } \Gamma_2.
\end{array} \right.
\] (6)
Obviously, it admits a unique solution $\phi^f \in V$ by the means of which we can write the solution $\mathbf{\sigma} \in \mathcal{X}^f$ of the Kirchhoff-Love problem as below:

$$\mathbf{\sigma} = \mathbf{\sigma}^0 + \phi^f L,$$

with $\mathbf{\sigma}^0 \in \mathcal{X}^0$ and with $\gamma_0(\phi^f L) = \phi^f$, $\gamma_1(\phi^f L) = 0$. Thanks to this decomposition, the previous formulation can now be written under the equivalent form:

$$\begin{cases}
\forall \mathbf{\tau} \in \mathcal{X}^0, & a(\mathbf{\sigma}^0, \mathbf{\tau}) + b(\mathbf{\tau}, (u_0, u_1)) = -a(\phi^f L, \mathbf{\tau}) \\
\forall (v_0, v_1) \in M \times N, & b(\mathbf{\sigma}^0, (v_0, v_1)) = \langle \phi^f, v_1 \rangle - \frac{1}{2} \mathbf{\Gamma}.
\end{cases} \tag{8}$$

The next result establishes the well-posedness of this problem, as well as its link with the initial Kirchhoff-Love model.

**Theorem 3.1** Problem (8) has a unique solution $(\mathbf{\sigma}^0, u_0, u_1)$, which moreover verifies:

$$(\mathbf{\sigma}^0 + \phi^f L, u_0, u_1) = (\mathbf{\sigma}, u |_{\Gamma}, \partial_n u |_{\Gamma}) \tag{9}$$

where $(\mathbf{\sigma}, u)$ satisfies the Kirchhoff-Love equations (1).

**Proof.** We apply the Babuška-Brezzi’s theory for mixed formulations in order to obtain the existence and the uniqueness of the solution, see [2]. Firstly, we show that the form $a(\cdot, \cdot)$ is coercive on $\mathcal{X}^0 \times \mathcal{X}^0$ and secondly, we verify the inf-sup condition for the bilinear form $b(\cdot, \cdot)$. The interpretation is obtained by the means of the Green’s formula (3) and by adequate choices for the test-functions.

**Remark 3.1** This result allows us to obtain the deflection $u$ of the plate, too. Indeed, $u$ is the unique solution of the biharmonic problem:

$$\begin{cases}
\Delta^2 u = f & \text{in } \Omega \\
u = u_0 & \text{on } \Gamma \\
\partial_n u = u_1 & \text{on } \Gamma.
\end{cases} \tag{10}$$

However, in the sequel we will calculate the displacement more directly, as the solution of a second order elliptic problem.

Let us now look at the Reissner-Mindlin model (2). We need to introduce two new bilinear forms $a_0(\cdot, \cdot)$ and $c(\cdot, \cdot)$ defined on $\mathcal{X}^e \times \mathcal{X}^e$, respectively on $\mathcal{X}^e \times L^2(\Omega)$ by:

$$a_0(\mathbf{\sigma}, \mathbf{\tau}) = \frac{1}{1 - \nu} \int_\Omega \text{div}\mathbf{\sigma} \cdot \text{div}\mathbf{\tau} \, d\Omega$$

$$c(\mathbf{\tau}, \mu) = \int_\Omega (\tau_{12} - \tau_{21}) \mu \, d\Omega, \tag{11}$$

where $\mathbf{\tau}$ is the unique solution of the biharmonic problem:

$$\begin{cases}
\Delta^2 u = f & \text{in } \Omega \\
u = u_0 + M & \text{in } \Omega \\
\partial_n u = u_1 & \text{on } \Gamma.
\end{cases}$$

Let us now look at the Reissner-Mindlin model (2). We need to introduce two new bilinear forms $a_0(\cdot, \cdot)$ and $c(\cdot, \cdot)$ defined on $\mathcal{X}^e \times \mathcal{X}^e$, respectively on $\mathcal{X}^e \times L^2(\Omega)$ by:

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$$c(\mathbf{\tau}, \mu) = \int_\Omega (\tau_{12} - \tau_{21}) \mu \, d\Omega, \tag{11}$$

where $\mathbf{\tau}$ is the unique solution of the biharmonic problem:
and we put:

\[ a^\varepsilon(\sigma, \tau) = a(\sigma, \tau) + \varepsilon^2 a_0(\sigma, \tau). \quad (12) \]

One can immediately notice that if \( \sigma \in X^\varepsilon \) is symmetric, then one has:

\[ c(\sigma, \mu) = 0, \quad \forall \mu \in L^2(\Omega). \quad (13) \]

So, the role of the new term \( c(\cdot, \cdot) \) is to dualize the symmetry of the bending tensor, while the bilinear form \( \varepsilon^2 a_0(\cdot, \cdot) \) takes into account the new variable which is the rotation vector \( \mathbf{r}^\varepsilon \). We propose the following variational formulation for the Reissner-Mindlin problem:

\[
\left\{ \begin{array}{l}
\text{find } \sigma^{\varepsilon,0} \in X^{\varepsilon,0}, \ (u_0^\varepsilon, r_0^\varepsilon) \in M \times N, \ \lambda^\varepsilon \in L^2(\Omega) \ \text{such that} \\
\forall \tau \in X^{\varepsilon,0}, \quad \forall (\zeta, \eta) \in M \times N, \quad \forall \mu \in L^2(\Omega), \\
a^\varepsilon(\sigma^{\varepsilon,0}, \tau) + b(\tau, (u_0^\varepsilon, r_0^\varepsilon)) + c(\tau, \lambda^\varepsilon) = -a^\varepsilon(\phi^f I, \tau) \\
b(\sigma^{\varepsilon,0}, (\zeta, \eta)) = \langle \phi^f, \eta \rangle - \frac{1}{2} \frac{1}{2} \Gamma \\
c(\sigma^{\varepsilon,0}, \mu) = 0,
\end{array} \right. \quad (14)
\]

which is a generalization of (8) introduced in the previous case. We show that:

**Theorem 3.2** Problem (14) has a unique solution. Its interpretation in terms of the Reissner-Mindlin boundary value problem (2) is given by the next relations:

\[
\left\{ \begin{array}{l}
\sigma^{\varepsilon,0} = \sigma^\varepsilon - \phi^f I \ \text{in } \Omega, \\
r_0^\varepsilon = \mathbf{r}^\varepsilon \cdot \mathbf{n} \ \text{on } \Gamma, \\
u_0^\varepsilon = u^\varepsilon \ \text{on } \Gamma, \\
\lambda^\varepsilon = -\frac{1}{2} \operatorname{curl} \mathbf{r}^\varepsilon \ \text{in } \Omega,
\end{array} \right. \quad (15)
\]

where \( (\sigma^\varepsilon, \mathbf{r}^\varepsilon, u^\varepsilon) \) verifies the equations (2).

**Remark 3.2** In order to obtain the transverse displacement \( u^\varepsilon \), one can now solve the problem

\[
\left\{ \begin{array}{l}
\Delta u^\varepsilon = \frac{1}{1 + \nu} \text{tr} \sigma^\varepsilon - \frac{\varepsilon^2 f}{1 - \nu} \ \text{in } \Omega, \\
u^\varepsilon = 0 \ \text{on } \Gamma_0 \cup \Gamma_1, \\
u^\varepsilon = u_0^\varepsilon \ \text{on } \Gamma_2,
\end{array} \right. \quad (16)
\]

while the rotation vector \( \mathbf{r}^\varepsilon \) is given by the relation:

\[ \mathbf{r}^\varepsilon = \frac{\varepsilon^2}{1 - \nu} \text{div} \sigma^\varepsilon + \nabla u^\varepsilon. \quad (17) \]
4 EQUIVALENT MIXED FORMULATIONS

4.1 Characterization of the space $\mathbf{X}^0$

We recall that for any $\mathbf{z} \in \mathbf{X}^0$ one has $D(\mathbf{z}) = 0$, that is $\text{div}(\text{div}\mathbf{z}) = 0$. Applying twice the Tartar’s lemma (see [3]), one gets that there exist unique functions $\varphi \in (H^1(\Omega))^2$ and $\rho \in L^2_0(\Omega)$ such that:

$$\mathbf{z} = \text{curl}\varphi + \rho \mathbf{J}, \text{ where } \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{curl}\varphi = \begin{pmatrix} \partial_2 \varphi_1 - \partial_1 \varphi_1 \\ \partial_2 \varphi_2 - \partial_1 \varphi_2 \end{pmatrix}. \quad (18)$$

By the means of this relation, we can now express the trace operators $\gamma_0$ and $\gamma_1$ in the following way:

$$\begin{align*}
\gamma_0(\mathbf{z}) &= -\partial_1 \varphi \cdot \mathbf{n} \\
\gamma_1(\mathbf{z}) &= -\partial_1(\partial_1 \varphi \cdot \mathbf{t}).
\end{align*} \quad (19)$$

If, moreover, the tensor $\mathbf{z} \in \mathbf{X}^0$ is symmetric, then it comes that $2\rho = \text{div}\varphi$ and in this case we shall write $\mathbf{z}$ as below:

$$\mathbf{z} = \begin{pmatrix} \partial_2 \varphi_1 & (\partial_2 \varphi_2 - \partial_1 \varphi_1)/2 \\ (\partial_2 \varphi_1 - \partial_1 \varphi_1)/2 & -\partial_1 \varphi_2 \end{pmatrix}, \quad (20)$$

with a unique function $\varphi$ now belonging to $\mathbf{H} = H^1(\Omega)^2_{K}$, where:

$$K = \left\{ \begin{pmatrix} ax + b \\ ay + c \end{pmatrix}; \ a, b, c \in \mathbb{R} \right\}. \quad (21)$$

4.2 New formulation for the Kirchhoff-Love problem

Since we know that the bending moment $\mathbf{g}$ is symmetric, we will use the decomposition (20) for the symmetric elements of $\mathbf{X}^0$ in order to write down and to study a new equivalent formulation of problem (8), whose main unknown will now be a function $\psi$ belonging to $\mathbf{H}$. Thus, we will avoid the discretization of the constraint $D(\mathbf{z}) = 0$ imposed on the test-functions.

To do that, let us define a bilinear symmetric and continuous form $A(\cdot, \cdot)$ on the space $\mathbf{H} \times \mathbf{H}$ by putting:

$$A(\psi, \varphi) = a(\text{curl}\psi + \frac{1}{2}(\text{div}\psi)\mathbf{J}, \text{curl}\varphi + \frac{1}{2}(\text{div}\varphi)\mathbf{J})$$

$$= \frac{1}{1-\nu} \int_{\Omega} [\partial_2 \psi_1 \partial_2 \varphi_1 + \partial_1 \psi_2 \partial_1 \varphi_2 + \frac{1}{2} (\partial_2 \psi_2 - \partial_1 \psi_1) (\partial_2 \varphi_2 - \partial_1 \varphi_1)] d\Omega$$

$$- \frac{\nu}{1-\nu^2} \int_{\Omega} (\partial_2 \psi_1 - \partial_1 \psi_2) (\partial_2 \varphi_1 - \partial_1 \varphi_2) d\Omega. \quad (22)$$

Let us also consider, for any $(v_0, v_1) \in M \times N$, a lifting $w \in H^2(\Omega)$ verifying: $w = v_0$ on $\Gamma$ and $\partial_n w = v_1$ on $\Gamma$. Then we notice that we have:

$$b(\text{curl}\varphi + \frac{1}{2}(\text{div}\varphi)\mathbf{J}, (v_0, v_1)) = -\langle \partial_1 \nabla w, \varphi \rangle_{\frac{1}{2}, \frac{1}{2}, \Gamma}. \quad (23)$$
which leads us to introduce a new bilinear form $B(\cdot, \cdot)$ on $H \times Z$ by setting:

$$B(\varphi, q) = -\langle \partial_t q, \varphi \rangle - \frac{1}{2} \int_{\Gamma} q \cdot t \, d\Gamma = 0, \quad (24)$$

where:

$$Z = \left\{ q \in H^{1/2}(\Gamma)^2; \ q = 0 \ on \ \Gamma_0, \ q \cdot t = 0 \ on \ \Gamma_1, \ \int_{\Gamma} q \cdot t \, d\Gamma = 0 \right\}. \quad (25)$$

It is obvious that to any $q \in Z$, one can now associate a unique couple $(v_0, v_1) \in M \times N$ by putting:

$$q = (\partial_t v_0)t + v_1 \mathbf{n}. \quad (26)$$

We also introduce the linear continuous forms $F(\cdot)$ and $G(\cdot)$ defined on $H$, respectively on $Z$ by:

$$F(\varphi) = -a(\phi^f L \text{curl} \varphi + \frac{1}{2}(\text{div} \varphi) \mathbf{L}) = -\frac{1}{1 + \nu} \int_{\Omega} \phi^f (\partial_2 \varphi_1 - \partial_1 \varphi_2) \, d\Omega, \quad (27)$$

$$G(q) = \int_{\Gamma} \phi^f q \cdot \mathbf{n} \, d\Gamma. \quad (28)$$

We are now able to write the following mixed variational problem:

$$\begin{cases} 
\text{find } \psi \in H, \ p \in Z \text{ such that } \\
\forall \varphi \in H, \ A(\psi, \varphi) + B(\varphi, p) = F(\varphi) \\
\forall q \in Z, \ B(\psi, q) = G(q) 
\end{cases} \quad (29)$$

and to prove, in a quite technical manner, that it admits a unique solution. This is achieved by the means of the Babuška-Brezzi’s theory. The link between the solution of (27) and the solution $(\sigma, u)$ of the initial Kirchhoff-Love model (1) is stated in the next theorem.

**Theorem 4.1** Let $(\psi, p) \in H \times Z$ be the unique solution of (27). Then one has:

$$\begin{cases} 
\sigma = \text{curl} \psi + \frac{1}{2}(\text{div} \psi) \mathbf{L} + \phi^f L \text{ in } \Omega \\
\nabla u = p \text{ on } \Gamma. 
\end{cases} \quad (30)$$

Therefore, it is sufficient to solve the mixed problem (27) in order to get, thanks to the relation above, the tensor $\sigma^0 = \text{curl} \psi + \frac{1}{2}(\text{div} \psi) \mathbf{L}$ and implicitly the bending moment $\sigma = \sigma^0 + \phi^f L$ solution of the initial Kirchhoff-Love model.

**Remark 4.1** Let us notice that we can now calculate the displacement $u$, as the unique solution of the following second order elliptic problem:

$$\begin{cases} 
\Delta u = \frac{1}{1 + \nu} (\text{tr} \sigma) \text{ in } \Omega \\
u = 0 \text{ on } \Gamma_0 \cup \Gamma_1 \\
\partial_n u = p \cdot \mathbf{n} \text{ on } \Gamma_2. \quad (31)$$

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4.3 New formulation for the Reissner-Mindlin model

In the same manner as for the Kirchhoff-Love model, we show that to any \( \tau \in X^\varepsilon \) we can associate a unique couple \((\varphi, \rho)\) with \( \rho \in H^1(\Omega)_{\mathbb{R}}^2 \) and \( \varphi \in (H^1(\Omega)_{\mathbb{R}})^2 \) such that:

\[
\tau = \text{curl} \varphi + \rho J.
\] (30)

Let us consider the space

\[
W^\varepsilon = \left\{ \rho \in L^2_0(\Omega) ; \varepsilon \text{curl} \rho \in L^2(\Omega)^2 \right\},
\] (31)

endowed with the weighted norm \( (\|\cdot\|^2_{0,\Omega} + \varepsilon |\cdot|_{1,\Omega}^2)^{1/2} \) and let us define the Hilbert space:

\[
Y^\varepsilon = (H^1(\Omega)_{\mathbb{R}})^2 \times W^\varepsilon.
\]

Next, we introduce the following bilinear form on \( Y^\varepsilon \times Y^\varepsilon \):

\[
A^\varepsilon((\psi, \xi) , (\varphi, \rho)) = A((\psi, \xi) , (\varphi, \rho)) + \varepsilon^2 A_0((\psi, \xi) , (\varphi, \rho)),
\] (32)

where:

\[
A((\psi, \xi) , (\varphi, \rho)) = \frac{1}{1 - \nu} \int_{\Omega} \left[ \partial_2 \psi_1 \partial_2 \varphi_1 + \partial_1 \psi_2 \partial_1 \varphi_2 \right] d\Omega + \frac{1}{1 - \nu} \int_{\Omega} \left[ (\xi - \partial_1 \psi_1)(\rho - \partial_1 \varphi_1) + (\xi - \partial_2 \psi_2)(\rho - \partial_2 \varphi_2) \right] d\Omega
\]

\[
- \frac{\nu}{1 - \nu^2} \int_{\Omega} \left( \partial_2 \psi_1 - \partial_1 \psi_2 \right) \left( \partial_2 \varphi_1 - \partial_1 \varphi_2 \right) d\Omega
\] (33)

and

\[
A_0((\psi, \xi) , (\varphi, \rho)) = \frac{1}{1 - \nu} \int_{\Omega} \text{curl} \xi \cdot \text{curl} \rho d\Omega.
\]

We equally define the bilinear form \( C(\cdot , \cdot) \) on \( Y^\varepsilon \times L^2(\Omega) \) by setting:

\[
C((\varphi, \rho) , \lambda) = \int_{\Omega} \lambda \left( 2\rho - \text{div} \varphi \right) d\Omega,
\] (34)

as well as the linear continuous form \( F^\varepsilon(\cdot) \) on \( Y^\varepsilon \) by:

\[
F^\varepsilon((\varphi, \rho)) = F((\varphi, \rho)) - \varepsilon^2 a_0(\phi^I, \text{curl} \varphi + \rho J).
\] (35)

We are now able to give an equivalent variational formulation to problem (14), which writes as below:

\[
\begin{aligned}
\text{find} \quad (\psi^\varepsilon, \xi^\varepsilon) \in Y^\varepsilon, \quad p^\varepsilon \in Z, \quad \lambda^\varepsilon \in L^2(\Omega) \quad \text{such that} \\
\forall (\varphi, \rho) \in Y^\varepsilon, \quad A^\varepsilon((\psi^\varepsilon, \xi^\varepsilon) , (\varphi, \rho)) + B((\varphi, \rho) , p^\varepsilon) + C((\varphi, \rho) , \lambda^\varepsilon) = F^\varepsilon((\varphi, \rho)) \quad \text{(36)}\\
\forall q \in Z, \quad B((\psi^\varepsilon, \xi^\varepsilon) , q) = G(q) \\
\forall \mu \in L^2(\Omega), \quad C((\psi^\varepsilon, \xi^\varepsilon) , \mu) = 0,
\end{aligned}
\]

and to state the next result:
Theorem 4.2 Problem (36) has a unique solution and moreover, the following relations hold:

\[
\begin{align*}
\sigma^\varepsilon &= \text{curl}\psi^\varepsilon + \xi^\varepsilon\mathbf{J} + \phi^f\mathbf{I} \quad \text{in } \Omega \\
\mathbf{r}^\varepsilon &= \mathbf{p}^\varepsilon \quad \text{on } \Gamma \\
-\frac{1}{2}\text{curl}\mathbf{r}^\varepsilon &= \lambda^\varepsilon \quad \text{in } \Omega,
\end{align*}
\]

where \((\sigma^\varepsilon, u^\varepsilon, \mathbf{r}^\varepsilon)\) represents the solution of the initial Reissner-Mindlin problem (2).

The previous statement gives us directly the bending moment; in order to recover the transverse displacement, it is sufficient to solve:

\[
\begin{align*}
\Delta u^\varepsilon &= \frac{1}{1 + \nu} tr\sigma^\varepsilon - \frac{\varepsilon^2}{1 - \nu} f \quad \text{in } \Omega \\
u^\varepsilon &= 0 \quad \text{on } \Gamma_0 \cup \Gamma_1 \\
\partial_n u^\varepsilon &= \mathbf{p}^\varepsilon \cdot \mathbf{t} \quad \text{on } \Gamma_2,
\end{align*}
\]

while the rotation \(\mathbf{r}^\varepsilon\) is given by (17).

So from now on, we shall study the variational problem (27) and its generalization (36) to the Reissner-Mindlin case. More precisely, we are interested in their finite element approximation, which will be presented in the next section.

5 FINITE ELEMENT DISCRETIZATION

Let \((T_h)_{h>0}\) be a regular family of triangulations of the polygonal domain \(\overline{\Omega}\), each \(T_h\) consisting of triangles \(K : \overline{\Omega} = \cup_{K \in T_h} K\). For every triangle \(K\) of \(T_h\), we denote by \(h_K\) its diameter and by \(m(K)\) its area. Finally, we define the discretization parameter \(h = \max_{K \in T_h} h_K\) and we introduce the set of edges of the triangulation \(T_h\) situated on \(\Gamma_1 \cup \Gamma_2\):

\[\partial T^1_h = \{T; T \text{ edge of } K \in T_h, T \subset (\Gamma_1 \cup \Gamma_2)\}.
\]

5.1 Approximation of the Kirchhoff-Love problem

Let us begin by presenting a conforming low-order approximation of the continuous problem (27). For that, we consider two finite dimensional spaces \(H_h \subset \mathbf{H}\) and \(Z_h \subset \mathbf{Z}\), which we take as below:

\[
H_h = \left\{ \varphi_h \in H^1(\Omega)^2_K; \forall K \in T_h, \varphi_h|_K \in P_1(K)^2 \text{ if } \partial K \cap \partial T^1_h = \emptyset \right\},
\]

and \(\varphi_h|_K \in P_2(K)^2 \text{ if } \partial K \cap \partial T^1_h \neq \emptyset\).

\[
Z_h = \left\{ q_h \in \mathbf{Z}; q_h \in C^0(\Gamma)^2 \text{ and } \forall T \in \partial T^1_h, q_h|_T \in P_1(T)^2 \right\}.
\]

The degrees of freedom of \(\varphi_h \in H_h\) are the values of \(\varphi_h\) at the nodes of the triangulation \(T_h\), to which we add the values at the midpoints of the edges situated on \(\Gamma_1 \cup \Gamma_2\) (i.e. the bubble-functions on the boundary \(\Gamma_1 \cup \Gamma_2\)).
We can now write down the discrete problem as follows:

\[
\begin{aligned}
\text{find } & \psi_h \in H_h, \ p_h \in Z_h \ 	ext{ such that } \\
\forall \varphi_h \in H_h, & \quad A(\psi_h, \varphi_h) + B(\varphi_h, p_h) = F_h(\varphi_h) \quad (41) \\
\forall q_h \in Z_h, & \quad B(\psi_h, q_h) = G_h(q_h).
\end{aligned}
\]

Numerical integration is used on the linear forms \( F(\cdot) \) and \( G(\cdot) \), which are replaced in the discrete case by:

\[
\begin{aligned}
F_h(\varphi_h) &= -\frac{1}{1 + \nu} \int_\Omega \phi_h^f (\partial_2 \varphi_{1h} - \partial_1 \varphi_{2h}) \, d\Omega, \\
G_h(q_h) &= \int_\Gamma \phi_h^f q_h \cdot n \, d\Gamma.
\end{aligned}
\]

The discrete function \( \phi_h^f \) is a \( P_1 \)-continuous finite element approximation of \( \phi_f \), the solution of the auxiliary problem (6).

With the above choice for the finite element spaces, we can show that the variational inf-sup condition of Babuška-Brezzi holds uniformly with respect to \( h \), which allows us to establish the following error estimate cf. [2]:

\[
\begin{aligned}
|\psi - \psi_h|_{1,\Omega} + \| p - p_h \|_{\frac{\nu}{\nu+1},\Gamma} &\leq c \{ \inf_{\varphi_h \in H_h} |\psi - \varphi_h|_{1,\Omega} + \inf_{q_h \in Z_h} \| p - q_h \|_{\frac{\nu}{\nu+1},\Gamma} \\
+ \sup_{\varphi_h \in H_h} \frac{F_h(\varphi_h) - F(\varphi_h)}{|\varphi_h|_{1,\Omega}} + \sup_{q_h \in Z_h} \frac{G_h(q_h) - G(q_h)}{\| q_h \|_{\frac{\nu}{\nu+1},\Gamma}} \}.
\end{aligned}
\]

The last two terms can be bounded as below:

\[
\begin{aligned}
\sup_{\varphi_h \in H_h} \frac{F_h(\varphi_h) - F(\varphi_h)}{|\varphi_h|_{1,\Omega}} + \sup_{q_h \in Z_h} \frac{G_h(q_h) - G(q_h)}{\| q_h \|_{\frac{\nu}{\nu+1},\Gamma}} &\leq c |\phi_f - \phi_h^f|_{1,\Omega} \leq c \inf_{v_h \in V_h} |\phi_f - v_h|_{1,\Omega}.
\end{aligned}
\]

where

\[
V_h = \{ v_h \in V; \forall K \in T_h, \ v_h|_K \in P_1(K) \}.
\]

In order to give the convergence rate of the discretization method, we assume that the solution (\( \sigma, u \)) of the initial Kirchhoff-Love model (1) has the following smoothness:

\[
\begin{aligned}
\sigma &\in H^a(\Omega)^4, \quad u \in H^{2+a}(\Omega), \quad 0 < a \leq 1, \\
\| \sigma \|_{a,\Omega} + \| u \|_{2+a,\Omega} &\leq c \| f \|_{0,\Omega}.
\end{aligned}
\]

We also suppose that \( \phi_f \in H^{1+b}(\Omega) \) and \( \| \phi_f \|_{1+b,\Omega} \leq c \| f \|_{0,\Omega} \) with \( 0 < b \leq 1 \). Then we immediately obtain, by classical interpolation:
**Theorem 5.1** The approximation method (41) of the variational problem (27) is unconditionally convergent. Furthermore, under the previous regularity assumptions, one has that:

\[
|\psi - \psi_h|_{1,\Omega} + \|p - p_h\|_{1/2,\Gamma} \leq ch^{\min\{a,b\}} \|f\|_{0,\Omega},
\]

with a constant \(c\) independent upon the discretization.

### 5.2 Approximated bending tensor and displacement

It is now easy to come back to the calculus of the quantities which interested us at the beginning of this paper, that is the bending moment \(\sigma\) and the deflection of the plate \(u\). Concerning the tensor \(\sigma\), we already know from Theorem 4.1 that:

\[
\sigma = \text{curl} \psi + \frac{1}{2} \left( \text{div} \psi \right) J + \phi^f I,
\]

where \(\psi\) verifies the equations (27). Therefore we set:

\[
\sigma_h = \text{curl} \psi_h + \frac{1}{2} \left( \text{div} \psi_h \right) J + \phi^f_h I,
\]

with \(\psi_h\) the solution of (41). Then it is obvious that:

\[
\| \sigma - \sigma_h \|_{0,\Omega} \leq c |\psi - \psi_h|_{1,\Omega} + \| \phi^f - \phi^f_h \|_{0,\Omega} \leq ch^{\min\{a,2b\}} \|f\|_{0,\Omega}
\]

and equally that:

\[
\| D(\sigma) - D(\sigma_h) \|_{-1,\Omega} = \| \Delta(\phi^f - \phi^f_h) \|_{-1,\Omega} \leq ch^b \|f\|_{0,\Omega}.
\]

Concerning now the plate's deflection, we know that the continuous solution \(u\) verifies the boundary value problem (29). Then we calculate the approximation \(u_h\) of \(u\) as the solution of the next problem, which is a discretization of the variational formulation of (29):

\[
\begin{cases}
\text{find } u_h \in V_h \text{ such that } \\
\forall v_h \in V_h, \quad \int_{\Omega} \nabla u_h \cdot \nabla v_h \, d\Omega = -\frac{1}{1 + \nu} \int_{\Omega} (\text{tr} \sigma_h) v_h \, d\Omega + \int_{\Gamma_2} p_h \cdot n \, v_h \, d\Gamma,
\end{cases}
\]

where \(\sigma_h\) is of course given by (47) and \(p_h\) by the problem (41). This leads us to the following error estimate:

\[
\| u - u_h \|_{1,\Omega} \leq c \{ \inf_{v_h \in V_h} \| u - v_h \|_{1,\Omega} \} + \| \sigma - \sigma_h \|_{0,\Omega} \|f\|_{0,\Omega} \|
\]

As a conclusion to this problem, we finally state:
Theorem 5.2 Under the previous smoothness hypotheses on the solution \((\varphi, u)\), one has that:
\[
\|\varphi - \varphi_h\|_{0, \Omega} + \|u - u_h\|_{1, \Omega} + \|D(\varphi) - D(\varphi_h)\|_{-1, \Omega} \leq ch^{\min\{a,b\}} \|f\|_{0, \Omega},
\]
where the constant \(c\) is independent upon the discretization.

So, the approximation method of the Kirchhoff-Love model described in this paper is unconditionally convergent and is optimal whenever the solution \((\varphi, u)\) of (1), as well as the solution \(\phi^f\) of (6) are sufficiently smooth. More precisely, if \(\varphi \in H^1(\Omega)^4\) and \(u \in H^3(\Omega)\) and \(\phi^f \in H^2(\Omega)\) (which is the case, for instance, when \(\Omega\) is convex since \(a = b = 1\)), then the convergence rate is \(O(h)\).

### 5.3 Approximation of the Reissner-Mindlin problem

Let us now study the discretization of the variational formulation (36) which describes the Reissner-Mindlin problem (2). In order to approximate \(p^\varepsilon\), we shall use the same finite elements as in the Kirchhoff-Love case. The approximation of the additional unknowns \(\lambda^\varepsilon \in L^2(\Omega)\) and \(\xi^\varepsilon \in W^\varepsilon\) will be achieved in the following finite dimensional spaces:

\[
W^\varepsilon_h = \left\{ \rho_h \in W^\varepsilon; \forall K \in T_h, \rho_{h|K} \in P_1(K) \right\},
\]
\[
L_h = \left\{ \lambda_h \in L^2(\Omega); \forall K \in T_h, \lambda_{h|K} \in P_0(K) \right\},
\]
while for \(\psi^\varepsilon\) we employ the finite element space:

\[
M_h = \left\{ \varphi_h \in (H^1(\Omega)|_K)^2; \forall K \in T_h, \varphi_{h|K} \in P_2(K)^2 \right\}.
\]

For the sake of simplicity, we denote:

\[
Y^\varepsilon_h = M_h \times W^\varepsilon_h \subset Y^\varepsilon
\]

and we put:

\[
F^\varepsilon_h((\varphi_h, \rho_h)) = F_h((\varphi_h, \rho_h)) - \varepsilon^2 a_0(\phi^f_h, \text{curl}(\varphi_h + \rho_h)),
\]

Then the discrete version of (36) writes as below:

\[
\begin{cases}
\text{find } (\psi^\varepsilon_h, \xi^\varepsilon_h) \in Y^\varepsilon_h, \quad p^\varepsilon_h \in Z_h, \quad \lambda^\varepsilon_h \in L_h & \text{such that} \\
\forall (\varphi_h, \rho_h) \in Y^\varepsilon_h, \quad A^\varepsilon((\psi^\varepsilon_h, \xi^\varepsilon_h), (\varphi_h, \rho_h)) + B((\varphi_h, \rho_h), p^\varepsilon_h) + C((\varphi_h, \rho_h), \lambda^\varepsilon_h) \\
\quad = F^\varepsilon_h((\varphi_h, \rho_h)) \\
\forall q_h \in Z_h, \quad B((\psi^\varepsilon_h, \xi^\varepsilon_h), q_h) = G_h(q_h) \\
\forall \mu_h \in L_h, \quad C((\psi^\varepsilon_h, \xi^\varepsilon_h), \mu_h) = 0
\end{cases}
\]

and we can obtain, in a technical manner, the existence and the uniqueness of the solution of this mixed problem. Let us notice here that thanks to the above choice of finite dimensional spaces, we are able to prove that the discrete \(\inf-sup\) condition of Babuška-Brezzi holds uniformly with respect to both the discretization parameter \(h\) and the plate’s thickness \(\varepsilon\).
Remark 5.1 It is equally possible to approximate $\psi^\varepsilon$ by the same finite elements as in the Kirchhoff-Love case and $\lambda^\varepsilon$ by linear elements, and thus one will get a cheaper method. However, the discrete inf-sup condition will not be uniform with respect to $h$ this time and the convergence of the method will depend on $\varepsilon$.

One can now establish an error bound similar to the previous case, that is:

$$
|\psi^\varepsilon - \psi_h^\varepsilon|_{1,\Omega} + ||\varepsilon - \varepsilon_h^\varepsilon||_{0,\Omega} + \varepsilon |\varepsilon - \varepsilon_h^\varepsilon|_{1,\Omega} + ||p^\varepsilon - p_h^\varepsilon||_{2,\Gamma} + ||\lambda^\varepsilon - \lambda_h^\varepsilon||_{0,\Omega}
\leq c\{\inf_{\varphi_h \in M_h} |\psi^\varepsilon - \varphi_h|_{1,\Omega} + \inf_{\rho_h \in W_h^\varepsilon} (||\varepsilon - \rho_h||_{0,\Omega} + \varepsilon |\varepsilon - \rho_h|_{1,\Omega})
+ \inf_{q_h \in Z_h} ||p^\varepsilon - q_h||_{2,\Gamma} + \inf_{\mu_h \in L_h} ||\lambda^\varepsilon - \mu_h||_{0,\Omega} + |\phi^f - \phi_h^f|_{1,\Omega}\},
$$

(58)

where $c$ is a constant independent upon $h$ and $\varepsilon$. Therefore it comes that for any fixed $\varepsilon$, the approximation method proposed for the Reissner-Mindlin model is unconditionally convergent.

It is now sufficient to use Theorem 4.2 in order to obtain the approximated bending moment $\sigma_h^\varepsilon$:

$$
\sigma_h^\varepsilon = \text{curl} \psi_h^\varepsilon + \varepsilon J_h^L + \phi_h^f L,
$$

(59)

while a $P_1$ - continuous finite element discretization of the boundary value problem (38) will give us an approximation of the transverse displacement $u_h^\varepsilon$. The discrete rotation vector $r_h^\varepsilon$ will then be recovered by the means of relation (17):

$$
r_h^\varepsilon = \frac{\varepsilon^2}{1 - \nu} \text{div} \sigma_h^\varepsilon + \nabla u_h^\varepsilon.
$$

(60)

Let us finally notice that the finite element method employed gives us low-order approximations of the physical quantities in the following spaces: $\sigma_h^\varepsilon \in H(div; \Omega)^2$, $u_h^\varepsilon \in H^1(\Omega)$ and $r_h^\varepsilon \in L^2(\Omega)^2$, with also an approximation of $\text{curl} r^\varepsilon$ in $L^2(\Omega)$ thanks to the multiplier $\lambda^\varepsilon$. Moreover, they satisfy the following estimates:

$$
||\sigma^\varepsilon - \sigma_h^\varepsilon||_{0,\Omega} + \varepsilon ||\text{div}(\sigma^\varepsilon - \sigma_h^\varepsilon)||_{0,\Omega} + ||D(\sigma^\varepsilon) - D(\sigma_h^\varepsilon)||_{-1,\Omega} + |u^\varepsilon - u_h^\varepsilon|_{1,\Omega}
\leq c\{||\psi^\varepsilon - \psi_h^\varepsilon||_{1,\Omega} + ||\varepsilon - \varepsilon_h^\varepsilon||_{0,\Omega} + \varepsilon ||\varepsilon - \varepsilon_h^\varepsilon||_{1,\Omega} + ||p^\varepsilon - p_h^\varepsilon||_{2,\Gamma} + |\phi^f - \phi_h^f||_{1,\Omega}\},
$$

(61)

and

$$
||r^\varepsilon - r_h^\varepsilon||_{0,\Omega} \leq |u^\varepsilon - u_h^\varepsilon|_{1,\Omega} + \varepsilon^2 ||\text{div}(\sigma^\varepsilon - \sigma_h^\varepsilon)||_{0,\Omega}.
$$

(62)

In order to obtain the convergence rate of the discretization method, let us assume the following regularity for the solution of (2):

$$
\begin{align*}
\sigma^\varepsilon & \in H^{1+a}(\Omega)^2, \quad u^\varepsilon \in H^{1+a}(\Omega) \quad \text{with } a > 0, \\
||r^\varepsilon||_{1+a,\Omega} + ||u^\varepsilon||_{1+a,\Omega} + \varepsilon ||\text{div}\sigma^\varepsilon||_{a,\Omega} & \leq c \|f\|_{0,\Omega}.
\end{align*}
$$

(63)
This hypothesis is verified in convex domains with \( a = 1 \), at least for clamped plates (see [2]). Let us also notice that one can’t improve the estimate for \( \text{div}\mathbf{\varepsilon} \), even for a smooth domain and a smooth loading \( f \): indeed, \( \text{div}\mathbf{\varepsilon} \) is not uniformly bounded in \( H^1(\Omega)^2 \) because of boundary layers in the Reissner-Mindlin model. Now, the previous inequality implies:

\[
|\psi^\varepsilon|_{1+a,\Omega} + \|\xi^\varepsilon\|_{a,\Omega} + \varepsilon|\xi^\varepsilon|_{1+a,\Omega} + \|\mathbf{p}^\varepsilon\|_{1/2+a,\Gamma} + \|\lambda^\varepsilon\|_{a,\Omega} \leq c \|f\|_{0,\Omega},
\]

which together with the estimates (58) and (61) allows us to deduce the main result of the section:

**Theorem 5.3** Under the above regularity assumptions, the discretization method for the Reissner-Mindlin model is convergent of order \( O(h_{\min\{a,b\}}) \):

\[
\|\mathbf{\sigma}^\varepsilon - \mathbf{\sigma}_h^\varepsilon\|_{0,\Omega} + \varepsilon\|\text{div}(\mathbf{\sigma}^\varepsilon - \mathbf{\sigma}_h^\varepsilon)\|_{0,\Omega} + |u^\varepsilon - u_h^\varepsilon|_{1,\Omega} + \|\mathbf{r}^\varepsilon - \mathbf{r}_h^\varepsilon\|_{0,\Omega} \leq ch_{\min\{a,b\}}\|f\|_{0,\Omega},
\]

independently upon the plate’s thickness.

Therefore, our method is locking-free and, in the case of a convex polygon for instance (when \( a = b = 1 \)), it is also optimal.
REFERENCES

