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RECENT ACHIEVEMENTS ON THE USE OF PURE HYPERBOLIC CATTANEO-TYPE CONVECTION-DIFFUSION MODELS IN CFD

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ABSTRACT

From the theoretical point of view it seems quite obvious that diffusive processes can not take place with infinite velocity inside matter, since this fact would violate causality in the special relativity framework. Moreover, there is some experimental evidence in favor of this assertion [1]. However, the standard diffusion equation is based on Fick's law [2] in the case of mass transport, and on Fourier's law [3] in the case of heat conduction. Some other quite common constitutive laws of mathematical physics establish a similar time-independent functional relation between the flux and the spacial gradient of the state variable. This is the case of Ohm's law in electricity [4], of Darcy's law in fluid motion within porous media [5], etc. When combined with the adequate conservation principle (i.e.: the continuity equation), these constitutive laws lead to a pure parabolic mathematical model that predicts an infinite speed of propagation for the mass or energy being transported [6, 7]. For these reasons, the infinite speed paradox underlies many mathematical models that are frequently used in computational engineering and science. It happened to be ironical that the first comprehensive derivation for the standard diffusion equation was given by Einstein himself [8]. In fact, Einstein recognized this result to be physically unrealistic and pointed out the need to modify the diffusion equation in order to overcome the paradox.

This issue is just ignored in a number of applications in which the standard linear parabolic model is supposed to be accurate enough for practical purposes, although the simple idea of mass or energy being transported at infinite speed is disturbing. However, in some other applications it could be mandatory to take into account the wave nature of diffusive processes to perform accurate predictions [1, 9]. For the above stated reasons, a considerable effort has been historically devoted to remove the infinite speed paradox from the standard diffusion equation. Basically, two kind of approaches have been presented: on one side, some non-linear constitutive laws have been proposed, thus keeping the parabolic nature of the model [10]; on the other, some time-dependent constitutive laws have been proposed, what leads to a new class of hyperbolic-type diffusion models, as the one first proposed by Cattaneo [11].

On the other hand, diffusion is not the only transport mechanism in CFD problems, given that convection is always present (and frequently predominant). Since the speed of propagation for the mass or energy being transported by convection is always finite, a characteristic scale of time for the combined convection-diffusion transport mechanisms does not exist. Therefore, no stability conditions emanate from the essence of the problem and it looks like this fact is deeply related to the spurious oscillations that occur in the numerical treatment of the standard convection-diffusion equation. A number of stabilization techniques have been proposed (artificial diffusion, SUPG, TG, GLS, FIC, etc.) [12]. Essentially, all these techniques get to stabilize the problem by adding a certain amount of diffusion.

At present, methods like FV (Finite Volume) and DG (discontinuous Galerkin) are attracting increasing attention [12]. In these techniques emphasis is essentially focused towards satisfying the conservation principle, what requires tracking adequately the value of the flux. Anyway, stabilization is a major issue in CFD, since it is required in most of the problems. Furthermore, it is widely accepted that stabilizing properly the transport problem is an important keystone in the way to get a robust stabilizing procedure for the whole set of the Navier-Stokes equations. In the past, the study of hyperbolic diffusion had been limited to pure-diffusive problems [11]. Recently it has been proposed a generalization of the hyperbolic diffusion equation that can also be applied to convective cases [6, 7, 13, 14, 15].

In this paper we present the pure hyperbolic convection-diffusion model that has been proposed by the authors. The model yields stable results in convection dominated situations while it gives rise to simple stability conditions with physical meaning. On the other hand, the model also predicts new phenomena (diffusion waves) while the way to define properly the correct boundary conditions is made clear (being different for subcritical and supercritical flows) and the infinite speed paradox is overcome. A comprehensive study of the 1D steady-state convection-diffusion equation is performed. Thus, we can analyze whether the infinite speed paradox (inherent to the linear parabolic model) contributes or not to the non-physical oscillations that appear in convection dominated flows when the standard linear parabolic model is discretized with centered methods. Since the flux becomes a main variable of the problem, the proposed hyperbolic model is specially suitable for the discontinuous Galerkin method. Thus, a high-order upwind DG formulation is finally developed and applied to several application problems.

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