NOVEL FORM OPTIMIZATION CONCEPTS OF SHELL AND MEMBRANE STRUCTURES

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**Key words:** Optimization, Form Finding, Membranes, Shells.

**Abstract.** There exist two different approaches to the generation of structural form: the methods of form finding and structural optimization, respectively. As they differ with their scope of application the paper discusses possibilities to merge the advantages of each methods with respect to a robust and general method for the structural optimization of tensile structures.
1 INTRODUCTION

Two different lines of research have developed which deal with the generation of structural shapes: the fields of “form finding” and “structural optimization”, respectively. The methods of form finding are usually restricted to tensile structures (cables and membranes) whereas the methods of structural optimization are far more general and can be applied to any kind of structure\(^1\)\(^3\). The differences of the two approaches, however, are not only the level of specialization but also their aims. Form finding methods are designed to determine structural shape from an inverse formulation of equilibrium. Again, two different methods are known: the soap film analogy for the generation of structures acting in pure tension which are related to minimal surfaces, and hanging models to generate structures in compression by inversion of tensile structures. In the context of structural optimization other criteria or even a combination of several criteria can be chosen to define the shape with respect to the special problem under consideration. The most general approach of structural optimization, however, has to be paid by numerical expense and is therefore often restricted to rather few degrees of freedom. On the other hand, numerical simulations of form finding methods are comparatively robust, are suited for many unknowns but are designed for very special applications. After a short summary of basic strategies the paper focuses on several ideas of how to generalize form finding strategies and to combine them with themselves or structural optimization to get the best out of each method.

2 FORM FINDING

2.1 Pre-stressed membranes – soap film analogy

Form finding of pre-stressed membranes means: find the shape of equilibrium of a membrane which has after pre-stressing a defined state of surface stress. For constant surface stress intensity the shape will be a minimal surface which is expressed by the “soap film analogy”. This is an inverse problem, since the stresses in the deformed configuration after pre-stressing are given and not – as usual in structural mechanics – induced by deformation of material. As a consequence, the solution exposes some problems which stem from the fact that a tangential deformation of the shape of equilibrium does not imply any force or stress. As a further consequence the consistent linearization of the weak form of equilibrium with respect to discretization parameters like finite element nodal displacements leads to a singular stiffness matrix. Therefore, several special methods have been developed which are designed to overcome that problem by modified approximation or regularization schemes, e.g. \(^4\)\(^-\)\(^8\).

Starting from a rigorous mechanical formulation a numerical procedure is developed for the form finding of minimal surfaces and pre-stressed membrane structures. The method is based on an iteratively adapted reference configuration which defines a regularization of the stiffness matrix and gives the method its name: the updated reference strategy. From a mathematical point of view the method can be identified as a method of numerical continuation. The regularization term is formulated with respect to the deformation...
independent reference configuration. It augments the weak form by blending with the continuation factor $\lambda$. The augmented weak form of equilibrium states as:

$$\delta W_\lambda = \lambda t \int_A \text{det}(\mathbf{F}) (\mathbf{F} \cdot \mathbf{F}^T) : \nabla \mathbf{F} dA + (1 - \lambda) \int_A (\mathbf{F} : \mathbf{S}) : \nabla \mathbf{F} dA = 0$$  \hspace{1cm} (1)

where $\mathbf{S}$, $\mathbf{S}$ and $\mathbf{F}$ are the Cauchy-, 2nd Piola-Kirchhoff stress tensors and the deformation gradient, respectively. The membrane thickness is denoted by $t$ and $A$ is the reference surface area. It is assumed that Cauchy and 2nd Piola-Kirchhoff are equivalently distributed, however, with respect to the actual and reference configuration, respectively.

Fig. 1: application of the “Updated Reference Strategy” for the form finding of pre-stressed membranes
The method can be applied to any finite element discretization of pre-stressed cable and membrane structures. It is very robust and reliable as is shown by many illustrative examples. Further analysis states that the well known force density method is the special case of applying the updated reference strategy to cable nets.

2.2 Hanging models – inverse principle

The hanging chain and its inverse is one of the oldest methods which are known to generate the shape of an arch which is free of bending, subjected only to compressive axial force. The method has been used intensively during the centuries, e.g. by Antoni Gaudi, to give one well known name among all the others. Extended to two directions to define the shape of shells the hanging model concept has been brought to perfection by Heinz Isler.9

The numerical simulation of hanging models by finite element methods is a standard task of non-linear analysis considering large displacements. Insofar the methods are well established and readily available. The examples in the middle an at the bottom of Table 2 show some results which are determined using membrane finite elements with an isotropic St.-Venant-Kirchhoff-material allowing for large displacements and small strains.

If we trace the procedure of minimizing strain energy of a structure originally subjected to bending we realize that first the shape is modified to reduce bending which is energetically very inefficient. Looking at hanging models from the optimization point of view the bending reduction is implicitly fulfilled just by taking a structure which is not able to resist bending and shear, e.g. the chain or textile cloths in two directions. The goal of hanging models is to perform the transition from a ‘bending structure’ to a ‘membrane structure’. A further systematical search for the optimal structure among the class of ‘membrane structures’ is not performed.
3 GENERALIZATION OF SOAP FILM ANALOGY FOR STRUCTURAL OPTIMIZATION

Let us consider a standard shape optimization problem of a plate or shell with respect to minimum structural volume $V$ of the undeformed structure and some constraints on stresses and displacements:

$$
V(a_p) \rightarrow \min; \quad p = 1, \ldots, \text{no. of variables}
$$

$$
g_k(a_p) = 0; \quad k = 1, \ldots, \text{no. of eq. constraints}
$$

$$
g_k(a_p) \leq 0; \quad k = 1, \ldots, \text{no. of ineq. constraints}
$$

Again, only variations normal to the structural contour are relevant for volume modifications. For that reason shape parameters $a_p$ have to be chosen carefully or have to be linked at the boundaries into 1D move directions with normal components on the boundary. That is standard practice in shape optimal design. Alternatively, the updated reference strategy can be generalized omitting linking schemes which are necessary only from the reasons of improper shape discretization.

Imagine the generation of a new shape as a transformation or “deformation” of the initial shape. Now, we can define three distinct configurations: (i) the initial shape of structural optimization, (ii) the actual shape of optimization which is the reference configuration of mechanical deformation, and (iii) the deformed shape. The geometry of the deformed configuration is the result of two subsequent shape transformations (Fig. 3).

![Fig. 3. The configurations of optimal structural design.](image-url)
By $\Phi$ we denote the shape gradient which maps the initial shape into the reference shape of mechanical deformation. The shape gradient is the counterpart of the deformation gradient which maps the reference configuration into the actual configuration of mechanical deformation. If we apply an equivalent idea as above we can determine the variation of the volume of the undeformed structure by:

$$
\delta V = \int \left( \Phi \cdot det(\Phi^{-1} \cdot \Phi^T) \right) \delta \Phi \, dV
$$

(3)

The term $(det(\Phi^{-1} \cdot \Phi^T))$ is interpreted as “pull back” of the metric tensor $G$ from the reference shape to the initial shape. That is equivalent to the definition of the 2nd Piola-Kirchhoff stresses as a pull back transformation of the Cauchy stresses. Now, assume $(det(\Phi^{-1} \cdot \Phi^T))$ as constant we get after discretization with respect to the design parameters $a_p$ and $a_q$ the second order derivatives

$$
V_{(\lambda),pq} = \int \left( \Phi_{,p} \cdot det(\Phi^{-1} \cdot \Phi^T) \right) \delta \Phi_{,q} \, dV
$$

(4)

which is equivalent to the regularization term in (1). This idea the basis of an iterative approximation scheme for the volume or weight minimization of structures.

Figure 4 shows a typical example. The contour of a connecting rod has to be optimized with respect to weight and restricting maximum allowable stresses. Variables are the position of the nodes defining the contour B-splines.

Fig. 4: Connecting rod, design model and iteration history.
4 COMBINING SOAP FILM ANALOGY AND HANGING MODEL

The following idea was motivated by a practical implication. Originally, the shape of a concrete pillar was designed with respect to a soap film experiment. For the experiment, surface stresses and cable forces were considered with respect to formal and esthetical aspects of the resulting shape. As the design loads had been applied, the final structure appeared to be a very poor design since it showed a lot of bending. Obviously, during the form finding procedure the final loads had not been considered. Insofar the result was not surprising. The solution to the problem was to mix soap film analogy and hanging model such to arrive at a compromise of architectural appearance and mechanical needs. The following figure shows several states of optimization with respect to different degrees of mixture and load combinations. The strength of the approach is that it still very robust and gives fast results. The weakness is obviously its heuristic nature and the fact that the behavior of the structure cannot be controlled with respect to any further optimal criterion.

Fig. 5 shows results of the mentioned concrete pillar problem from different views. Row 1 displays the results of the misleading soap film approach which gives the most slender columns and a smooth overall shape. Row 3 is the pure hanging model result with respect to projected constant area load and row 2 shows a mixture of the methods. The amount of pre-stress is chosen such that the wrinkles which appear in row 3 disappear. The final quality of the structure, however, cannot be systematically checked by this procedure neither can the instability behavior be considered during form finding.

Front view  Bird view  Cross section

soap film analogy

combination

hanging model

Fig. 5: Form finding of a concrete pillar by soap film analogy and hanging model
5 MERGING HANGING MODEL AND STRUCTURAL OPTIMIZATION

In classical terms structural optimization is defined by the combination of three models: the design model, the analysis model and the optimization model. All of them have their origin in different fields of research. This is the reason for the general power of the approach but also for its weakness. Since the structural geometry is defined in mathematical terms it cannot reflect anything of the mechanical behavior. If instead of CAGD design model the shape of a related hanging model is used, the shape is a priori better with respect to the mechanical behavior of the structure. The optimization variables are now the load case parameters \( p \) as e.g. load intensity and distribution. The optimization problem states now as:

\[
\begin{align*}
    f[x(p)] & \rightarrow \min \\
    \text{w.r.t.} \\
    g_j[x(p)] & \leq 0; \quad j=1\ldots \text{num. of ineq. constr.} \\
    h_j[x(p)] & = 0; \quad j=1\ldots \text{num. of eq. constr}
\end{align*}
\]

(5)

where \( f, g, h \) are objective functions, inequality and equality constraints, respectively. The shape \( x \) is a function of the load parameters \( p \) and is defined by the displacements of a related hanging model. They are determined from the weak form of equilibrium which is defined by shape generating load case \( p \) which acts on the hanging membrane:

\[
\delta w = \int_{V} \sigma(x) \delta \varepsilon(x) \, dv - \int_{\partial V} p \, \delta x = 0
\]

(6)

Fig. 6 displays the results of a simple, principal example and compares the mentioned approach with a classical shape optimization. The shape of a shell is to be optimized with respect to minimal strain energy. The left row shows the iteration history of the presented method. The shape is defined by the deflection of hanging model. It is set up by some parameters of a St.-Venant-Kirchhoff material and a shape generating load case which reflects a constant projected area load. The load intensity is the only shape variable. In contrast, the right row shows the result of the conventional approach. Here the shape is defined by the combination of four \( G_1 \)-continuously coupled Bézier-patches. Shape variables are the vertical coordinates. With respect to the symmetry of the structures six variables remain. The final results are comparable. The free edge shows negative curvature. The merged method converges quite smooth whereas the subsequent iteration steps of the conventional method reflects the numerical properties of the chosen optimization strategy which was a SQP-method. Neglecting the discretization error the shape of the hanging model has higher order continuity and is more smooth the “conventional” shape which by definition cannot be more than \( G_1 \).
<table>
<thead>
<tr>
<th>Merged method</th>
<th>Conventional method</th>
<th>iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Three load steps to determine start geometry</td>
<td>Start geometry</td>
<td>1</td>
</tr>
<tr>
<td>![Load factor p = 2.25 image]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>![Load factor p = 2.025 image]</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>![Load factor p = 1.7036 image]</td>
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<td></td>
<td>6</td>
</tr>
<tr>
<td>![Load factor p = 1.7930 final image]</td>
<td></td>
<td>8</td>
</tr>
</tbody>
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Fig. 6: merged vs. conventional optimization strategy

6 CONCLUSIONS

The two methods of form finding are presented: the numerical simulation of the soap film analogy and the hanging model. Both methods are very powerful regarding the numerical efficiency and the effectiveness. However, they are restricted to their narrow field of application. The paper discusses several principal possibilities to overcome this shortcoming by combination of form finding methods with structural optimization strategies and by combination of the two form finding methods themselves. The presented examples encourage to conclude that by these combination the strengths of the different methods can be bundled allowing for the development of alternative, powerful and robust optimization strategies.
7 REFERENCES


