

## Implicit Time Integration Algorithms For High-Order Methods On Unstructured Tetrahedral Grids With $p$ –Multigrid Strategy

\* M. Parsani<sup>1</sup>, K. Van den Abeele<sup>2</sup> and C. Lacor<sup>3</sup>

<sup>1,2,3</sup> Vrije Universiteit Brussel, Department of Mechanical Engineering,  
Fluid Dynamics and Thermodynamics Research Group  
Triomflaan 43, B-1050 Brussels, Belgium

<sup>1</sup> mparsani@vub.ac.be, <sup>2</sup> kvdabeel@vub.ac.be, <sup>3</sup> chris.lacor@vub.ac.be

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### ABSTRACT

Computational fluid dynamics has undergone tremendous development as a discipline for three decades. This has been made possible by progresses in many fronts, including numerical algorithms, grid generation and adaptation, as well as the dramatic increase in computer CPU speeds. Moreover, in recent years there is a growing interest in multidisciplinary CFD applications such as aeroacoustics, turbulent combustion and biomedics. These applications are characterized by large scale disparities in the flow fields features, where an adequate and accurate discrete representation is needed. In addition, since CFD is more and more used as an industrial design and analysis tool the applications are often of industrial relevance, requiring unstructured grids for efficient meshing. High accuracy must therefore be achieved on such unstructured grids. Discontinuous Galerkin schemes, Spectral Volume and Spectral Difference schemes are especially suited for these purposes.

While relatively good computational efficiency has been attained for the Euler equations, there are still significant challenges remaining for the NavierStokes equations. A major obstacle in achieving such a goal is the geometrical stiffness of the discrete NavierStokes equations caused by the requirement to adequately resolve viscous boundary layers and unsteady propagating vortices with an economical distribution of grid points. In fact, when spatial numerical methods are combined with classical time integration algorithms, such as explicit Runge-Kutta solvers, there is a restrictive CFL condition, in particular when the cell aspect ratio is high [1]. Moreover, high-order methods are restricted to a small CFL number which can take an excessive amount of CPU-time to reach a state-steady solution with explicit solvers. Hence, with high-order methods, it is possible to achieve low error levels more efficiently than with traditional schemes, but efficient implicit solution approaches are necessary to fully fulfill this potential. In the present contribution an implicit LU-SGS approach is combined for the first time with the Spectral Volume approach and with a  $p$ –Multigrid strategy [2].

The spectral volume (SV) method consists of two basic components [3]. One is the data reconstruction, and the other is the (approximate) Riemann solver. In this method in order to perform a high-order polynomial reconstruction, instead of using a (large) stencil of neighboring cells, an unstructured grid cell,

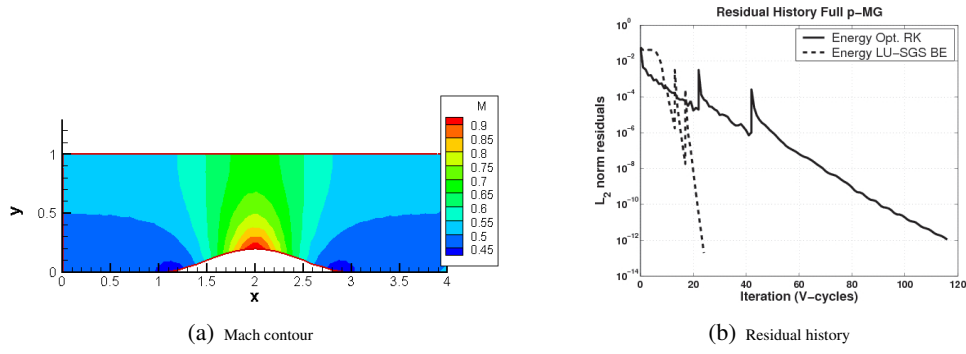


Figure 1: Mach contour (a) and convergence history with LU-SGS with Backward Euler difference and optimized explicit R-K smoothers for the inviscid-sine-bump flow (b);  $3^{rd}$ -order SV method; full  $p$ -multigrid calculation.

called a spectral volume (SV), is partitioned into a structured set of sub-cells, where the cell-averaged solutions are the degrees-of-freedom (DOFs). These DOFs are used to reconstruct a high-order polynomial inside the SV and the numerical flux can be computed with an approximate Riemann solver. The general LU-SGS formulation with Backward Euler difference applied to the spectral volume for the conserved variables  $\mathbf{Q}$  is indicated as follows

$$\left( \frac{I}{\Delta t} - \frac{\partial \mathbf{R}_{sv}}{\partial \mathbf{Q}_{sv}} \right) \Delta \widetilde{\mathbf{Q}}_{sv}^{(k+1)} = \mathbf{R}_{sv}(\mathbf{Q}^*) - \frac{I}{\Delta t} \Delta \mathbf{Q}_{sv}^{(k)} \quad (1)$$

where  $\mathbf{R}_{sv}$  is the residual. The indices  $k$  and  $k + 1$  refer to the previous and current SGS sweeps, respectively, and the symbol  $(. *)$  indicates the most recent solution in the neighbour cells.

As can be seen in Table 1 the LU-SGS with Backward Euler difference needs much few iterations and much less CPU-time, being more efficient than the R-K solvers.

In the full paper the efficiency of the method will also be discussed for 2D Navier-Stokes cases, showing its efficiency and stability.

Computation	CPU-time [s] (single grid)	CPU-time [s] (full $p$ -multigrid)
Optimized Explicit R-K	$\approx 258000$ (after $10^5$ iterations)	$\approx 1328$
LU-SGS with BE	$\approx 1801$	$\approx 414$

Table 1: CPU-times for the residual  $L_2$ -norm to drop 10 orders of magnitude;  $3^{rd}$ -order SV method.

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