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A thermoelectromechanical model for the analysis of smart structures

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ABSTRACT

Due to the increasing interest in integrating piezoelectric materials into structures for static and dynamic control, many models, describing their behaviour, have been developed with varying levels of simplification. The present paper deals with the fully coupled thermoelectromechanical analysis of piezolaminated structures in the sense mentioned above, including the geometrical and thermal non-linear effects. A thermodynamically consistent continuum mechanics based framework is developed, which includes the conservation of mass, linear and angular momentum and the conservation of energy. The second principle of thermodynamics is used to derive the restrictions for the constitutive equations using the Coleman-Noll analysis approach. The resulting set of equations is more general and valid for a wider class of problems than most models published in literature. The interested reader is referred to [1] and [2].

Due to the possibly large deformations it is important to distinguish between spatial and material quantities, which refer respectively to either the deformed or the initial configuration. Spatial quantities and the tensor elements expressed in the deformed base vector system $\bar{\boldsymbol{g}}_i$ or $\bar{\boldsymbol{g}}^i$, are denoted with a bar on top. Further, the investigated object is parameterised by Θ^1, Θ^2 and Θ^3, resulting in the base vectors $\boldsymbol{g}_i = \boldsymbol{r}_{,i}$, where $\boldsymbol{r}$ denotes the position vector and $[\,]_{,i}$ the partial derivative along Θ^i.
Due to the interaction between the polarisation and the electric field, body forces and moments emerge. With the definition of the polarisation $\bar{\boldsymbol{p}}\, d\bar{V} = dQ\, d\boldsymbol{k}$, it follows that

$$^{e}\bar{\boldsymbol{f}} = \left(\bar{\boldsymbol{p}} \cdot \bar{\boldsymbol{\nabla}}\right) \bar{\boldsymbol{\mathcal{E}}} \qquad \text{and} \qquad ^{e}\bar{\boldsymbol{m}} = \bar{\boldsymbol{p}} \times \bar{\boldsymbol{\mathcal{E}}}, \tag{1}$$

which denote the ponderomotoric force and the electrically induced body moment density. The *momentum conservation* can now be written as

$$\bar{\boldsymbol{\nabla}}\boldsymbol{\sigma} + {}^{e}\bar{\boldsymbol{f}} + \bar{\rho}\left(\bar{\boldsymbol{b}} - \ddot{\boldsymbol{v}}\right) = 0, \tag{2}$$

where $\boldsymbol{\sigma}$ includes the Cauchy and Maxwell stress tensor. The mass density $\bar{\rho}$, is determined by the *mass conservation* law

$$\bar{\rho}\sqrt{\bar{g}} = \rho\sqrt{g} \qquad \text{or} \qquad \left(\dot{\bar{\rho}} + \bar{\rho}\bar{\boldsymbol{\nabla}} \cdot \dot{\boldsymbol{v}}\right) d\bar{V} = 0. \tag{3}$$

Further, $\bar{\boldsymbol{b}}$, $\boldsymbol{\upsilon}$ and $\bar{g}$ are the spatial body forces, the displacement vector and the deformed metric determinant $|\bar{\boldsymbol{g}}|$, respectively. The symmetry of the Cauchy stress tensor is usually proven by the angular momentum balance, which is now disrupted by the body moment ${}^{e}\bar{\boldsymbol{m}}$. From the *angular momentum conservation* it follows that the skew-symmetric part of the stress tensor equals

$$\text{skew}\,(\boldsymbol{\sigma}) = -\text{skew}\left(\bar{\boldsymbol{p}} \otimes \bar{\boldsymbol{\mathcal{E}}}\right). \tag{4}$$

Two additional field equations have to be considered. Namely, *Gauss' law* and the heat energy equation

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{D}} = 0, \qquad \dot{Q} = \left(\bar{\rho}\,\bar{h} - \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{h}}\right) d\bar{V} \qquad \text{and} \qquad \bar{\boldsymbol{D}} = \varepsilon_0 \bar{\boldsymbol{\mathcal{E}}} + \bar{\boldsymbol{p}}, \tag{5}$$

where $\bar{D}$, $\dot{Q}$, $\bar{h}$ and $\bar{\boldsymbol{h}}$ denote dielectric displacement, the heat energy rate, the inner heat source and the heat flux, respectively.

The mechanical and electrical work rates are formulated as

$${}^{m}\dot{W} = \boldsymbol{\sigma}^{\mathrm{T}} : \left(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}\right) d\bar{V} + \left(\bar{\boldsymbol{\nabla}}\boldsymbol{\sigma} + \bar{\rho}\left(\bar{\boldsymbol{b}} - \ddot{\boldsymbol{\upsilon}}\right)\right) \cdot \dot{\boldsymbol{\upsilon}}\, d\bar{V} \qquad \text{and} \qquad {}^{e}\dot{W} = \left({}^{e}\bar{\boldsymbol{f}} \cdot \dot{\boldsymbol{\upsilon}} + \bar{\rho}\,\bar{\boldsymbol{\mathcal{E}}} \cdot \dot{\bar{\boldsymbol{\pi}}}\right) d\bar{V}, \tag{6}$$

which, under consideration of (2), sum up to

$$\dot{W} = \left(\boldsymbol{\sigma}^{\mathrm{T}} : \left(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}\right) + \bar{\rho}\,\bar{\boldsymbol{\mathcal{E}}} \cdot \dot{\bar{\boldsymbol{\pi}}}\right) d\bar{V}, \qquad \text{where} \qquad \bar{\boldsymbol{p}} = \bar{\rho}\,\bar{\boldsymbol{\pi}}. \tag{7}$$

After introducing a quadratic Gibbs free energy functional and performing the Coleman-Noll analysis, taking into account the Meixner inequality, the following constitutive laws evolve

$$\begin{aligned} S^{rs} &= \frac{1}{2}\left(\mathbb{C}^{rsij} + \mathbb{C}^{ijrs}\right) E_{ij} - \mathbb{E}^{irs}\mathcal{E}_i - \mathcal{B}^{rs}\left(\bar{T} - T\right) - p^r\,\mathcal{E}_k\,\bar{g}^{ks} \\ D^k &= \frac{1}{2}\left(\mathcal{D}^{ik} + \mathcal{D}^{ki} + 2|\boldsymbol{F}|\varepsilon_o\,\bar{g}^{ki}\right)\mathcal{E}_i + \mathbb{E}^{kij}E_{ij} + \lambda^k\left(\bar{T} - T\right) \\ s &= c_{E,\mathcal{E}}\ln\frac{\bar{T}}{T} + \frac{1}{\rho}\mathcal{B}^{ij}E_{ij} + \frac{1}{\rho}\lambda^i\mathcal{E}_i \quad \text{and} \quad h^i = \mathcal{L}^{ij}\bar{T}_{,j}, \end{aligned} \tag{8}$$

where $\boldsymbol{S}$ denotes the second Piola-Kirchhoff stress tensor.

Multiplying (2) scalar with $\delta\boldsymbol{\upsilon}$, $(5)_1$ and $(5)_2$ with $\delta\varphi$ and $\delta\bar{T}$, where φ denotes the electric potential, and integrating the results over the deformed volume, assuming no entropy production, and applying Gauss' divergence theorem, ultimately leads to the following weak formulations

$$\int_V \left(\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{\mathrm{T}}\right) : \left(\delta\boldsymbol{F}\boldsymbol{F}^{-1}\right) dV - \int_V \left(\left(\boldsymbol{p} \cdot \boldsymbol{\nabla}\right)\left(\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{\mathcal{E}}\right) + \rho\left(\boldsymbol{b} - \ddot{\boldsymbol{\upsilon}}\right)\right) \cdot \delta\boldsymbol{\upsilon}\, dV - \int_{\partial\bar{V}} \left(\boldsymbol{\sigma}\delta\boldsymbol{\upsilon}\right) \cdot \bar{\boldsymbol{n}}\, d\bar{A} = 0 \tag{9}$$

$$\int_V \boldsymbol{D} \cdot \delta\boldsymbol{\mathcal{E}}\, dV - \int_{\partial V} \boldsymbol{n} \cdot \boldsymbol{D}\,\delta\varphi\, dA = 0 \tag{10}$$

$$\int_V \left(\boldsymbol{h} \cdot \left(\boldsymbol{\nabla}\delta\bar{T}\right) - \left(\bar{T}\dot{s} - \rho h\right)\delta\bar{T}\right) dV - \int_{\partial V} \boldsymbol{n} \cdot \boldsymbol{h}\,\delta\bar{T}\, dA = 0, \tag{11}$$

where $\bar{\boldsymbol{n}}$ and $\boldsymbol{n}$ denote the surface normal vector in the deformed and the initial configuration, respectively, and $\bar{\boldsymbol{\mathcal{E}}} = -\bar{\boldsymbol{\nabla}}\varphi$.

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