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SYMBOLIC COMPUTATION OF PLANETARY GEAR TRAIN EFFICIENCY

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Abstract. *The determination of the mechanical efficiency of a planetary gear train has been the object of research in the past years. However, most of the research done is oriented to obtain in a systematic manner the numerical value of the efficiency. The procedure presented in this work makes it possible to deduce symbolically the efficiency. The analytical expression of the efficiency is not unique for a given planetary gear train, since it depends on the power flow structure. The number of different cases for this structure increases rapidly with the complexity of the gear train. In spite of this, it is shown that it is possible to easily derive an analytical expression thanks to the special nature of the torque and power equations satisfied by the gear train. The procedure is applied to several types of planetary gear trains. Finally, the relation between efficiency and transmission ratio is studied.*

1 INTRODUCTION AND DEFINITIONS

Gear trains are widely used to transmit power between two shafts, the input shaft and the output one, and to get a desired output shaft speed. The reduction of the output shaft speed with respect to that of the input shaft, is the most common requirement. The ratio between the input and output shaft speed is called *transmission ratio*. There are an enormous variety of possible gear train configurations for such applications as machinery and gas turbine transmissions, automobile and helicopter gearboxes, and robotic wrist mechanisms. The most important classification of gear trains according to their structure is that in *ordinary gear trains* and *planetary gear trains*. In an ordinary gear train the shafts of the gears are fixed in space and therefore each gear rotates only around its shaft. A planetary gear train consists of one or more central gears rotating around a fixed shaft with gears revolving around them, like planets moving around the sun. Due to this analogy, the gears which rotate only around the fixed shaft are called *suns* and gears which rotate around the moving axis are called *planets*. Points on the planet gears trace out epicyclic curves. For this reason, planetary gear trains are also called 'epicyclic gear trains' or 'epicyclic drives'.

Each meshing pair of gears has associated with it a *gear carrier* or *arm* which ensures that the distance between the centers of the two gears remains constant. From a constructive point of view, there can be only three types of entities in a planetary gear trains: suns, planets and arms. We shall call any of them *element*. Each set of meshing sun, planet and its arm, constitutes a *fundamental circuit* or simply circuit, from now on. Another important concept is that of *link*. A link is a set of elements which are held together. Then, each link has its own rotating speed and all its elements have the same speed. The elements of a link may be held by means of screws, keys or other similar ways. It is also possible that a link is manufactured from a single piece. According to the type of elements of a link, there may be two types of links:

a) links composed of an arbitrary number of suns and arms, including only a single sun or a single arm, and

b) links composed of an arbitrary number of planets.

It is easy to understand that the former rotate around the fixed shaft of the planetary gear train, whereas the latter have a planet-like movement. There are also two types of kinematic connections or *kinematic pairs*:

a) gear pairs between meshing gears and

b) turning or rotating pairs between the fixed shaft and a sun or between a planet and its arm.

Rotating pairs are built with rolling or sliding bearings. It has been shown[1] that the following equation holds between the number of links N, the number of gear pairs J and the number of degrees of freedom (d.o.f.) F of a planetary gear train:

$$F = N - J - 1 \tag{1}$$

Bearing in mind that each circuit is uniquely defined by a gear pair, the number of gear pairs equals the number of circuits. Moreover, a circuit is composed of three different elements, each of them being part of a different link.

2 A BRIEF LITERATURE REVIEW

Planetary gear trains offers the possibility of achieving a given transmission ratio with smaller weight and size than those that would be required with an ordinary gear train. Therefore, the analysis of planetary gear trains presents a considerable interest in Mechanical Engineering. In an ordinary gear train the power loss due to gear teeth friction is usually about one or two percent of the transmitted power. On the contrary, the power loss in planetary gear drives can be surprisingly high leading to a very low efficiency. Therefore, the analysis of the mechanical efficiency is one of the more important steps when designing a planetary gear drive. The analysis should not only give a numerical estimate of the efficiency, but also explain the manner in which the power is transmitted throughout the mechanical device. The former is obviously important for the designer and the latter is also important because the causes of a poor efficiency and the solutions for improving it, can be found by analysing the power-flow.

Early studies on the efficiency of single degree of freedom planetary gear trains goes back to[2]. The efficiency of multiple degree of freedom planetary gear trains and variable speed drives is treated in[3], [4] and[5]. The works of Tuplin[6], Glover[7] and Jensen[8] remark the importance of the efficiency as a performance criterion and try to provide some guidelines for the design of planetary gear trains from this point of view. Namely, some relatively simple planetary gear trains are studied in[7] and ready-to-use formulas for the efficiency are given. The relationship between transmission ratio and efficiency is studied in[8]. Sanger[9] analyzes the problem of the determination of power flow in complex planetary gear trains. This problem is intimately linked to that of the efficiency calculation. Looman[10] has also addressed the problem of self-locking for a particular type of planetary gear train. The approach given in[11] tries to be more general.

In general, all the previously cited works are applied to particular types of planetary gear trains. In principle, they could be applied for very complex ones. However, such methods are manual methods and are conceived for being applied directly by the engineer. Unfortunately, a manual analysis of complex gear trains becomes lengthy and tedious. Very high and enduring mental concentration is required to avoid errors. The development of general and systemathic methods for determining the efficiency of planetary gear trains has drawn the interest of several authors. In particular, the methods proposed in[12], [13] and[14] are conceived for being implemented on a computer program. From all these three methods, that developed in[14] is the one that can handle more general planetary gear train structures. The method developed by[13] is the first one in applying symbolic computation to the problem. However, it is oriented to the analysis of split-path transmissions. This is also the case of[15] for a split-path continuously variable transmission. Symbolic Mathematics software has also been applied in[16] for the determination of speeds and torques in frictionless planetary gear trains.

Krstich[17] analyzes the efficiency of a two-d.o.f. differential gear train for automotive applications. In spite ot the title of his work, it is focused on a particular type of differential gear train.

The method proposed in this work is conceived for its implementation on a symbolic mathematical computation program (SMCP). Any type of planetary gear train can be treated. In fact, other type of power drives such as split-path transmissions can be handled. The method yields not only the expression of the efficiency, but also gives those of the torques exerted on the elements of the gear drive. A nice feature of the proposed method is explained in section 6. The analytical expression of the efficiency is not unique and there are many variants. The number of variants increases with the complexity of the train. Unlike the method explained in[13], the present method makes it unnecessary to derive all the variants of the efficiency expression.

3 SPEED EQUATIONS

It has been previously explained that the number of distinct turning speeds is equal to the number of links. Moreover, any circuit is composed by a sun, a planet and an arm. We shall use the subscripts i, j and r for these elements. Therefore, for each of the J circuits one can write the following speed equation:

$$\frac{\omega_j - \omega_r}{\omega_i - \omega_r} + Z_{ij} = 0 \tag{2}$$

In the above expression, Z_{ij} is the ratio betwen the number of teeth of gear i to that of gear j. For (2) to be correct, Z_{ij} should be taken positive for outside engaged gears, and negative for inside engaged gears.

In a planetary gear train with F degrees of freedom, there must be a link held fixed or *ground link*, an input link and F output links. The speed of the ground link is zero and there are F speeds which can be arbitrarily chosen. In a real application of such a planetary gear train, these speeds are the input speed and those of $F-1$ output links. Then, the number of unknown speeds is $N-F-1$ which after (1), equals the number of speed equations like (2). The object of this work are single degree of freedom planetary gear trains. In this particular case, the unknown speeds are the only output speed and the $N-3$ speeds of the links excluding the ground and the input link. Moreover, without loss of generality, the input speed can be made equal to one. The symbolic solution of the resulting system of $N-2$ equations (2), is straightforward. The solution actually yields the speed ratios respect to the input speed.

4 TORQUE AND POWER EQUATIONS

In this section, we will derive the equations that determine the torques exerted on the elements of a planetary gear train and the power transmitted to them. In the rest of this work, it will be assumed that the gear train is in equilibrium and consequently the inertia forces do not appear. First of all, let us define the following variables:

a) T_i^{ext} = torque externally applied on the element i of a given circuit, and

b) T_i = torque exerted on the element i of a given circuit by the remaining elements of that circuit.

From the equilibrium condition of each element, we should have $T_i^{ext} + T_i = 0$. The equilibrium condition is also satisfied by the circuit itself. Denoting with the subscripts i, j and r the sun, planet and arm of the circuit, one should have that $T_i^{ext} + T_j^{ext} + T_r^{ext} = 0$. Finally, combining these expressions, we arrive at the equilibrium equation for the circuit:

$$T_i + T_j + T_r = 0 \tag{3}$$

Figure 1 shows the torques and forces applied on the elements of a circuit. For the purpose of clarity, only the tangential component of the force exerted on the teeth gears, F, has been depicted. The forces that keeps the sun and the arm in equilibrium are not shown.

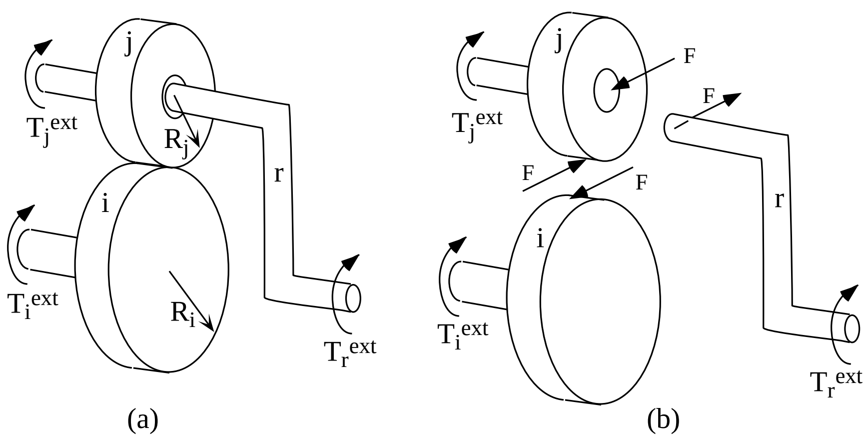

Figure 1: Torques and forces applied to the elements of a circuit.

In order to distinguish the circuit for which the equilibrium equation is writen, we will use a second subscript. Then, the equilibrium equation for circuit n is:

$$T_{in} + T_{jn} + T_{rn} = 0 \tag{4}$$

where T_{in} is the torque exerted on the element i of the circuit n by the remaining elements of that circuit. If we neglect the power loss caused by the friction at the gear contact, the power balance in the elements of the circuit is:

$$T_{in}\omega_i + T_{jn}\omega_j + T_{rn}\omega_r = 0 \tag{5}$$

and taking into account the speed equation (2), we get finally:

$$T_{in} = T_{jn}Z_{ij} \tag{6}$$

This equation could be also derived from Figure 1 (b), since the torque equilibrium requires $T_i = -FR_i$ and $T_j = -FR_j$. The number of equations like (3) and (6) is equal to the number of circuits or equivalently, to the number of gear pairs J. Another set of equations that should be satisfied is obtained by considering the torque equilibrium in each link of the gear train. Then, considering an arbitrary link k, we can write:

$$\sum_{n=1}^{J} T_{kn} = 0 \tag{7}$$

where it is implicitly assumed that $T_{kn} = 0$ if the link k does not form part of circuit n. Equation (7) takes a particular form for the following three cases: when k is the input link, the ground link or any of the output links. In the first case, we shall write:

$$T_{input} + \sum_{n=1}^{J} T_{kn} = 0 \tag{8}$$

where T_{input} is the input torque. In the second case, denoting by T_{reac} the reaction torque applied to the ground link, we shall write:

$$T_{reac} + \sum_{n=1}^{J} T_{kn} = 0 \tag{9}$$

Finally, if the planetary gear train has F degrees of freedom, it will be possible to write F equations like:

$$T_{l,out} + \sum_{n=1}^{J} T_{kn} = 0 \tag{10}$$

where $T_{l,out}$ is the output torque delivered by the l output link. Table 1 enumerates the number and types of equations and unknowns. It is assumed that the input torque is known. Therefore, remenbering that planetary gear trains satisfy (1), the number of unknown torques is equal to the number of equations. The above set of equations can be readily solved using a SMCP.

unknown torques	number	equations	number
torques on elements: $T_{in}, T_{jn}, T_{rn} \quad n = 1, \dots J$	$3J$	equilibrium on circuits	J
output torques: $T_{l,out} \quad l = 1, \dots F$	F	power balance on circuits	J
reaction torque: T_{reac}	1	equilibrium on links	N
TOTAL:	$3J + F + 1$	TOTAL:	$2J + N$

Table 1: Equations and unknown torques on a planetary gear train.

5 CONSIDERATION OF GEAR FRICTION LOSSES

Equations (6) have been obtained neglecting the power loss due to the friction. Let us denote by η_n the efficiency of a gear pair corresponding to the circuit n, and let i and j be the gears of that circuit. Then, if gear j is driven by gear i, we will have:

$$\eta_n = -\frac{T_{jn}(\omega_j - \omega_r)}{T_{in}(\omega_i - \omega_r)} \tag{11}$$

In other words, η_n would be the efficiency achieved by the gear pair if the arm were held and the planetary gear pair transformed to an ordinary gear pair. For this reason, we shall call this efficiency, *ordinary efficiency*. It takes values close to one. Introducing (2) in the above expression, one gets:

$$T_{in} = T_{jn}\frac{Z_{ij}}{\eta_n} \tag{12}$$

Inversely, if gear i is driven by j, the right expression should be:

$$T_{in} = T_{jn} Z_{ij} \eta_n \tag{13}$$

We can take into account simultaneously both cases by defining the *effective gear teeth ratio* Z^*_{ij} as:

$$Z^*_{ij} = \begin{cases} \dfrac{Z_{ij}}{\eta_n} & \text{if } i \text{ drives } j \\ Z_{ij}\eta_n & \text{if } j \text{ drives } i \end{cases} \tag{14}$$

Then, when considering the friction, the equations of the power balance in circuits (6) should be replaced by:

$$T_{in} = T_{jn} Z^*_{ij} \tag{15}$$

For the correct application of the above expression, it is necessary to determine which of the gears of the circuit is the driver one. A gear of a circuit is the driving gear if the externally applied torque performs a positive work in the circuit resulting from holding the arm. In terms of instantaneous work, this condition is:

$$T_{in}^{ext}(\omega_i - \omega_r) > 0 \tag{16}$$

Writting the above equation in terms of the torque exerted by the remaining elements of the circuit on i, we conclude that gear i drives j if:

$$T_{in}(\omega_i - \omega_r) < 0 \tag{17}$$

since $T_{in}^{ext} + T_{in} = 0$.

6 SYMBOLIC DERIVATION OF THE EFFICIENCY

This section explains how to derive the symbolic expression of the efficiency for a one-d.o.f. planetary gear train. The results of the last section show that the expression of the efficiency is not unique, since it depends on the power flow sense over the circuits. Unfortunately, the determination of the power transmission requires the calculation of the speeds and torques. Moreover, we should replaced equations (6) by equations (15) for the calculation of the torques, but if we still do not know the power flow sense we cannot determine the right expression of the Z_{ij}^*. To escape from this devil's circle, the calculation of the torques regarding the friction proceeds as follows:

I) calculate the speeds from (2);

II) calculate the torques neglecting the friction losses, that is from the circuits equations (4) and (6) and the links equations (7), (8), (9) and (10);

III) determine on each circuit the driving and driven gear, according to the criterion given by (17);

IV) replace the actual gear teeth ratios Z_{ij} by the effective ratios Z_{ij}^*, after (14).

V) calculate again the torques but now replacing equations (6) by their counterparts (15).

Finally, for a one-d.o.f. planetary gear train, the efficiency is given by:

$$\eta = -\frac{T_{out}\omega_{out}}{T_{in}\omega_{input}} \tag{18}$$

In the above expression the speeds are those of the frictionless train, since the speed equations obviously remains unchanged when the friction is considered. On the contrary, the output torque T_{out} is that calculated in the last step, that is, from the equations with the effective gear teeth ratios. The above procedure lies on the assumption that the friction does not modify the direction of the power flow in the gear train. This is a reasonable assumption, see[9]. It allows identifying the driving gears by neglecting the friction losses.

The previously explained procedure has been implemented on a computer program in[14]. This implementation has the drawback that it can only perform a numerical evaluation of the efficiency for a planetary gear train with given gear ratios. The first obstacle for the symbolic evaluation of the efficiency is the fact that for a planetary gear train with J circuits or gear pairs, there may be 2^J different analytical expressions of it. Therefore, the number of possible variants of the expression grows rapidly as the complexity of the train increases.

This might seem an unsurmountable obstacle for the symbolic evaluation. However, it is possible to obtain any of the 2^J variants, since for each of them, the Z_{ij} are replaced by the Z^*_{ij}. Then, the resulting expression of the torques will be those obtained by simply doing the same replacement in the expressions of the torques obtained without considering the friction losses. In summary, this means that we do not need to evaluate the 2^J expressions. This allows the practical use of SMCP for the desired purpose.

7 EFFICIENCY OF THE FOUR LINK TRAINS

The minimum number of links of a one-d.o.f. planetary gear train is four, see[18]. The 4-link planetary gear train is the simplest one that can be built. Many commercial planetary gear drives are 4-link trains or are composed by several stages of 4-link trains, the input shaft of a stage being the output shaft of the previous one. In this section, the efficiency of the 4-link planetary gear trains is derived by the procedure previously presented.

According to (1), a 4-link 1-d.o.f. planetary gear train has two circuits or two gear pairs, that is $J = 2$. Figure 2 (a) shows a schematic representation of a 4-link train. In that figure, gears have been represented as vertically white elongated boxes. Arms and shafts are represented by thick solid lines. The links are numbered from 1 to 4. Links 1, 2, 3 are composed of only one element. The elements of links 1 and 2 are suns. Actually, the gear of link 2 is a ring gear with internal teeth, whereas the gear of link 1 is a sun gear with external teeth. The ground link is 2 and this is represented by the diagonal short lines on its base. Link 3 is the only arm of the train and link 4 is made up of two planets. The left hand side planet meshes with 1 and the right hand side planet with 2. The two circuits of this planetary gear train are characterized by the combination $i-j-r$ or sun-planet-arm. In what follows we define circuits a and b as those whose elements combination are $1-4-3$ and $2-4-3$, respectively.

The configuration shown in Figure 2 (a) for the 4-link train is not the only possible. It is also possible to built link 4 from a single planet, like Figure 2 (b) shows. Likewise, it is also possible to choose link 2 as a gear with external teeth and link 1 as a ring gear. This kind of constructive configurations do not affect the symbolic evaluation of the efficiency.

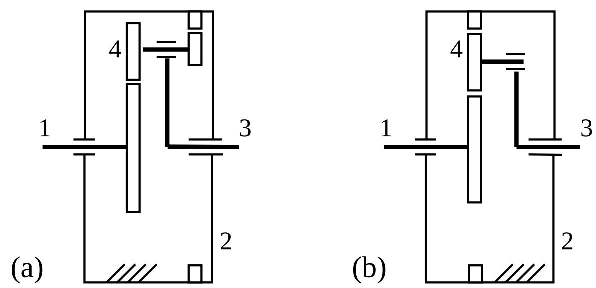

Figure 2: Schematic representation of two different configurations of a 4-link planetary gear train.

This is an importannt advantage of the procedure presented in this work, in comparison to the method proposed in[7]. The configuration influences the numerical value of the efficiency through the numerical values of the gear ratios Z_{ij}. Particularly, for the trains of Figure 2, one should take Z_{14} positive and Z_{24} negative.

Moreover, there are two possible inversions of the 4-link train. One corresponds to link 1 as input link and 3 as output link and the other to the inverse choice. We shall refer to these inversions as 4-link train with *input to sun* and *input to arm*, respectively.

7.1 4-link planetary gear train, input to sun.

The speed equations (2) are:

$$\frac{\omega_1 - \omega_3}{\omega_4 - \omega_3} + Z_{41} = 0 \qquad \frac{\omega_4 - \omega_3}{\omega_2 - \omega_3} + Z_{24} = 0 \tag{19}$$

where ω_2 should be set equal to zero, since the ground link is 2. The equilibrium equations (4) on the circuits a and b are:

$$T_{1a} + T_{4a} + T_{3a} = 0 \qquad T_{2b} + T_{4b} + T_{3b} = 0 \tag{20}$$

Next, the power balance equations (6) on the circuits for the frictionless train become:

$$T_{4a} = T_{1a} Z_{41} \qquad T_{2b} = T_{4b} Z_{24} \tag{21}$$

Finaly, the equilibrium equations (7), (8), (9) and (10) applied to the links 1 to 4, take the form:

$$\left.\begin{array}{ll} T_{input} + T_{1a} = 0 & T_{reac} + T_{2b} = 0 \\ T_{out} + T_{3a} + T_{3b} = 0 & T_{4a} + T_{4b} = 0 \end{array}\right\} \tag{22}$$

In the above equations, tne input shaft speed ω_1 and the input torque T_{input} can be made equal to one without loss of generality. Then, all the speeds and torques are actually the

link l	speeds ω_l $\omega_{out} = \omega_3$	torques in circuit n $i-j-r=$ sun-planet-arm a: 1-4-3	b: 2-4-3
1	1	-1	
2	0		$Z_{24}Z_{41}$
3	$\dfrac{1}{1-Z_{24}Z_{41}}$	$1+Z_{41}$	$-Z_{41}(1+Z_{24})$
4	$\dfrac{1+Z_{24}}{1-Z_{24}Z_{41}}$	$-Z_{41}$	Z_{41}
$T_{jn}(\omega_j-\omega_r)$		$\dfrac{-Z_{24}Z_{41}}{1-Z_{24}Z_{41}}$	$\dfrac{Z_{24}Z_{41}}{1-Z_{24}Z_{41}}$
$T_{out} = -1 + Z_{24}Z_{41}$		$T_{reac} = -Z_{24}Z_{41}$	

Table 2: Speeds and torques for the 4-link planetary gear train with input to the sun.

ratio to the input speed and input torque. Bearing in mind (18), the efficiency is then given by $-T_{out}\omega_{out}$. Table 2 contains all the variables resulting from the symbolic solution of equations (19) to (22).

The information provided in Table 2 is manifold and allows to evaluate all the magnitudes associated with the planetary gear train, that is, speeds, torques and efficiency. The first group of rows in Table 2 shows the speed of the links ω_l and the torques exerted on the elements, T_{in}. In this case i can be 1 to 4 and n can be a or b, for the two existing circuits. There are two blank cells in the group of cells of the torques. These are those corresponding to T_{1b} and T_{2a}. These torques can be considered zero since link 1 is not part of circuit b and 2 is not part of circuit a. Actually, these torques do not even appear in the equations. For this reason, their cells have been left blank. The heading of the torque cells, the upper right corner cell, gives also information of the structure of the circuits. Namely, it gives the links that composed the circuit in the following order: sun, planet and arm. In the 4-link planetary gear train, links 1 and 2 are just suns. This is not the case in more complex trains and a link that is the sun of a circuit, can be the arm in other circuit. Hence, the information provided in the upper right corner cell is very useful.

The second group of cells has only one row and this corresponds to the powers of the fixed-shaft circuit. Namely, the magnitude shown in this row is the power exerted on the planet of the circuit. This has been indicated by denoting this row with its analytical expression $T_{jn}(\omega_j - \omega_r)$, where n is the circuit and j is the planet of the circuit. Therefore, the two cells of this row are $T_{4a}(\omega_4 - \omega_3)$ and $T_{4b}(\omega_4 - \omega_3)$. The third group of rows show the output and the reaction torques, as it is indicated. The format given to Table 2, makes it possible to provide in a compact manner all the relevant magnitudes of a planetary gear train. Finally, the efficiency of the train is given by $-T_{out}\omega_{out}$. According to the procedure explained in section 6, the expression of T_{out} is that of the Table 2 obtained by replacing Z_{24} and Z_{41} by Z_{24}^* and Z_{41}^*, respectively. The expression of these two last gear ratios is obtained after (14) and once the sign of the powers $T_{4a}(\omega_4 - \omega_3)$ and $T_{4b}(\omega_4 - \omega_3)$ becomes known. For this 4-link planetary gear train, there are only two variants of the efficiency expression. This is because the above powers have equal absolute values and opposite signs. Finally, we can summarize the results for the efficiency in Table 3.

condition	efficiency: $\eta = -T_{out}\omega_3$	torques are obtained by replacing
if $\dfrac{Z_{24}Z_{41}}{1 - Z_{24}Z_{41}} > 0$	$\eta = \dfrac{1 - Z_{24}Z_{41}/(\eta_a\eta_b)}{1 - Z_{24}Z_{41}}$	Z_{41} by Z_{41}/η_a Z_{24} by Z_{24}/η_b
if $\dfrac{Z_{24}Z_{41}}{1 - Z_{24}Z_{41}} < 0$	$\eta = \dfrac{1 - Z_{24}Z_{41}\eta_a\eta_b}{1 - Z_{24}Z_{41}}$	Z_{41} by $Z_{41}\eta_a$ Z_{24} by $Z_{24}\eta_b$

Table 3: Expressions of the efficiency for the 4-link planetary gear train with input to the sun.

In Table 3, the ordinary efficiencies of circuits a and b have been denoted by η_a and η_b, respectively. It is important to remember that the expressions of the torques exerted on the elements when the friction is considered, are obtained from those shown in Table 2 once the gear ratios are replaced by the effective gear ratios. The right replacement is indicated in Table 3.

7.2 4-link planetary gear train, input to arm.

The application of the procedure for this case leads to the results shown in Tables 4 and 5. For the same reason as in the previous case, there are only two variants of the efficiency expression. The input speed and torque have been set equal to one. Since the output shaft is now link 1, the efficiency is given by $-T_{out}\omega_1$.

link l	speeds ω_l $\omega_{out} = \omega_1$	torques in circuit n $i - j - r =$ sun-planet-arm a: 1-4-3	b: 2-4-3
1	$1 - Z_{24}Z_{41}$	$\dfrac{1}{1 - Z_{24}Z_{41}}$	
2	0		$\dfrac{1}{1 - Z_{14}Z_{42}}$
3	1	$\dfrac{1 + Z_{41}}{Z_{24}Z_{41} - 1}$	$\dfrac{1 + Z_{42}}{Z_{14}Z_{42} - 1}$
4	$1 + Z_{24}$	$\dfrac{1}{Z_{14} - Z_{24}}$	$\dfrac{1}{Z_{24} - Z_{14}}$
$T_{jn}(\omega_j - \omega_r)$		$\dfrac{Z_{24}Z_{41}}{1 - Z_{24}Z_{41}}$	$\dfrac{Z_{24}Z_{41}}{Z_{24}Z_{41} - 1}$
$T_{out} = \dfrac{1}{Z_{24}Z_{41} - 1}$		$T_{reac} = \dfrac{1}{Z_{42}Z_{14} - 1}$	

Table 4: Speeds and torques for the 4-link planetary gear train with input to the arm.

8 EFFICIENCY OF THE FIVE LINK PLANETARY GEAR TRAIN

There are two types of 5-link planetary gear trains. One of them has a gear pair connecting two planets and the other one has only gear pairs connecting a planet to a sun. This last one is the object of this section, because it is more interesting from a practical point of view. Furthermore, there is only one inversion of this planetary gear train without idle gears. Figure 3 (a) shows a schematic representation of this train. Links 1 and 3 can be indistinctly considered as the input and output links. Link 4 is an arm and link 5 is made of two planets. Finally, 2 is the ground link. A real model having the scheme of Figure 3 (a) is shown in Figure 3 (b). The corresponding links have been marked with the same number. One should note that the arm 4 cannot be selected as the input or output link, because then either link 1 or 3 would become an idle gear. Therefore, without loss of generality, one may consider as the only inversion of this train that having 1 as the input link and 3 as the output link. This is the choice adopted for

condition	efficiency: $\eta = -T_{out}\omega_1$	torques are obtained by replacing
if $\dfrac{Z_{24}Z_{41}}{1-Z_{24}Z_{41}} > 0$	$\eta = \dfrac{Z_{24}Z_{41}-1}{Z_{24}Z_{41}\eta_a\eta_b - 1}$	Z_{41} by $Z_{41}\eta_a$ Z_{24} by $Z_{24}\eta_b$
if $\dfrac{Z_{24}Z_{41}}{1-Z_{24}Z_{41}} < 0$	$\eta = \dfrac{Z_{24}Z_{41}-1}{Z_{24}Z_{41}/(\eta_a\eta_b) - 1}$	Z_{41} by Z_{41}/η_a Z_{24} by Z_{24}/η_b

Table 5: Expressions of the efficiency for the 4-link planetary gear train with input to the arm.

the evaluation of the efficiency. The results are shown in Table 6 and presented in the same way as in the previous section.

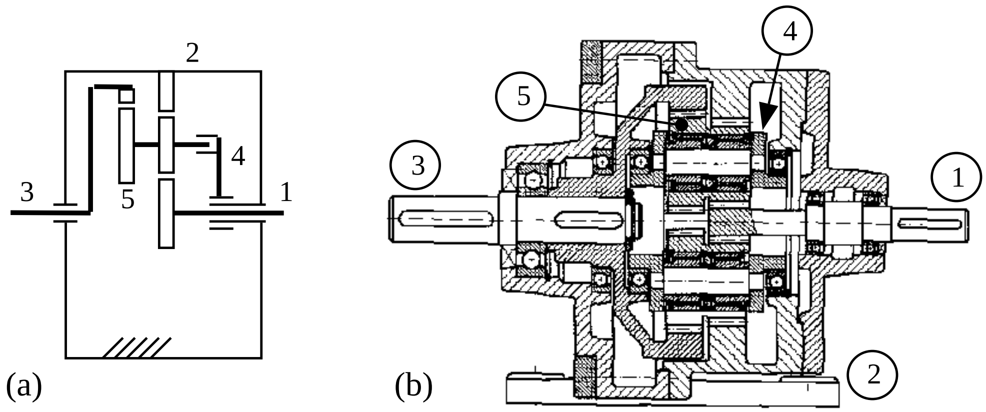

Figure 3: Schematic representation and real model of a 5-link planetary gear train.

The 1-d.o.f. 5-link planetary gear train has 3 gear pairs. This means that there may be up to $2^3 = 8$ different variants of the efficiency expression. However, two of them cannot arise, since the following holds:

$$T_{5a}(\omega_5 - \omega_4) + T_{5b}(\omega_5 - \omega_4) + T_{5c}(\omega_5 - \omega_4) = 0 \tag{23}$$

Thus, the cases with all the above powers having the same sign are not possible. Let us recall that a planet link is a link whose elements are only planets. Then, it is easy to show that for an arbitrary planetary gear train with N_p planet links, the number of different variants of the efficiency expression is given by:

$$\Pi_{i=1}^{N_p}(2^{J_i} - 2) \tag{24}$$

link l	speeds ω_l $\omega_{out} = \omega_3$	torques in circuit n, $i-j-r =$ sun-planet-arm — a: 1-5-4	b: 2-5-4	c: 3-5-4
1	1	-1		
2	0		$\dfrac{Z_{25}(Z_{51}-Z_{53})}{1-Z_{25}Z_{53}}$	
3	$\dfrac{1-Z_{25}Z_{53}}{1-Z_{25}Z_{51}}$			$\dfrac{1-Z_{25}Z_{51}}{1-Z_{25}Z_{53}}$
4	$\dfrac{1}{1-Z_{25}Z_{51}}$	$1+Z_{51}$	$\dfrac{(1+Z_{25})(Z_{53}-Z_{51})}{1-Z_{25}Z_{53}}$	$\dfrac{(Z_{25}Z_{51}-1)(1+Z_{53})}{1-Z_{25}Z_{53}}$
5	$\dfrac{1+Z_{25}}{1-Z_{25}Z_{51}}$	$-Z_{51}$	$\dfrac{Z_{51}-Z_{53}}{1-Z_{25}Z_{53}}$	$\dfrac{Z_{53}(1-Z_{25}Z_{51})}{1-Z_{25}Z_{53}}$
$T_{jn}(\omega_j-\omega_r)$		$\dfrac{Z_{51}Z_{25}}{Z_{25}Z_{51}-1}$	$\dfrac{(Z_{51}-Z_{53})Z_{25}}{(1-Z_{25}Z_{53})(1-Z_{25}Z_{51})}$	$\dfrac{Z_{53}Z_{25}}{1-Z_{25}Z_{53}}$
$T_{out} = \dfrac{Z_{25}Z_{51}-1}{1-Z_{25}Z_{53}}$			$T_{reac} = \dfrac{Z_{25}(Z_{53}-Z_{51})}{1-Z_{25}Z_{53}}$	

Table 6: Speeds and torques for the 5-link planetary gear train.

where J_i is the number of gear pairs of the planet link i. The factor within parenthesis in the above expression is the number of binary combinations of the power flow on the gear pairs, except the two cases in which all the power flows have the same sign. For the 4-link and 5-link planetary gear trains, we have $N_p = 1$, $J_1 = 2$, and $N_p = 1$, $J_1 = 3$, respectively. The resulting expressions for the efficiency of the 5-link gear train are enumerated in Table 7. The torques when the friction is considered, are obtained from Table 6 after replacing the gear ratios by the effective gear ratios according to criterion (17) and expression (14).

$\frac{Z_{51}Z_{25}}{Z_{25}Z_{51}-1}$	$\frac{(Z_{51}-Z_{53})Z_{25}}{(1-Z_{25}Z_{53})(1-Z_{25}Z_{51})}$	$\frac{Z_{53}Z_{25}}{1-Z_{25}Z_{53}}$	efficiency: $\eta = -T_{out}\omega_3$
$+$	$+$	$-$	$\frac{Z_{52}\eta_b - Z_{51}\eta_a}{Z_{52}\eta_b - Z_{53}/\eta_c} \cdot \frac{Z_{52}-Z_{53}}{Z_{52}-Z_{51}}$
$+$	$-$	$+$	$\frac{Z_{52}/\eta_b - Z_{51}\eta_a}{Z_{52}/\eta_b - Z_{53}\eta_c} \cdot \frac{Z_{52}-Z_{53}}{Z_{52}-Z_{51}}$
$+$	$-$	$-$	$\frac{Z_{52}/\eta_b - Z_{51}\eta_a}{Z_{52}/\eta_b - Z_{53}/\eta_c} \cdot \frac{Z_{52}-Z_{53}}{Z_{52}-Z_{51}}$
$-$	$+$	$+$	$\frac{Z_{52}\eta_b - Z_{51}/\eta_a}{Z_{52}\eta_b - Z_{53}\eta_c} \cdot \frac{Z_{52}-Z_{53}}{Z_{52}-Z_{51}}$
$-$	$+$	$-$	$\frac{Z_{52}\eta_b - Z_{51}/\eta_a}{Z_{52}\eta_b - Z_{53}/\eta_c} \cdot \frac{Z_{52}-Z_{53}}{Z_{52}-Z_{51}}$
$-$	$-$	$+$	$\frac{Z_{52}/\eta_b - Z_{51}/\eta_a}{Z_{52}/\eta_b - Z_{53}\eta_c} \cdot \frac{Z_{52}-Z_{53}}{Z_{52}-Z_{51}}$

Table 7: Expressions of the efficiency for the 5-link planetary gear train.

9 EFFICIENCY vs. TRANSMISSION RATIO

For the 4-link planetary gear train, it is possible to find a relationship between efficiency and transmission ratio. In this section, this relationship is derived when the 4-link planetary gear train is a speed reduction drive, that is, when the output shaft speed is less than that of the input shaft. In this case, and for the gear train with the input to sun we have:

$$R = \frac{\omega_3}{\omega_1} = \frac{1}{1 - Z_{24}Z_{41}} \tag{25}$$

Since the gear train is a speed reducer, $|R| < 1$ and therefore:

$$\frac{Z_{24}Z_{41}}{1 - Z_{24}Z_{41}} = R - 1 < 0 \tag{26}$$

In this case, the expression of the efficiency according to the results shown in Table 3 is given by:

$$\eta = \frac{1 - Z_{24}Z_{41}\eta_a\eta_b}{1 - Z_{24}Z_{41}} = R\left[1 - \left(1 - \frac{1}{R}\right)\eta_a\eta_b\right] = R + (1 - R)\eta_a\eta_b \tag{27}$$

For the input to arm 4-link planetary gear train, the transmission ratio is:

$$R = \frac{\omega_1}{\omega_3} = 1 - Z_{24}Z_{41} \tag{28}$$

Now, for $|R| < 1$ we should distinguish between two cases: $R > 0$ and $R < 0$. According to Table 5, the power flow is:

$$\frac{Z_{24}Z_{41}}{1 - Z_{24}Z_{41}} = \frac{1}{R} - 1 \tag{29}$$

Then, if the input and output shafts have the same rotation sense, the efficiency is:

$$\eta^+ = \frac{Z_{24}Z_{41} - 1}{Z_{24}Z_{41}\eta_a\eta_b - 1} = \frac{R}{1 - (1 - R)\eta_a\eta_b} \tag{30}$$

On the contrary case, we have:

$$\eta^- = \frac{Z_{24}Z_{41} - 1}{Z_{24}Z_{41}/(\eta_a\eta_b) - 1} = \frac{R}{1 - (1 - R)/(\eta_a\eta_b)} \tag{31}$$

Figure 4 plots the efficiency of the 4-link planetary gear reducer as a function of the transmission ratio. The solid lines correspond to an ordinary efficiency $\eta_a = \eta_b$ of 0.98 and the dashed lines to a value of 0.75. The straight lines correspond to the input to sun case. The curves having two branches for R positive and negative and dropping to zero, correspond to the input to arm case. The most interesting conclusion is the fact that the design of a 4-link planetary gear reducer with input to the sun is more efficient than that with input to the arm.

10 CONCLUSIONS

In this work, a systemathic procedure for the evaluation of planetary gear trains efficiency has been presented. It has been shown that the analytical expression of the efficiency can be easily obtained using a SMCP for solving the equations. The procedure has been applied to the simplest planetary gear trains: those having four and five links. The procedure can be applied to more complex planetary gear trains. Likewise, split-path transmissions can be also treated. From a kinematic and power flow point of a view, they are just inversions of planetary gear trains having an arm as the ground link.

The advantages of the proposed procedure over other methods proposed by different authors are manifold:

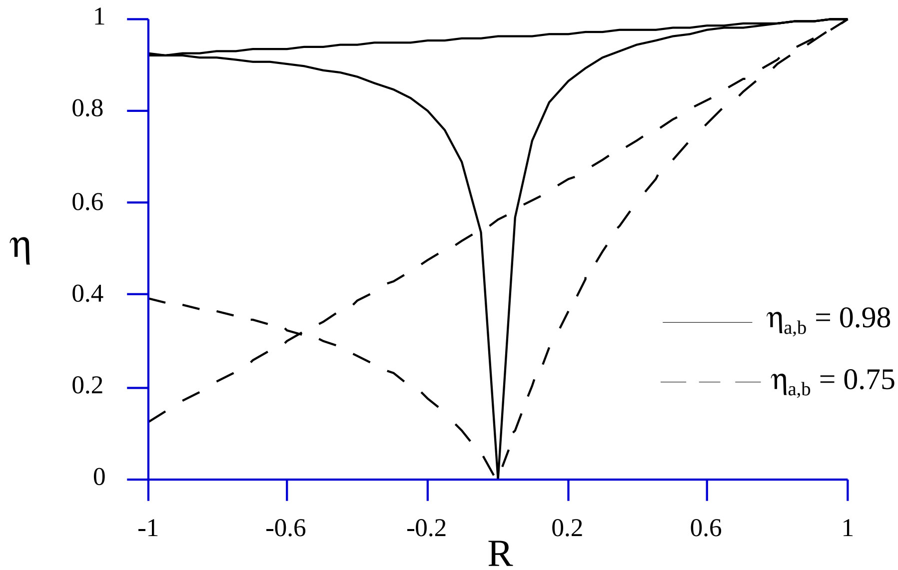

Figure 4: Efficiency of the 4-link planetary gear reducer as a function of the transmission ratio.

a) it can be easily implemented on a computer and fully automatized;

b) the only required information is the structure of the circuits of the planetary gear train, namely, the number and composition of them;

c) it gives not only the efficiency but also the torques exerted on the elements, which is a magnitude of paramount importance for the design of the train;

d) it does not rely on any analytical skill or knowledge of the engineer;

e) it is not necessary to derive all the variants of the efficiency expression.

f) it can handle any other type of one-d.o.f. power drive of known ordinary efficiency.

The analytical expression of the efficiency is not unique. It depends on the direction of the power flow on the gear pairs. Therefore, this information should also be given for the final determination of the efficiency. The number of different variants of the efficiency expression has also been deduced. Finally, the relationship between transmission ratio and efficiency has been derived for 4-link planetary gear reducers. It has been shown that a speed reducer with input to the sun is preferable to that with input to the arm.

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